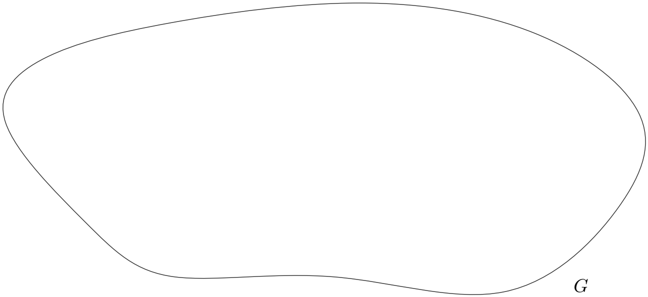
The combinatorics of monadic stability, monadic dependence, and related notions

Algomanet, Warsaw, September 9-13, 2024

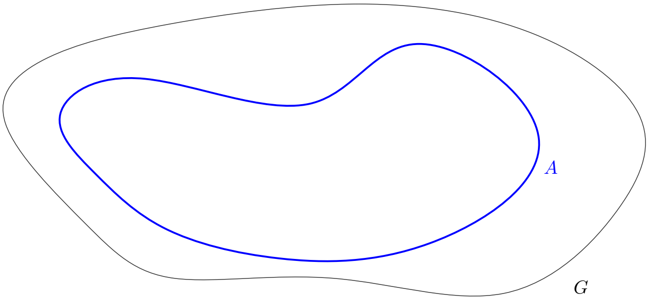
Jan Dreier, TU Wien

Thanks to Nikolas Mählmann for the following slides and figures.

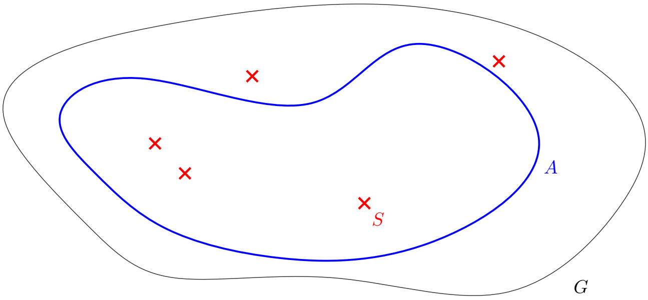
Characterizing Nowhere Denseness: Deletion-Flatness



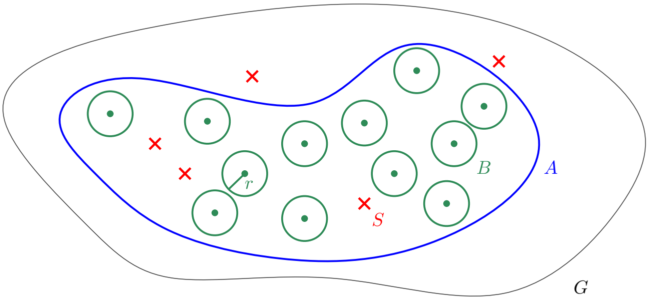
Characterizing Nowhere Denseness: Deletion-Flatness



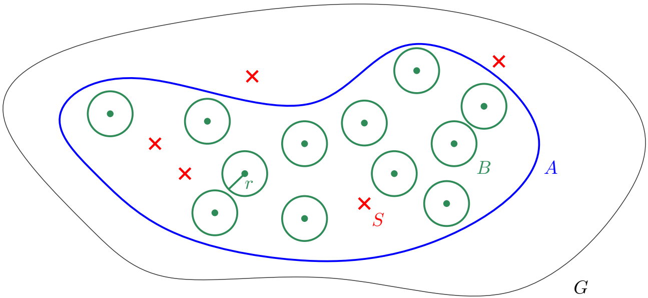
Characterizing Nowhere Denseness: Deletion-Flatness



Characterizing Nowhere Denseness: Deletion-Flatness



Characterizing Nowhere Denseness: Deletion-Flatness



Deletion-Flatness (Uniform Quasi-Wideness) (slightly informal)

A class $\mathcal{C}$ is *deletion-flat* if for every radius r, in every large set A we find a still-large set B that is r-independent after removing a set S of constantly many vertices.

Characterizing Nowhere Denseness: Deletion-Flatness

Theorem [Něsetřil, Ossona de Mendez, 2011]

A class $\mathcal{C}$ is deletion-flat if and only if it is nowhere dense.

Characterizing Nowhere Denseness: Deletion-Flatness

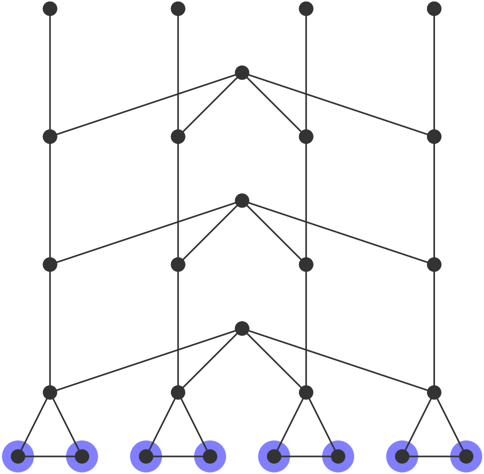
Theorem [Něsetřil, Ossona de Mendez, 2011]

A class $\mathcal{C}$ is deletion-flat if and only if it is nowhere dense.

