The combinatorics of monadic stability, monadic dependence, and related notions

Algomanet, Warsaw, September 9-13, 2024

Jan Dreier, TU Wien

Understanding **Monadic Stability** and **Monadic Dependence** via

- logic,
- combinatorics, and
- algorithms.

Map of the Universe

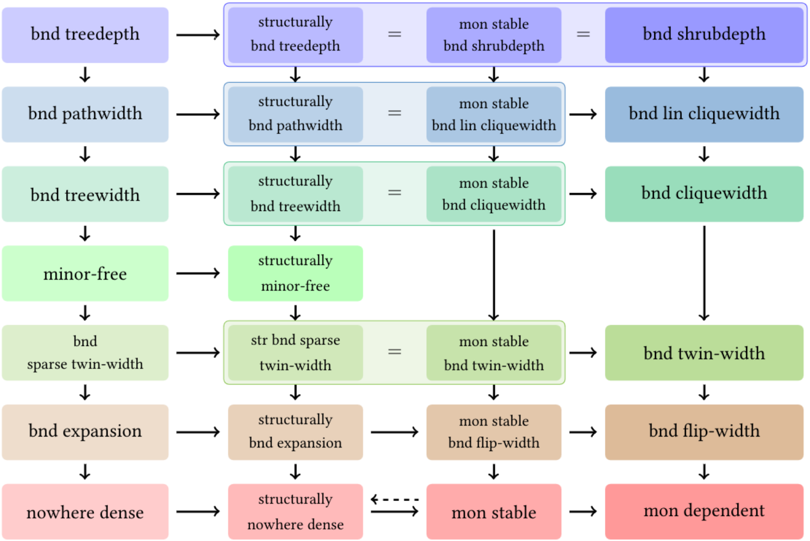

Figure by Michał Pilipczuk

Schedule

- Monday morning: meta-theorems, logic, nowhere dense (+exercises)

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Motivation

An Example

INDEPENDENTSET

- Input: Graph G and number k
- Question: Are there k pairwise non-adjacent vertices in G?

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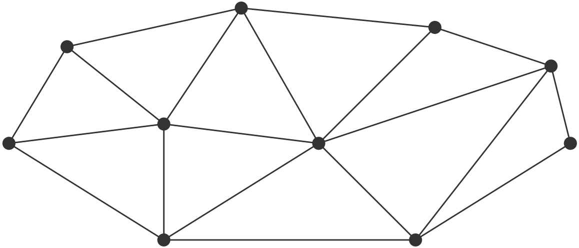
INDEPENDENTSET is NP-complete

Planar Graphs

Does it help to restrict the input to certain "well-behaved" graphs?
For example planar graphs?



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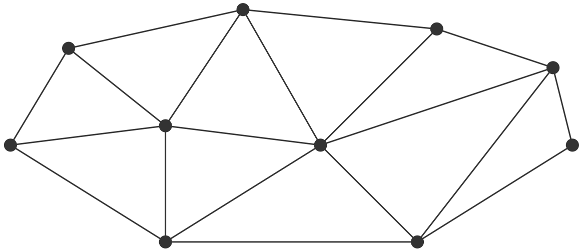
> INDEPENDENTSET is NP-complete on planar graphs.



Parameterized Complexity

Assign each instance a number, called the *parameter*. We hope that

- we can solve the instance if the parameter is small,
- interesting instances have a small parameter.

NP-hard problems may still be tractable for small parameter values!

Parameterized Independent Set

PARAMETERIZED INDEPENDENTSET

Input:	Graph G and integer k
Parameter:	k
Question:	Does G have an independent set of size k?

A parameterized problem is *fixed parameter tractable* (fpt) if instances with parameter k and size n can be solved in time $f(k)n^c$ (for some fixed function f and constant c).

Is PARAMETERIZED INDEPENDENTSET fixed parameter tractable?

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The best known algorithm takes time $n^{\Theta(k)}$.

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for $v_1 \in V$
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                check if $v_1, \ldots, v_k$ is an IS of size k

This is optimal (under certain complexity assumptions).

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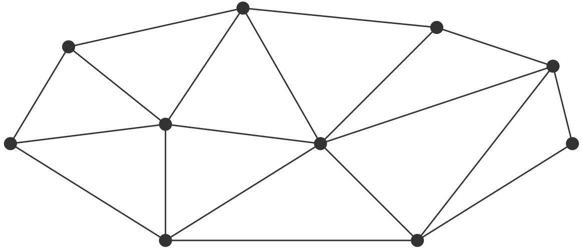
Parameterized IndependentSet is not fixed parameter tractable (unless FPT $=$ W[1]).

Independent Set on Planar Graphs

How about parameterized independent set on planar graphs?



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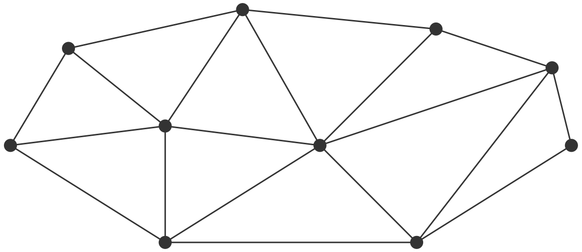

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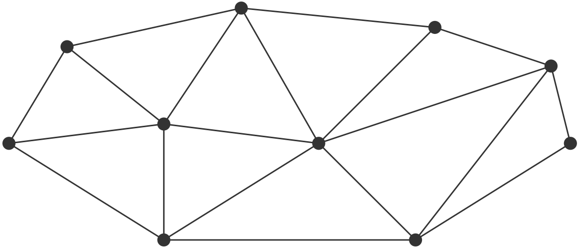

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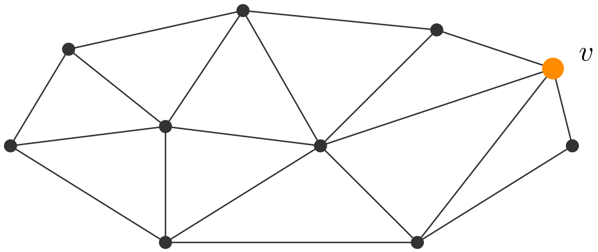
- In planar graphs there is always a vertex v with degree ≤ 5.