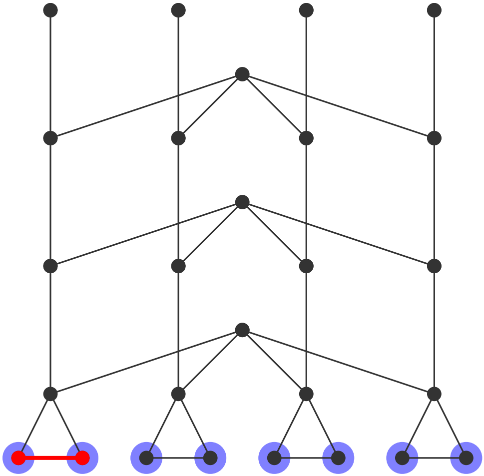
Deletion-Flatness (formal)

A class $\mathcal{C}$ is *deletion-flat* if for every radius r there exists a constant k such that in every large set $A \subseteq V(G)$ with $G \in \mathcal{C}$ one can find a still-large set B, $|B| \geq U_{r,\mathcal{C}}(|A|)$ with the follwing property. After removing at most k vertices,
$\forall b_1, b_2 \in B \;\; N_r(b_1) \cap N_r(b_2) = \emptyset$.

Deletion-Flatness: Example

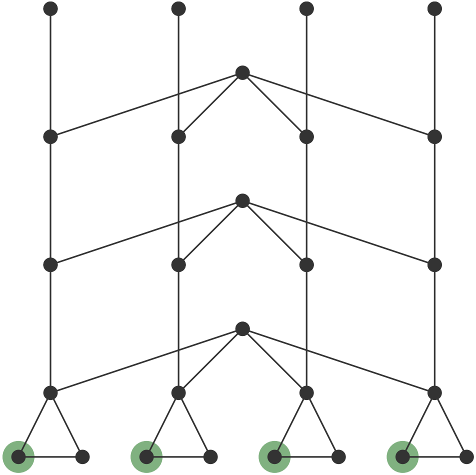


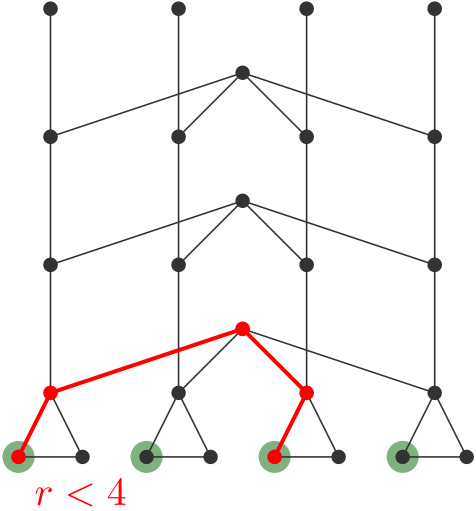
Deletion-Flatness: Example



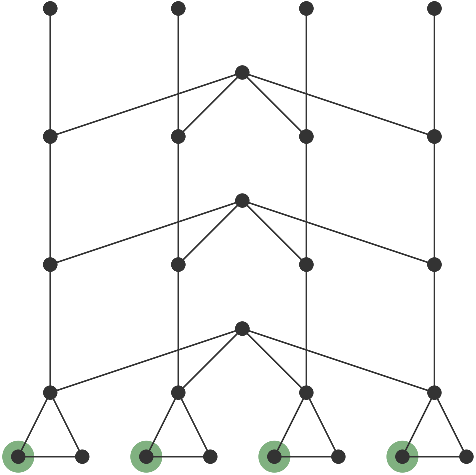
$r < 1$

Deletion-Flatness: Example

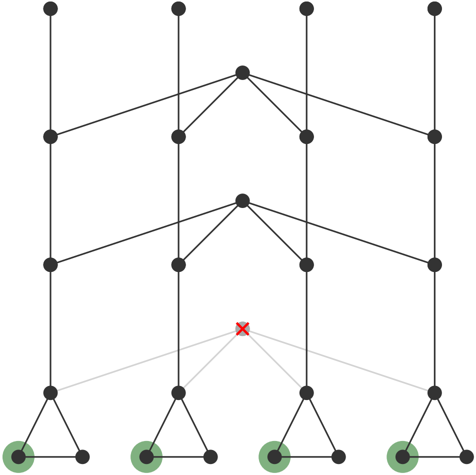



$r < 4$

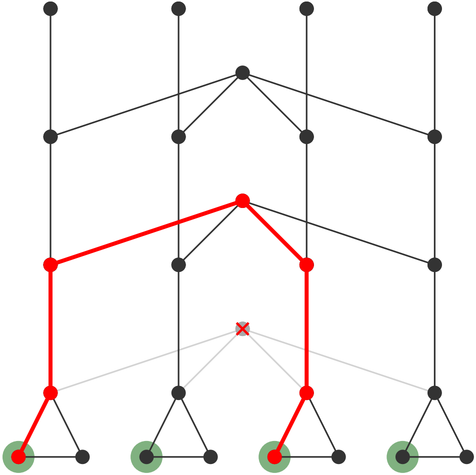
Deletion-Flatness: Example



Deletion-Flatness: Example

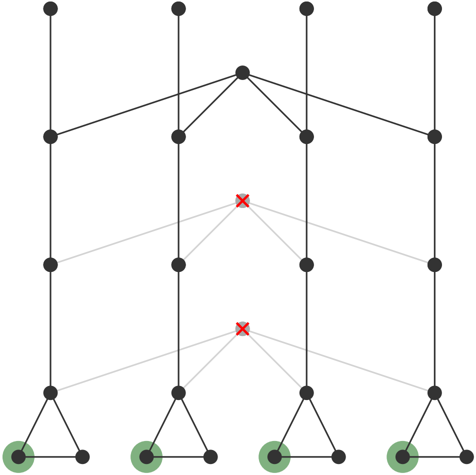


Deletion-Flatness: Example