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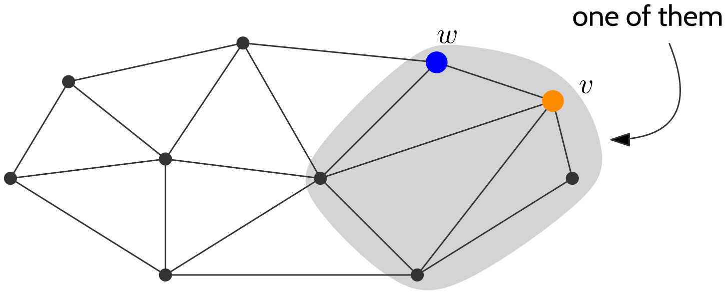

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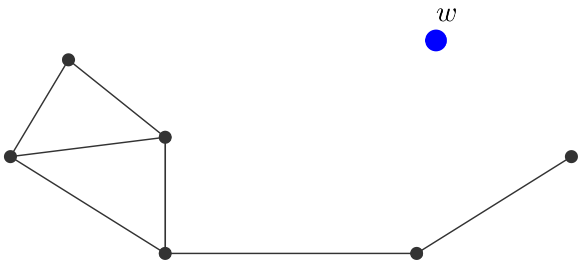

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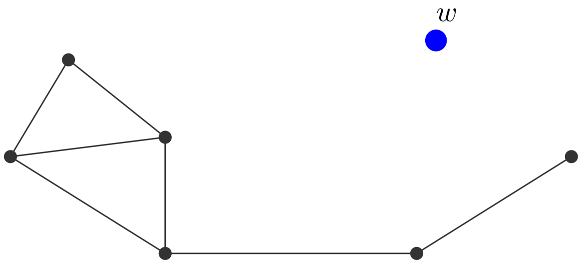
- Then find a solution of size $k-1$ in remaining graph.



Algorithm

IS(G, k):

  if G is empty return $k == 0$

  find vertex v with degree ≤ 5 in G

  for all $w \in N(v)$:

    if IS($G \setminus N(w)$, $k - 1$) return True

  return False

This solves PARAMETERIZED INDEPENDENTSET on planar graphs in time $O(6^k n)$.

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But the problem becomes tractable if we both

- consider only planar graphs, and
- parameterize by the solution size.

Other Problems?

- Is parameterized Dominating Set FPT on planar graphs?
- Is parameterized Clique FPT on bounded genus graphs?
- ...

We would like a single mechanism that answers these and similar questions.

Algorithmic Meta-Theorems

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"All Problems expressible in Logic L *can be solved efficiently on graph classes with property* P*"*

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Our Goal:

- L is first-order logic
- P are monadically dependent graph classes

Logic vs Graph-Theory

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- adjacency $\leftrightarrow$ binary relation
- colors $\leftrightarrow$ unary relation

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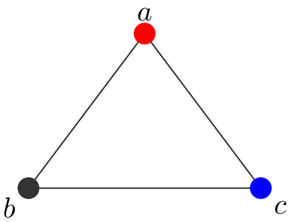
  - the universe are the vertices
  - E denotes the binary adjacency relation between vertices
  - c_i denotes the unary relation “the vertex is colored with color i”

Example

This graph is a structure G with

- universe $V = \{a, b, c\}$
- symmetrical binary relation
  $E := \{(a, b), (b, a), (b, c), (c, b), (a, c), (c, a)\}$
- unary relations $c_1 := \{a\}$, $c_2 := \{c\}$



First-Order Logic

For a given signature τ, first-order logic has ...

- element-variables ($x, y, z, \dots$)
- the equality relation $=$ as well as the relations from τ.
- quantifiers $\exists$ and $\forall$, as well as operators $\wedge$, $\vee$ and $\neg$

We mostly work on colored undirected graphs with $\tau = \{E, c_1, c_2, \dots\}$.

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Central Problem

First-Order Model-Checking

Input:	Graph G and first-order sentence φ
Question:	$G \models \varphi$?

Complexity

Theorem (Vardi 1982)

The model-checking problem is PSPACE-complete.

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It is reasonable to assume that the length of the formula is small compared to the size of the graph. Parameterize by $|\varphi|$.

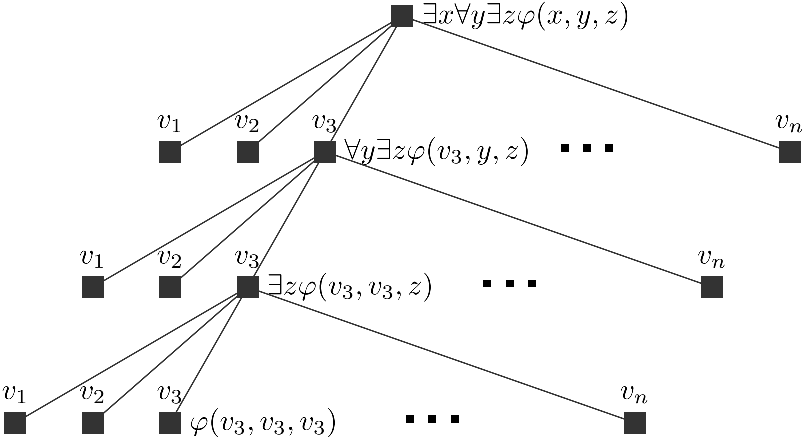
Parameterized Complexity (Upper Bound)

Theorem

One can decide whether $G \models \varphi$ in time $O(|G|^{|\varphi|})$.

Evaluation Trees

Proof: Construct an evaluation tree of size $O(|G|^{|\varphi|})$.



Parameterized Complexity (Lower Bound)

Conjecture (based on SETH)

One cannot decide whether $G \models \varphi$ in time $O(|G|^{q-1-\varepsilon})$ for any $\varepsilon > 0$ where q is the number of quantifiers of φ.

The previous algorithm is probably more or less optimal.

A faster model-checking algorithm would lead to an unexpected faster algorithm for many hard problems.

On certain graph classes, we can do much better though.

First-Order Logic

Target Statement

Let $\mathcal{C}$ be a "well-behaved" graph class. For an FO formula φ and graph $G \in \mathcal{C}$ one can decide whether $G \models \varphi$ in time $f(|\varphi|)n^{10}$ for some function f.

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Examples of "well-behaved" classes:

- bounded degree
- planar graphs
- ...

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…can be solved in time $f(k) \cdot n^{10}$ on well-behaved graph classes.

Exercise

Does the algorithic meta-theorem give an fpt algorithm for the following problem?

- Input: SAT-instance with planar incidence graph, $k \in \mathbb{N}$.
- Parameter: k.
- Question: is there a satisfying assignment with at most k variables set to true?

Exercise

Does the algorithic meta-theorem give an fpt algorithm for the following problem?

- Input: a planar graph G, and $k \in \mathbb{N}$.
- Parameter: k.
- Is there a dominating set of size at most k that induces a connected subgraph?

Exercise

Does the algorithic meta-theorem give an fpt algorithm for the following problem?

- Input: a planar graph G, and $k, d \in \mathbb{N}$.
- Parameter: k.
- Is it possible to remove k vertices such that every vertex has degree at most d?

What other graph classes are “well-behaved”?