$r < 6$

Deletion-Flatness: Example



What is it good for?

deltion-flatness $\Rightarrow$ Splitter game

Dense Graphs

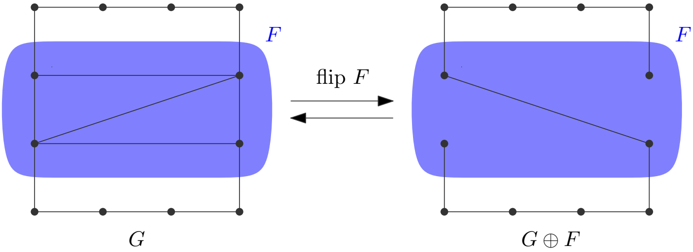
Can deletion-flatness handle cliques?

Dense Graphs

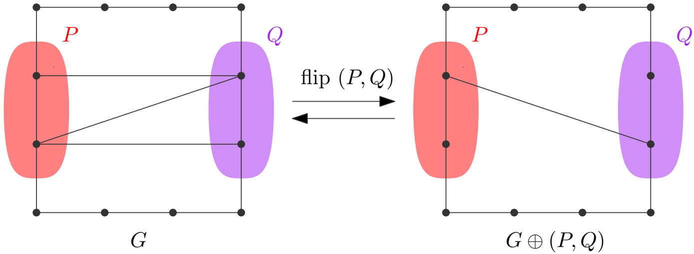
Can deletion-flatness handle cliques? How can we lift this notion to dense graphs?

Flips

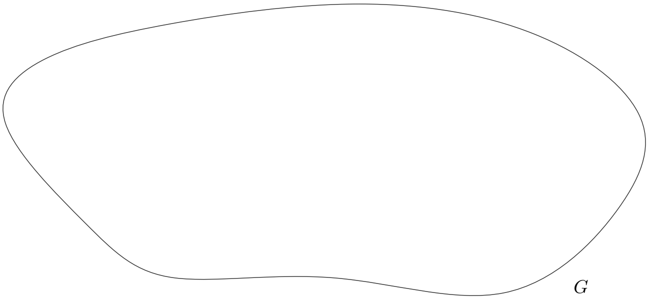
Denote by $G \oplus F$ the graph obtained from G by complementing edges between pairs of vertices from F.



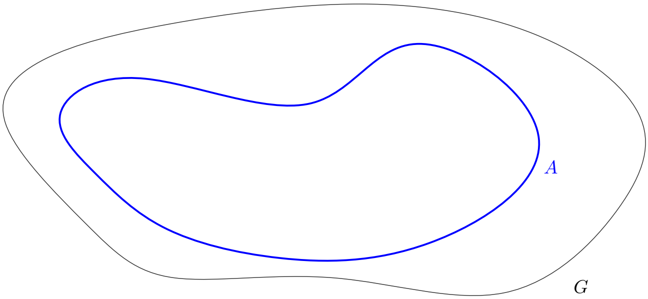
Denote by $G \oplus (P, Q)$ the graph obtained from G by complementing edges between pairs of vertices from $P \times Q$.



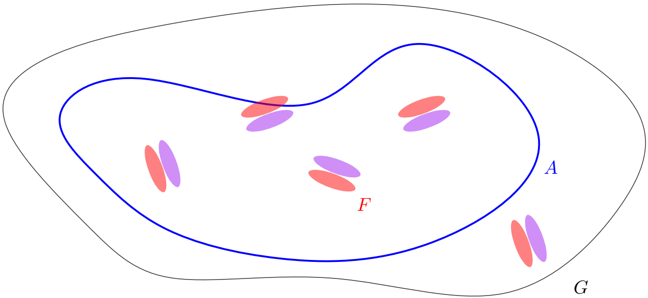
Characterizing Monadic Stability: Flip-Flatness



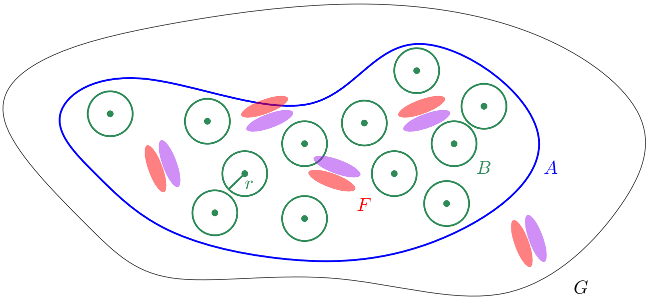
Characterizing Monadic Stability: Flip-Flatness



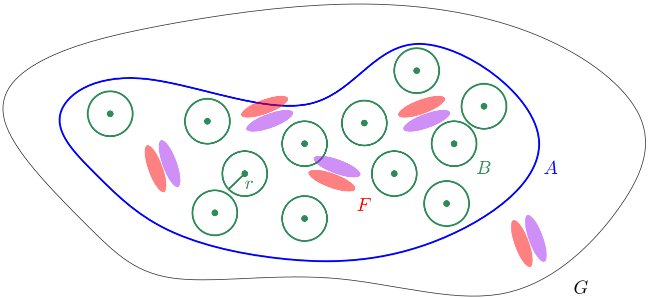
Characterizing Monadic Stability: Flip-Flatness



Characterizing Monadic Stability: Flip-Flatness



Characterizing Monadic Stability: Flip-Flatness



Flip-Flatness (slightly informal) [Gajarský, Kreutzer]

A class $\mathcal{C}$ is *flip-flat* if for every radius r, in every large set A we find a still-large set B that is r-independent after performing a set F of constantly many flips.

Characterizing Nowhere Denseness: Deletion-Flatness

Theorem [D, Mählmann, Siebertz, Toruńczyk, 2022]

A class $\mathcal{C}$ is flip-flat if and only if it is monadically stable.

Characterizing Nowhere Denseness: Deletion-Flatness

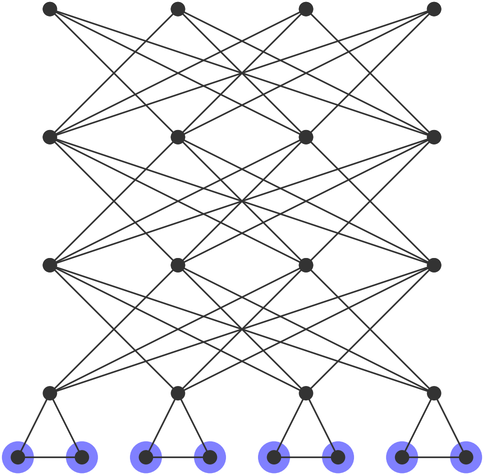
Theorem [D, Mählmann, Siebertz, Toruńczyk, 2022]

A class $\mathcal{C}$ is flip-flat if and only if it is monadically stable.

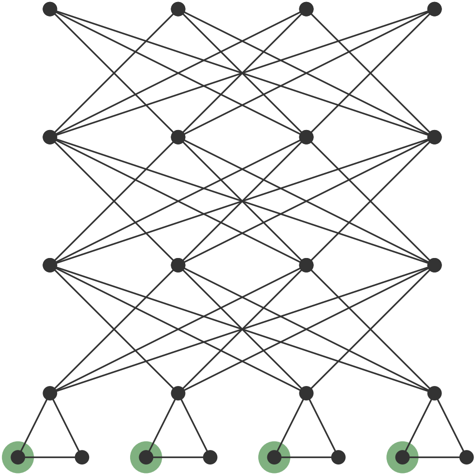
Flip-Flatness (formal)

A class $\mathcal{C}$ is *flip-flat* if for every radius r there exists a constant k such that in every large set $A \subseteq V(G)$ with $G \in \mathcal{C}$ one can find a still-large set B, $|B| \geq U_{r,\mathcal{C}}(|A|)$ with the follwing property. After performing at most k flips, $\forall b_1, b_2 \in B \;\; N_r(b_1) \cap N_r(b_2) = \emptyset$.