Many Sparse Graph Classes

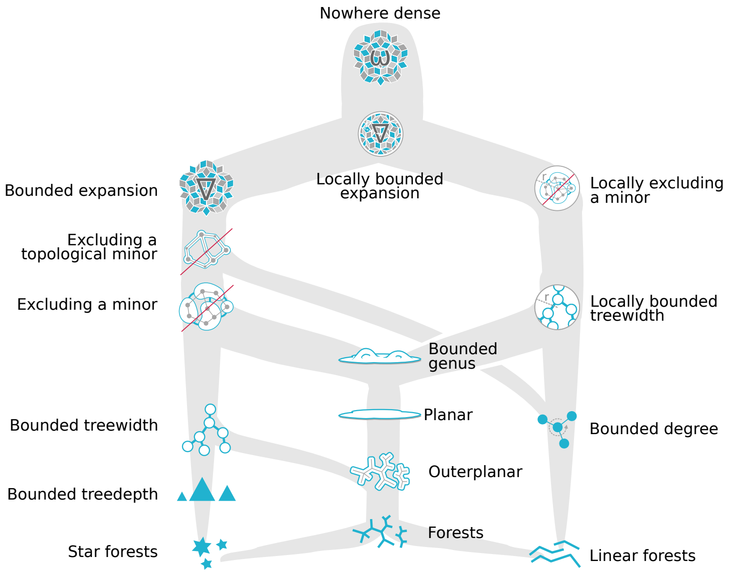

Figure by Felix Reidl

Milestones

Theorem (Grohe, Kreutzer, Siebertz 2017)

For a graph class $\mathcal{C}$ that is closed under subgraphs holds:
$\mathcal{C}$ is nowhere dense iff the first-order model-checking problem on $\mathcal{C}$ is FPT (assuming FPT $\neq$ AW[$*$]).

Bounded Degree Model Checking: Seese, 1996
Planar Model Checking: Flum, Grohe 2001
Bounded Expansion Model Checking: Dvořák, Král, Thomas, 2010
Nowhere Dense Model Checking: Grohe, Kreutzer, Siebertz, 2017

Nowhere Dense

Nešetřil, Ossona de Mendez

A graph class $\mathcal{C}$ is *nowhere dense* if for every $r \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that no graph in $\mathcal{C}$ contains the r-subdivided clique of size k as a subgraph.

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Let $\mathcal{C}$ be nowhere dense. Prove that there exists t such that no graph in $\mathcal{C}$ contains the biclique $K_{t,t}$ as a subgraph.

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Prove that the class of trees is nowhere dense

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Prove that the class of trees is nowhere dense

Prove that every class of bounded degree is nowhere dense.

Many Sparse Graph Classes

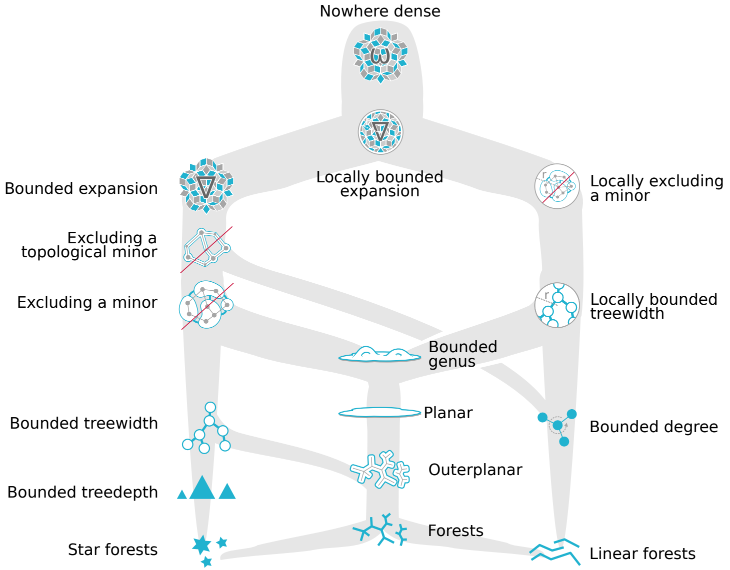
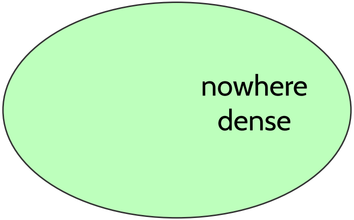


Figure by Felix Reidl

Can we go beyond nowhere dense?

Classes with FPT first-order model-checking?



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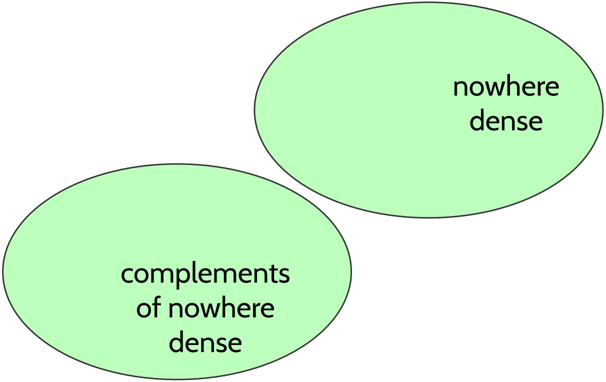
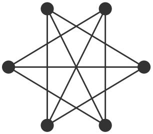


Complements

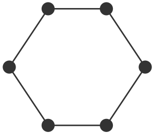
First-order model-checking is fpt on complements of nowhere dense classes by reduction.



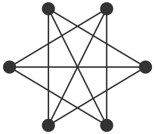
$\bar{G} \models \varphi$

Complements

First-order model-checking is fpt on complements of nowhere dense classes by reduction.



$$\Leftrightarrow$$



$G \models \bar{\varphi}$

$\bar{G} \models \varphi$

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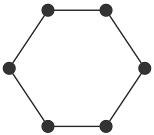
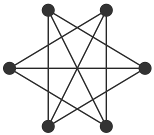


$G \models \bar{\varphi}$

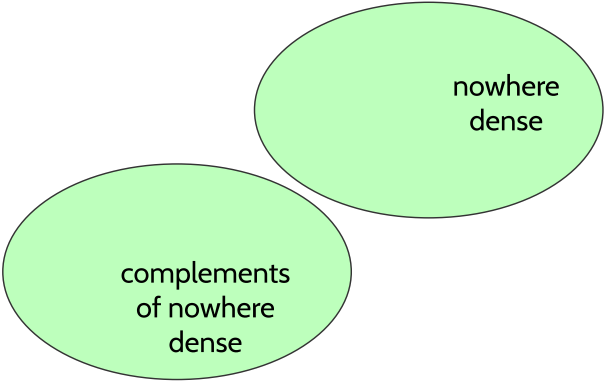
obtain $\bar{\varphi}$ from φ by swapping $\sim$ and $\not\sim$

$\Leftrightarrow$



$\bar{G} \models \varphi$

Classes with FPT first-order model-checking?



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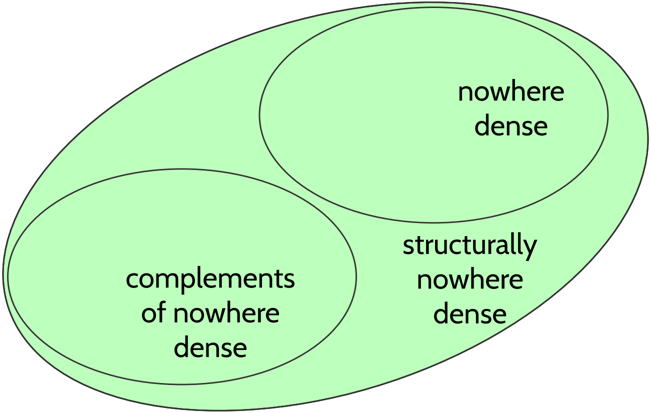
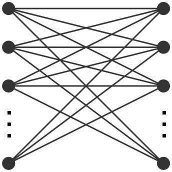


Fully Bipartite Graphs

Can we do fpt model-checking on the class of fully bipartite graphs?



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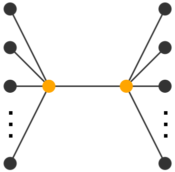
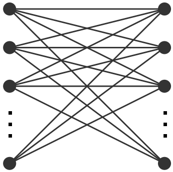


$G' \models \varphi'$



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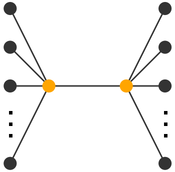
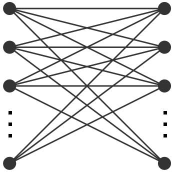


$G' \models \varphi'$

obtain φ' from φ by replacing $x{\sim}y$ with dist$(x, y) = 3$

$\Leftrightarrow$



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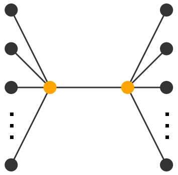
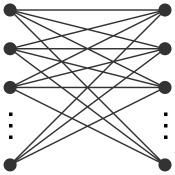


$G' \models \varphi'$

obtain φ' from φ by replacing $x{\sim}y$ with $\mathrm{dist}(x,y) = 3$

$\Longleftrightarrow$

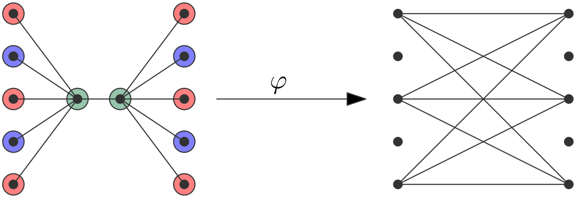
also restrict quantifiers to black vertices



$G \models \varphi$

Transductions (see board)

φ-transduction: color vertices + apply φ + take induced subgraph

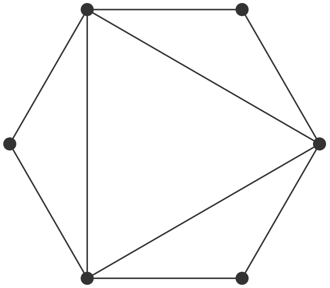


$\varphi(x, y) :=$ Red$(x) \wedge$ Red$(y) \wedge$ dist$(x, y) = 3$

A class $\mathcal{D}$ is a *transduction* of a class $\mathcal{C}$ if there exists φ such that every graph in $\mathcal{D}$ is a φ-transduction of some graph in $\mathcal{C}$.

Transductions

The class of subdivided cliques transduces the class of all graphs.



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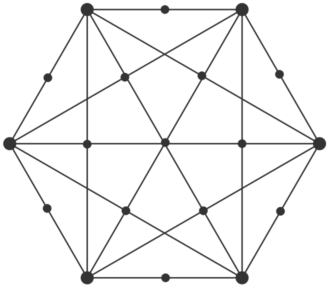

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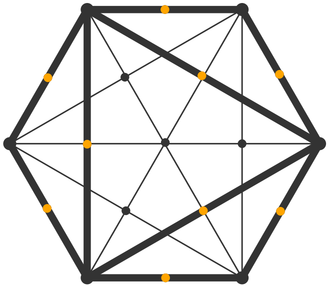

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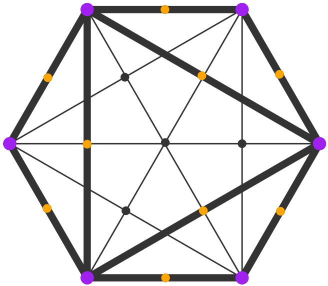

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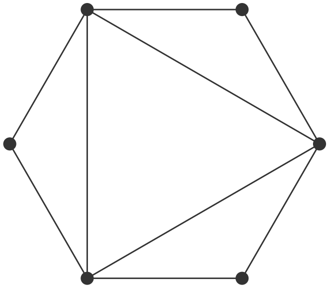


Tractable Classes

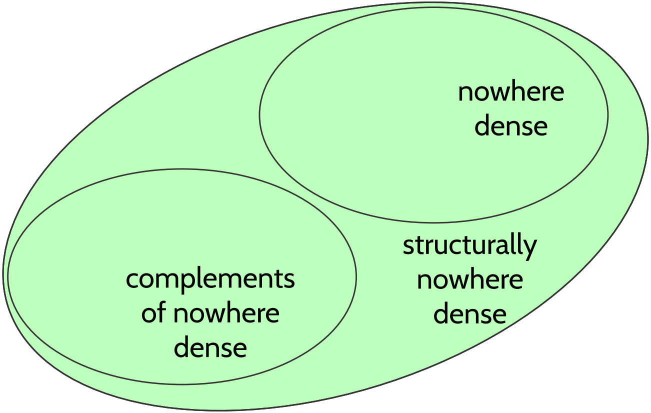
Gajarský, Kreutzer, Něsetřil, Ossona de Mendez, Pilipczuk, Siebertz, Toruńczyk, 2018.
Něsetřil, Ossona de Mendez, 2016

A class is *structurally nowhere dense*, if it is a transduction of a nowhere dense graph class.

D, Mählmann, Siebertz, 2023

The first-order model-checking problem on $\mathcal{C}$ is FPT on structurally nowhere dense graph classes.

Classes with FPT first-order model-checking?