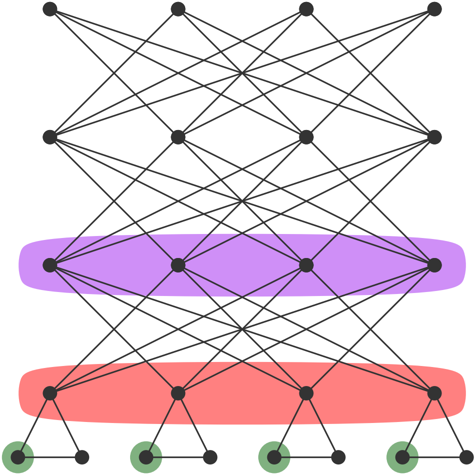
Flip-Flatness: Example



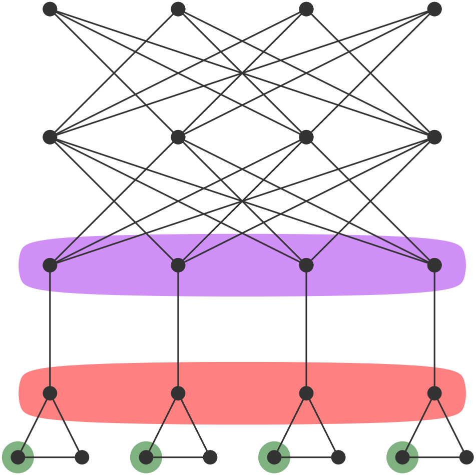
Flip-Flatness: Example



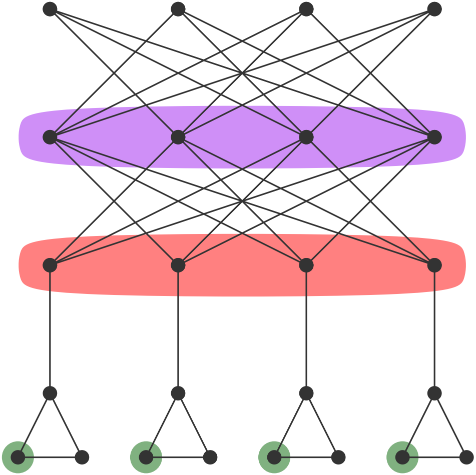
Flip-Flatness: Example



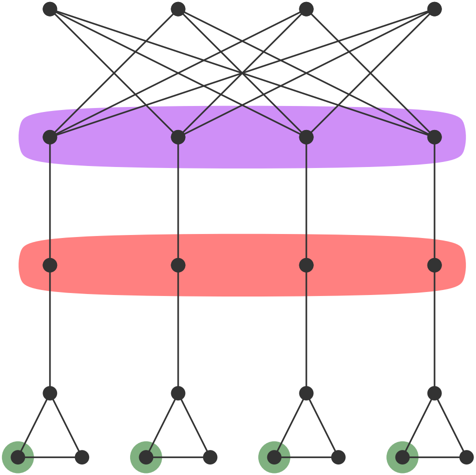
Flip-Flatness: Example



Flip-Flatness: Example



Flip-Flatness: Example



What is it good for?

flip-flatness $\Rightarrow$ Flipper game

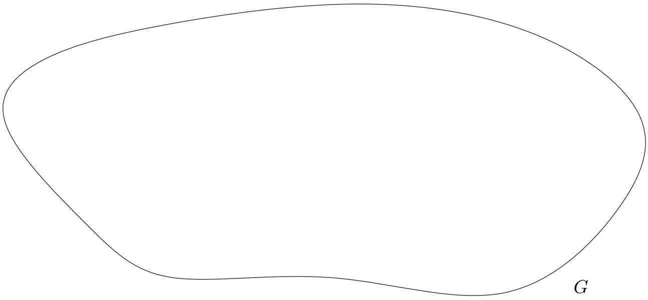
Monadic Dependence

Can flip-flatness handle ladders?

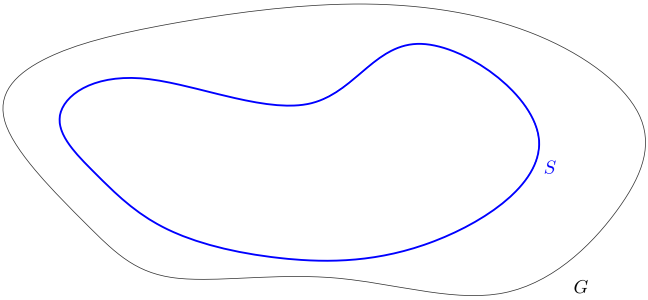
Monadic Dependence

Can flip-flatness handle ladders? How can we lift the notion to monadically dependent classes?

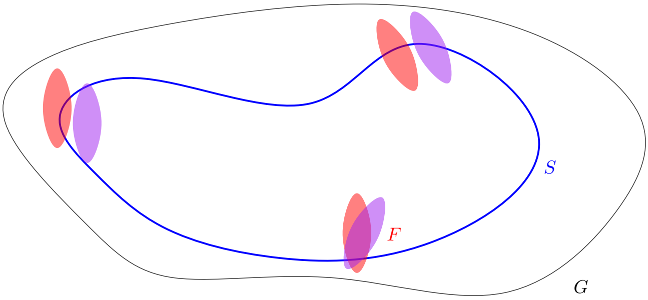
Characterizing Monadic Dependence: Flip-Breakability



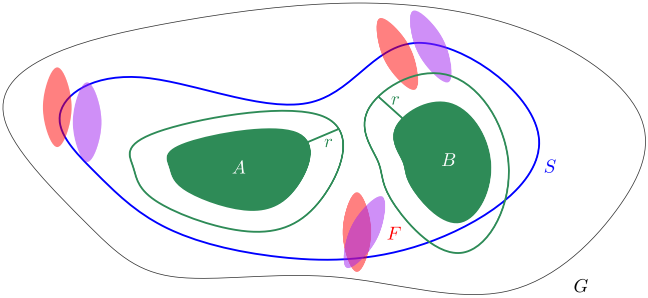
Characterizing Monadic Dependence: Flip-Breakability



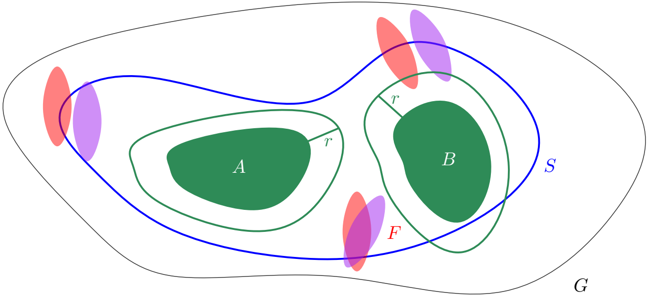
Characterizing Monadic Dependence: Flip-Breakability



Characterizing Monadic Dependence: Flip-Breakability

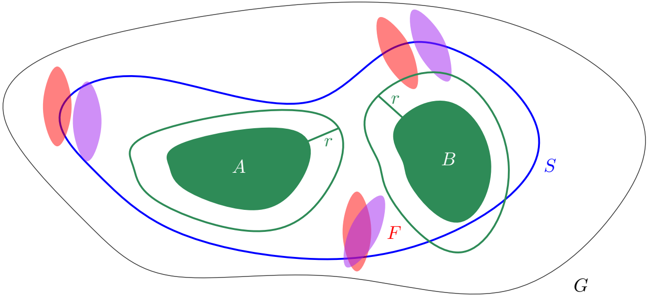


Characterizing Monadic Dependence: Flip-Breakability



Flip-Breakability (slightly informal)

A class $\mathcal{C}$ is *flip-breakable* if for every radius r, in every large set S we find two large sets A and B such that after k flips, $N_r(A) \cap N_r(B) = \varnothing$.