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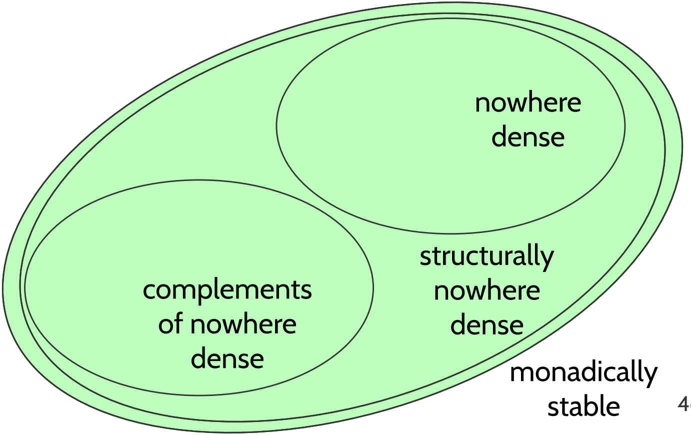
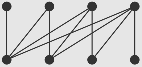


Monadic Stability/Dependence

Baldwin, Shelah, 1985

A class is *monadically stable*, if it does not transduce the class of all half-graphs.



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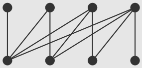

A class is *monadically dependent*, if it does not transduce the class of all graphs.

Adler, Adler

Every structurally nowhere dense class is monadically stable.

Monadic Stability/Dependence

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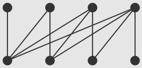

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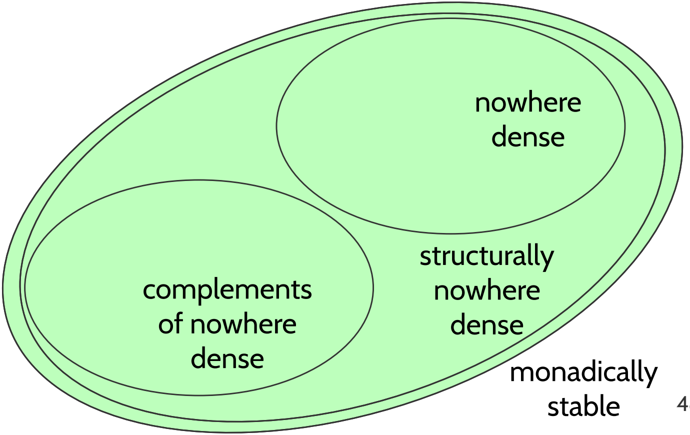
Adler, Adler

Every structurally nowhere dense class is monadically stable.

D, Eleftheriadis, Mählmann, McCarty, Pilipczuk, Toruńczyk 2024

Let $\mathcal{C}$ be monadically stable. The first-order model-checking problem is FPT on $\mathcal{C}$.

Classes with FPT first-order model-checking?



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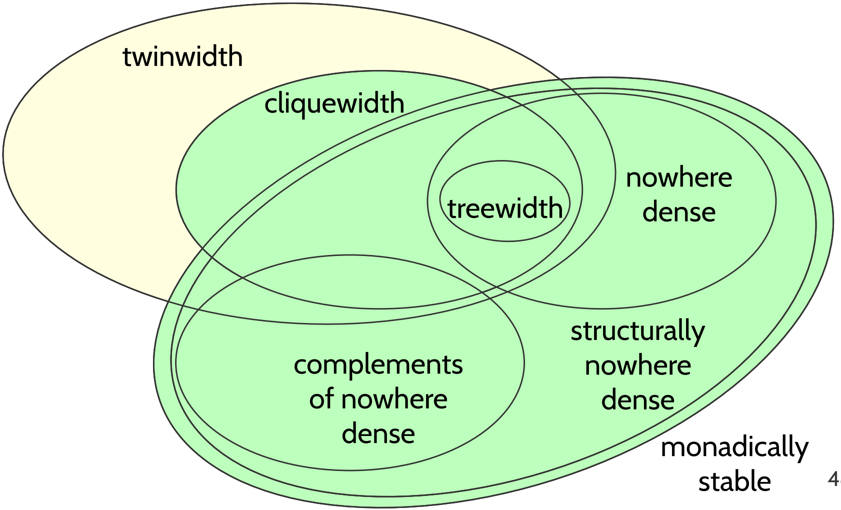
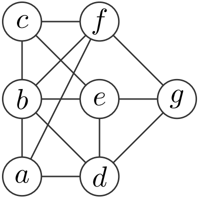


You already know normal graphs.



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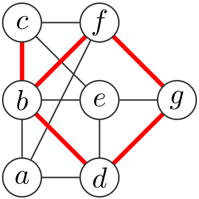
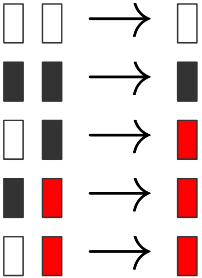
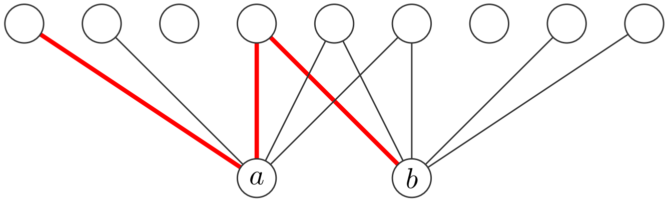


In *trigraphs* there are additional red error edges.

Contraction



We can contract two (not neccessarily adjacent) vertices a and b. The edges of the new vertex ab follow this table.



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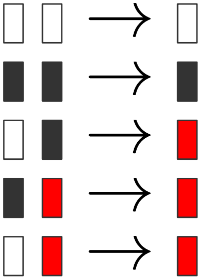
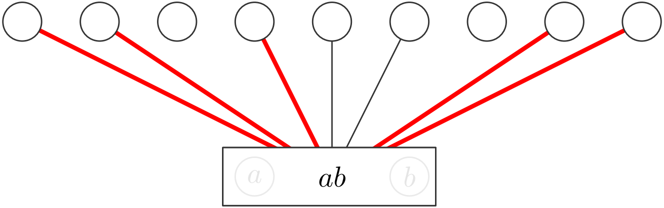
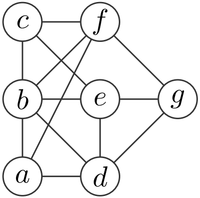
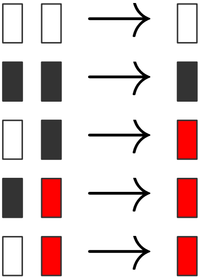




Contraction Sequences

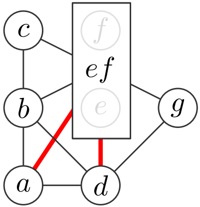


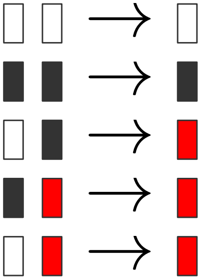


A *contraction sequence* is a sequence of contractions until only a single vertex is left.