Flip-Breakability (slightly informal)

A class $\mathcal{C}$ is *flip-breakable* if for every radius r, in every large set S we find two large sets A and B such that after k flips, $N_r(A) \cap N_r(B) = \varnothing$.

Characterizing Monadic Dependence: Flip-Breakability

Theorem [Dreier, Mählmann, Toruńczyk, 2024]

A class $\mathcal{C}$ is flip-breakable if and only if it is monadically dependent.

Characterizing Monadic Dependence: Flip-Breakability

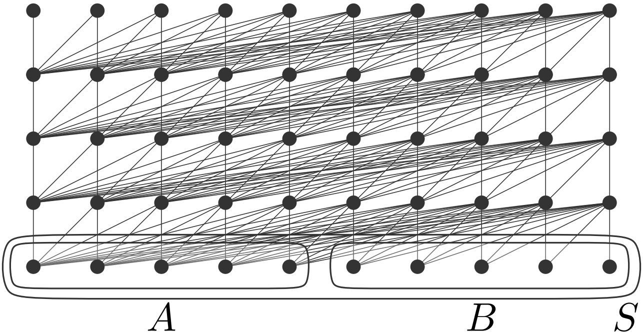
Theorem [Dreier, Mählmann, Toruńczyk, 2024]

A class $\mathcal{C}$ is flip-breakable if and only if it is monadically dependent.

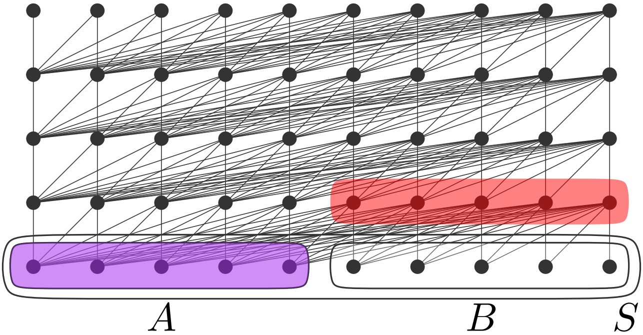
Flip-Breakability (formal)

A class $\mathcal{C}$ is *flip-breakable* if for every radius r there exists a constant k such that in every large set $S \subseteq V(G)$ with $G \in \mathcal{C}$ one can find a still-large sets A, B, $|A|, |B| \geq U_{r,\mathcal{C}}(|S|)$ with the follwing property. After performing at most k flips, $N_r(A) \cap N_r(B) = \emptyset$.

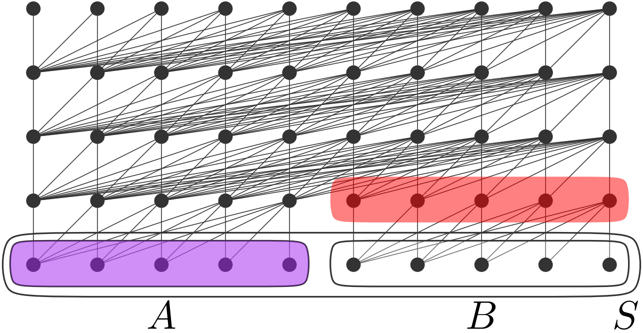
Flip-Breakability: Example



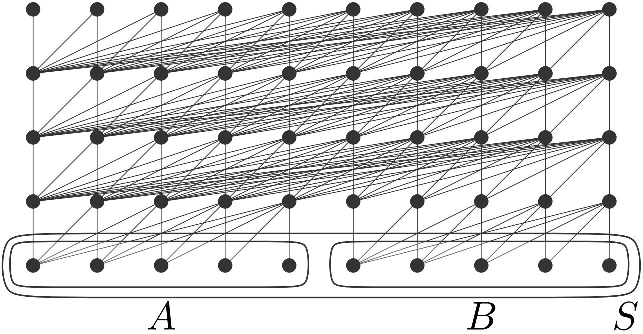
Flip-Breakability: Example



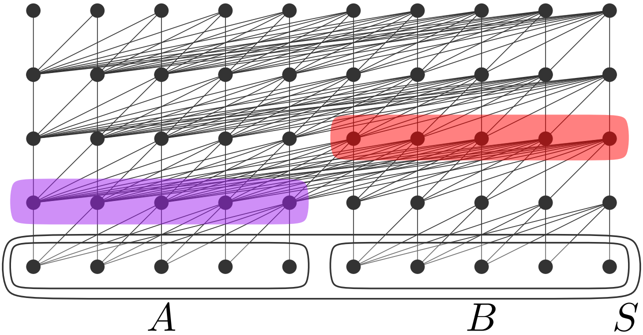
Flip-Breakability: Example



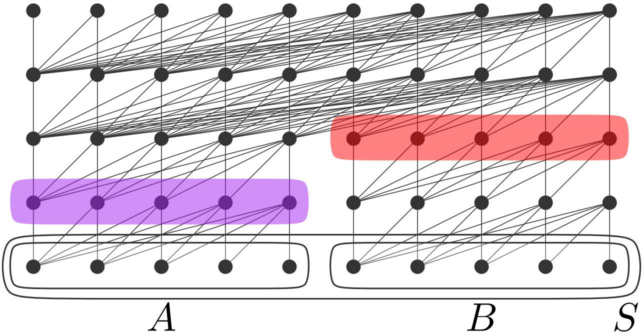
Flip-Breakability: Example



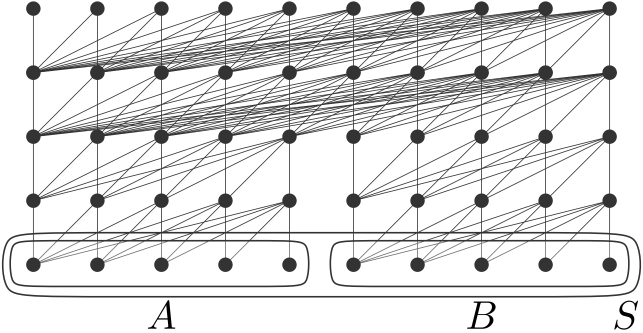
Flip-Breakability: Example



Flip-Breakability: Example



Flip-Breakability: Example



Variants of Flip-Breakability

1. We modify a graph using either flips or vertex deletions.

Variants of Flip-Breakability

1. We modify a graph using either flips or vertex deletions.
2. We demand our resulting set is either flat or broken.
   flat: pairwise separated; broken: separated into two large sets

Variants of Flip-Breakability

1. We modify a graph using either flips or vertex deletions.
2. We demand our resulting set is either flat or broken.
   flat: pairwise separated; broken: separated into two large sets
3. Separation means either distance-r or distance-∞.

Variants of Flip-Breakability

1. We modify a graph using either flips or vertex deletions.
2. We demand our resulting set is either flat or broken.
   flat: pairwise separated; broken: separated into two large sets
3. Separation means either distance-r or distance-∞.

		flatness	breakability
dist-r	flip-	**monadic stability**	**mon. dependence**
	deletion-	**nowhere denseness**	
dist-∞	flip-		
	deletion-		

Variants of Flip-Breakability

1. We modify a graph using either flips or vertex deletions.
2. We demand our resulting set is either flat or broken.
   flat: pairwise separated; broken: separated into two large sets
3. Separation means either distance-r or distance-∞.

		flatness	breakability
dist-r	flip-	**monadic stability**	**mon. dependence**
	deletion-	**nowhere denseness**	**nowhere denseness**
dist-∞	flip-		
	deletion-		

Variants of Flip-Breakability

1. We modify a graph using either flips or vertex deletions.
2. We demand our resulting set is either flat or broken.
   flat: pairwise separated; broken: separated into two large sets
3. Separation means either distance-r or distance-∞.

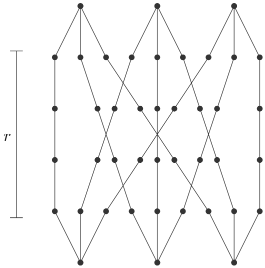
		flatness	breakability
dist-r	flip-	**monadic stability**	**mon. dependence**
	deletion-	**nowhere denseness**	**nowhere denseness**
dist-∞	flip-	**bd. shrubdepth**	**bd. cliquewidth**
	deletion-		

Variants of Flip-Breakability

1. We modify a graph using either flips or vertex deletions.
2. We demand our resulting set is either flat or broken.
   flat: pairwise separated; broken: separated into two large sets
3. Separation means either distance-r or distance-∞.

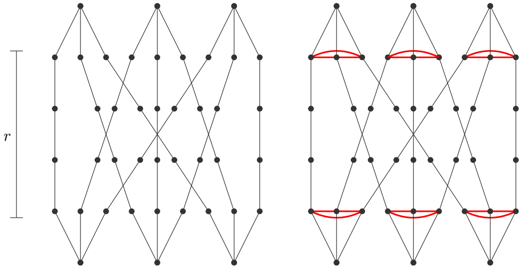
		flatness	breakability
dist-r	flip-	**monadic stability**	**mon. dependence**
	deletion-	**nowhere denseness**	**nowhere denseness**
dist-∞	flip-	**bd. shrubdepth**	**bd. cliquewidth**
	deletion-	**bd. treedepth**	**bd. treewidth**