Example by Édouard Bonnet

Contraction Sequences





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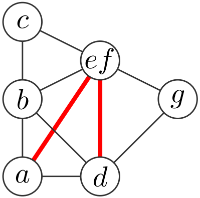

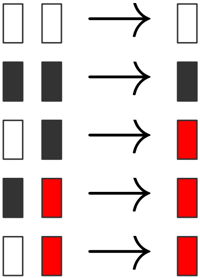

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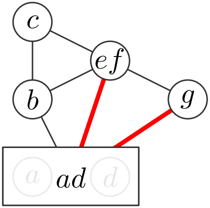

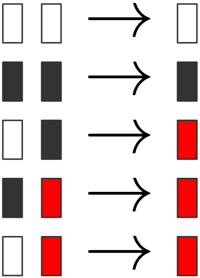

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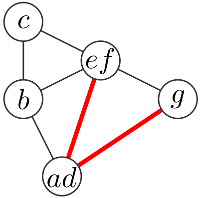

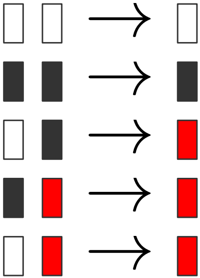

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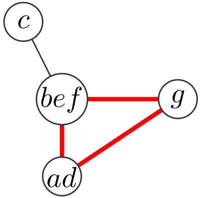

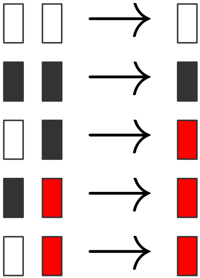

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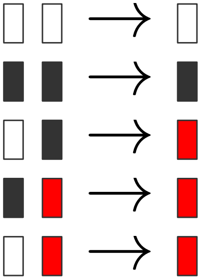

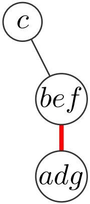

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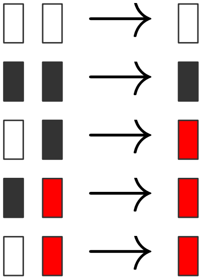

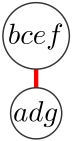

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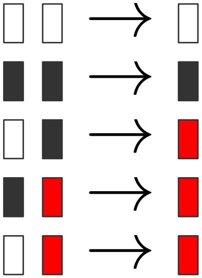

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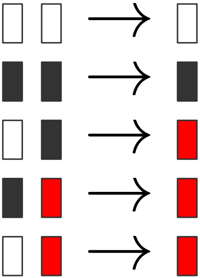
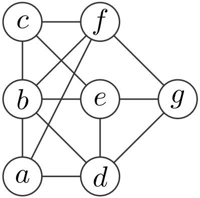
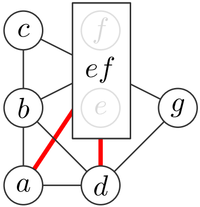


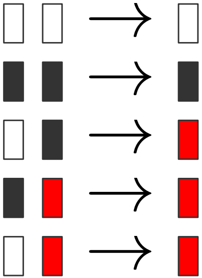


Bonnet, Kim, Thomassé, Watrigant 2021

Twinwidth: Smallest integer d such there is a contraction sequence where the red degree is *at all times* at most d.

Contraction Sequences





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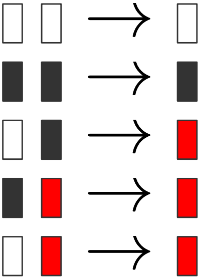

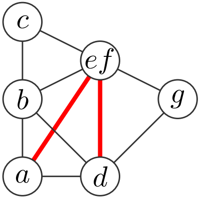

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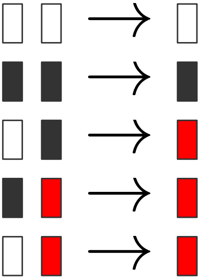

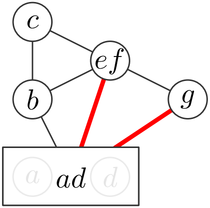

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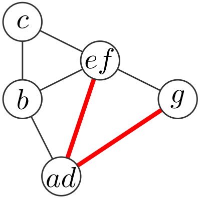

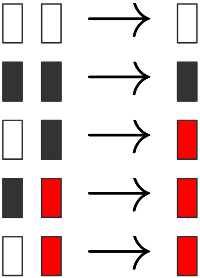

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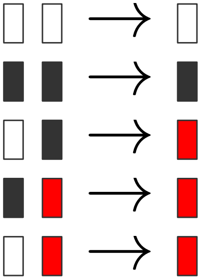

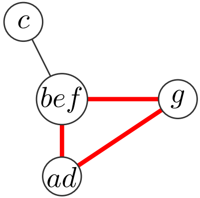

Bonnet, Kim, Thomassé, Watrigant 2021

Twinwidth: Smallest integer d such there is a contraction sequence where the red degree is *at all times* at most d.

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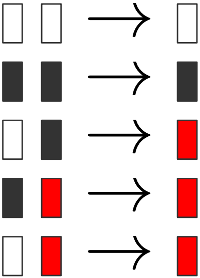

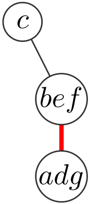

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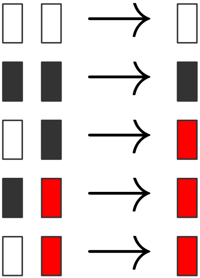

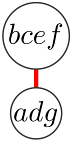

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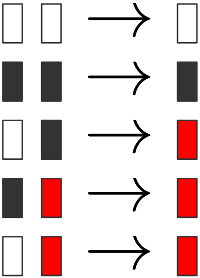

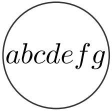

Bonnet, Kim, Thomassé, Watrigant 2021

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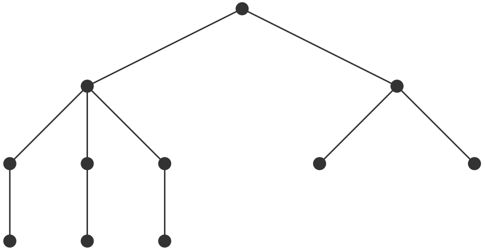

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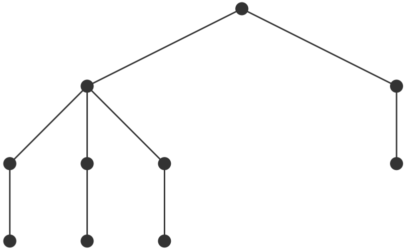

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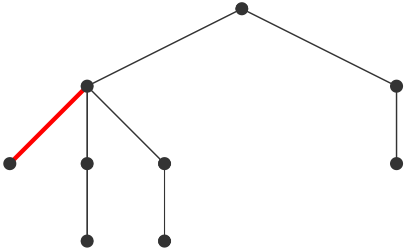

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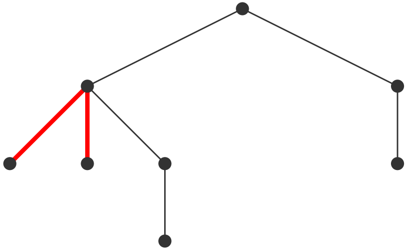

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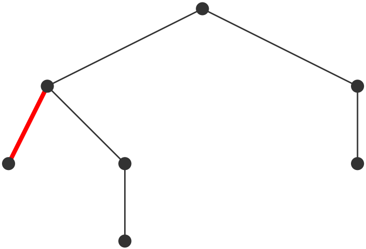

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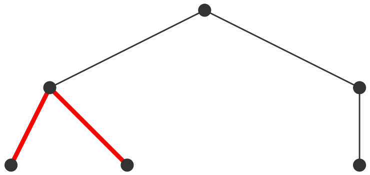

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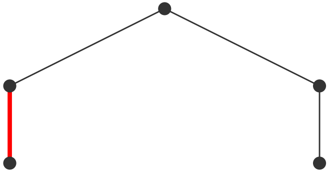

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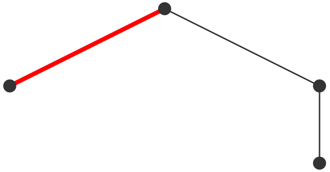

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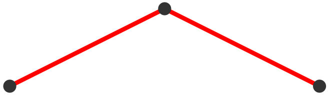

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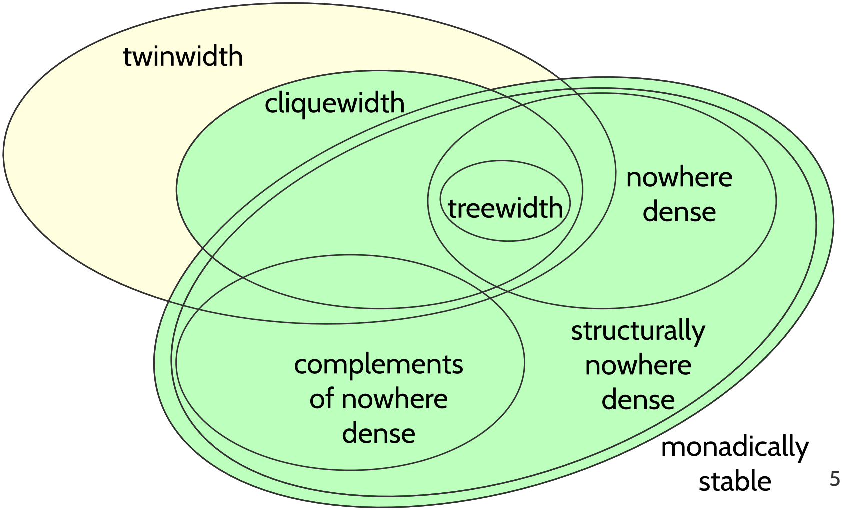
Bonnet, Kim, Thomassé, Watrigant 2021

Let $\mathcal{C}$ be a class of bounded twinwidth. Then first-order model-checking is fpt on $\mathcal{C}$, if one is additionally provided a contraction sequence of bounded twinwidth.

Classes with FPT first-order model-checking?



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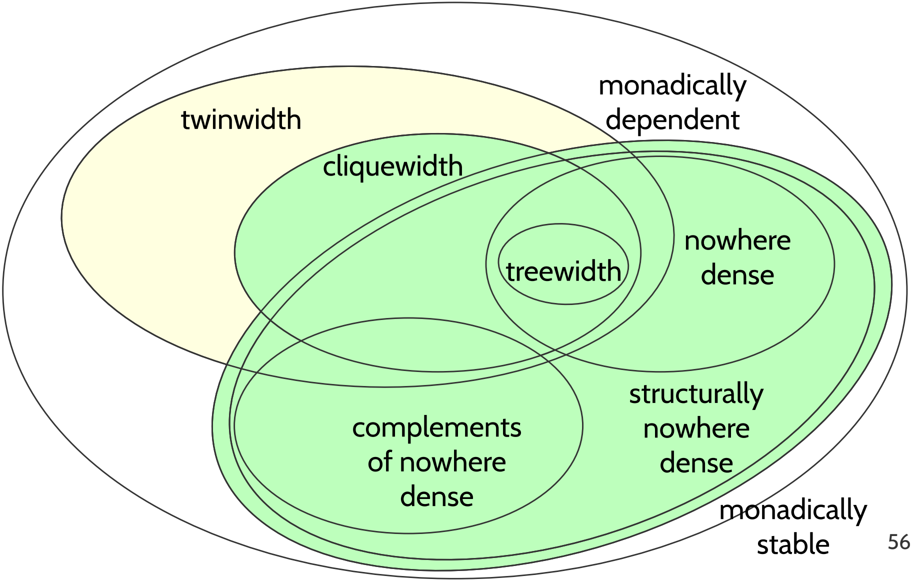

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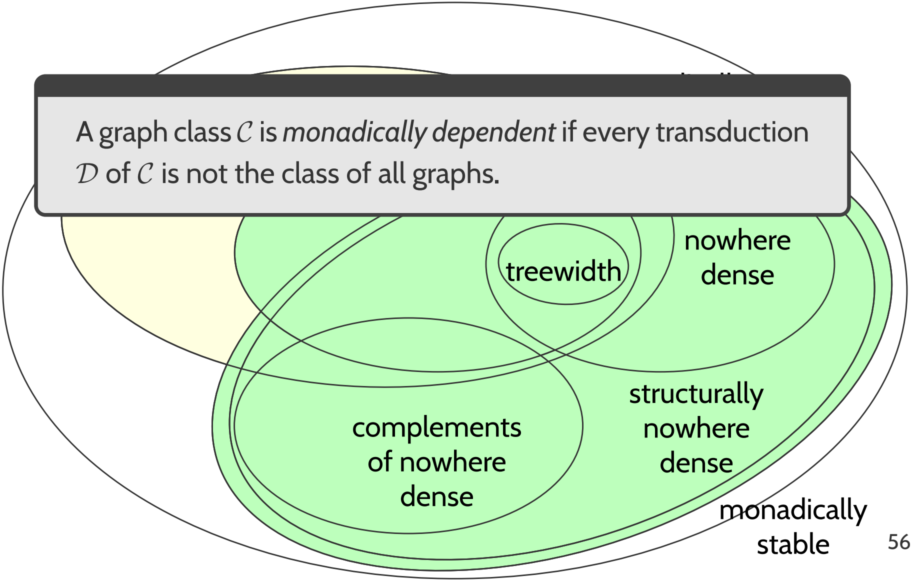