Monadic Dependence and Forbidden Induced Subgraphs



star r-crossing
$= r$-subdivided biclique

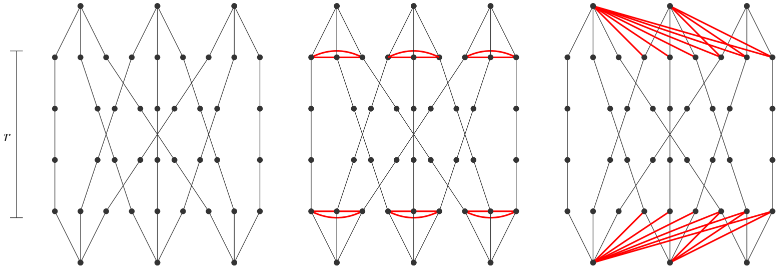
Monadic Dependence and Forbidden Induced Subgraphs



star r-crossing
$= r$-subdivided biclique

clique r-crossing

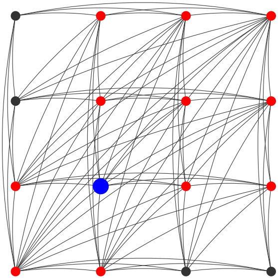
Monadic Dependence and Forbidden Induced Subgraphs



star r-crossing
$= r$-subdivided biclique

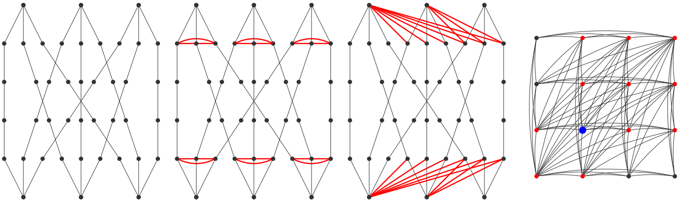
clique r-crossing

half-graph r-crossing



comparability grid

Monadic Dependence and Forbidden Induced Subgraphs

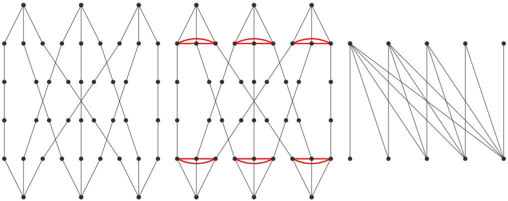


Theorem [Dreier, Mählmann, Toruńczyk, 2024]

Let $\mathcal{C}$ be a graph class. Then $\mathcal{C}$ is monadically dependent if and only if for every $r \geq 1$ there exists $k \in \mathbb{N}$ such $\mathcal{C}$ excludes as induced subgraphs

- all layerwise flipped star r-crossings of order k,
- all layerwise flipped clique r-crossings of order k,
- all layerwise flipped half-graph r-crossings of order k,
- the comparability grid of order k.

Subgraphs

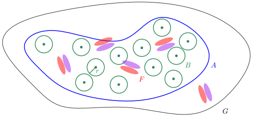
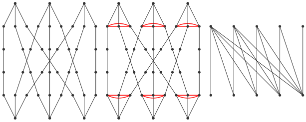
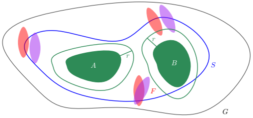
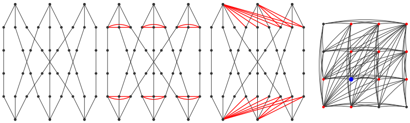


Theorem [Dreier, Eleftheriadis, Mählmann, McCarty, Pilipczuk, Toruńczyk, 2023]

Let $\mathcal{C}$ be a graph class. Then $\mathcal{C}$ is monadically stable if and only if for every $r \geq 1$ there exists $k \in \mathbb{N}$ such $\mathcal{C}$ excludes as induced subgraphs

- all layerwise flipped star r-crossings of order k,
- all layerwise flipped clique r-crossings of order k,
- all semi-induced halfgraphs of order k

Summary

	structure	non-structure
m. stable		
m. dep.		

Exercises

Let $\mathcal{C}$ be a graph class satisfying for some k the "structure side" of the dichotomy we proved today.

Show that $\mathcal{C}$ is radius-1 flip-breakable.

Let $\mathcal{C}$ be a graph class satisfying for some k the “structure side” of the dichotomy we proved today.

Show that $\mathcal{C}$ is radius-1 flip-breakable.

A stronger structure property can be derived for monadically stable classes, which implies radius-1 flip-flatness.

Let $\mathcal{C}$ be a graph class. We say $\mathcal{C}$ is *weakly sparse* if there exists t such that no graph in $\mathcal{C}$ contains $K_{t,t}$ as a subgraph.

Let $\mathcal{C}$ be a graph class. We say $\mathcal{C}$ is *weakly sparse* if there exists t such that no graph in $\mathcal{C}$ contains $K_{t,t}$ as a subgraph.

Prove for a weakly sparse graph class $\mathcal{C}$:

$\mathcal{C}$ is nowhere dense if and only if $\mathcal{C}$ is monadically dependent.

Let $\mathcal{C}$ be a graph class. We say $\mathcal{C}$ is *weakly sparse* if there exists t such that no graph in $\mathcal{C}$ contains $K_{t,t}$ as a subgraph.

Prove for a weakly sparse graph class $\mathcal{C}$:

$\mathcal{C}$ is nowhere dense if and only if $\mathcal{C}$ is monadically dependent.

Break the statement down as follows:

- If $\mathcal{C}$ is not monadically dependent, then $\mathcal{C}$ is not nowhere dense.
- If $\mathcal{C}$ is not nowhere dense and weakly sparse, then $\mathcal{C}$ is not monadically dependent.