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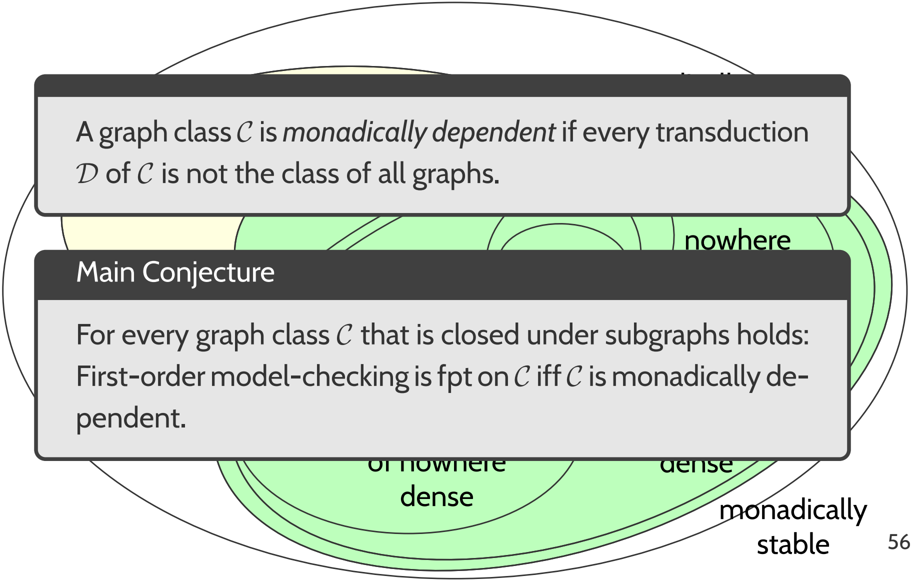
Main Conjecture

For every graph class $\mathcal{C}$ that is closed under subgraphs holds: First-order model-checking is fpt on $\mathcal{C}$ iff $\mathcal{C}$ is monadically dependent.



Map of the Universe

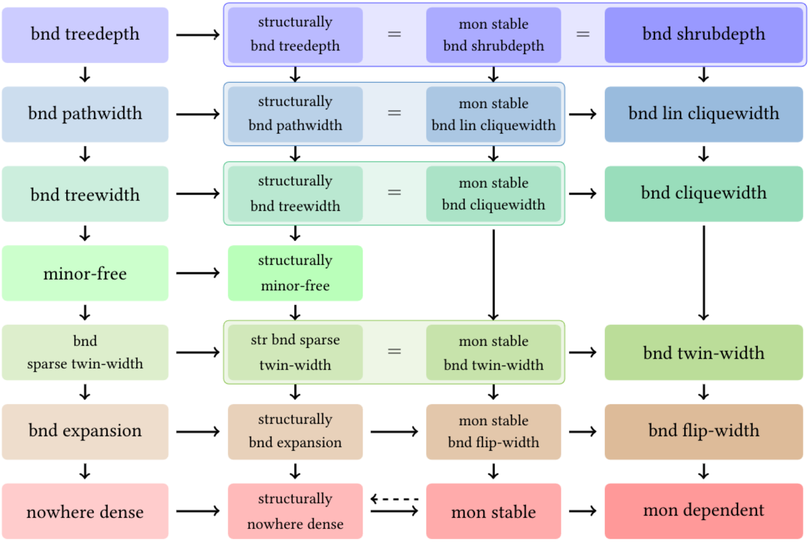


Figure by Michał Pilipczuk

Exercises

Exercise

Show that the class of empty graphs is monadically stable/dependent.

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Argue: If $\mathcal{C}$ transduces $\mathcal{D}$ and $\mathcal{D}$ transduces $\mathcal{E}$, then $\mathcal{C}$ transduces $\mathcal{E}$.

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Argue that therefore monadically stable/dependent classes are closed under transductions.

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Let $\mathcal{C}$ be the class of graphs of degree at most three. Show that $\mathcal{C}$ is monadically stable/dependent.

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Use the following theorem:

Corollary of Gaifman's Theorem

Let φ be a first-order formula. There is a number k with the following property. For every graph G there is a coloring $c : V(G) \to [k]$ such that for all $u, v \in V(G)$ with distance larger than k, the fact whether $(u, v) \in E(G)$ depends only on $(c(u), c(v))$.

Exercise

Show that the class of “star matchings” and the class of “comparability grids” are not monadically dependent.

Appendix

Restating the Milestones

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Grohe, Kreutzer, Siebertz 2017

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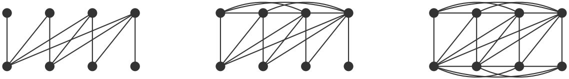
A graph class $\mathcal{C}$ is *unordered* if for some k it excludes the following graphs of order k as induced subgraphs.



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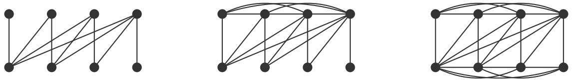

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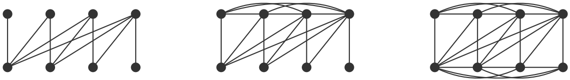

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D, Eleftheriadis, Mählmann, McCarty, Pilipczuk, Toruńczyk 2024

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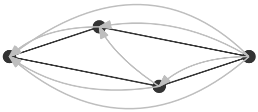

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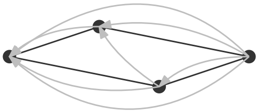

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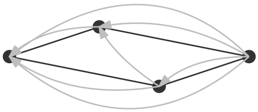

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Bonnet, Giocanti, Ossona de Mendez, Simon, Thomassé, Toruńczyk 2021

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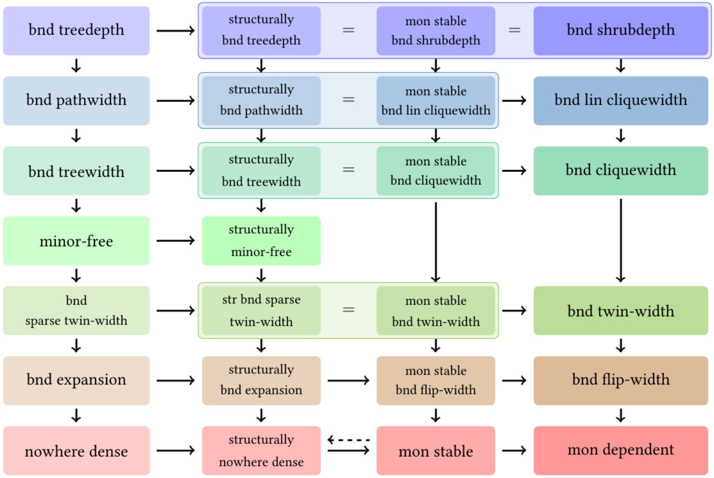

Figure by Michał Pilipczuk