Problems in NP can Admit Double-Exponential Lower Bounds when Parameterized by Treewidth or Vertex Cover

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Part 1.

(In)tractability and Treewidth

Intractable problems and approaches

Fixed-parameter tractability is a framework to deal with intractable problems:

- Choose a complexity parameter k independent of the input size n
- Find an OPT solution in time $f(k) \cdot n^{\mathcal{O}(1)}$ for some function f

Develop algorithms for graphs which are large but have a small solution size

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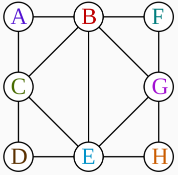
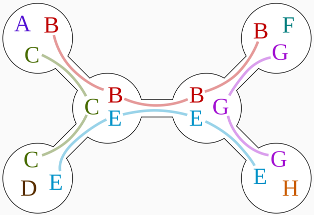
Treewidth

Def. A tree decomposition of G is a pair $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$, where T is a tree whose every node t is assigned a vertex subset $X_t \subseteq V(G)$, called a bag, with following conditions:



- $\mathcal{T}1$. $\bigcup_{t \in V(T)} X_t = V(G)$;
- $\mathcal{T}2$. For every $vw \in E(G)$, there exists a node t of T such that bag X_t contains both v and w;
- $\mathcal{T}3$. For every $v \in V(G)$, the set $T_v = \{t \in V(T) | v \in X_t\}$ induces a connected subtree of T.



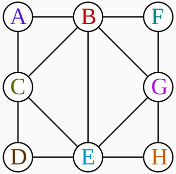
Def. The width of $\mathcal{T}$ is $\max_{t \in V(T)} |X_t| - 1$.

Def. The treewidth $\text{tw}(G)$ is the **minimum** width over all tree decompositions of G.

Treewidth

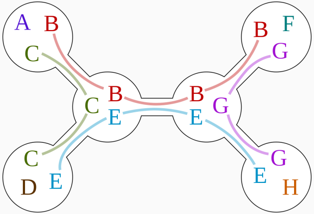
The treewidth of a graph G is

$$\min\{\omega(G^+) - 1 : G^+ \supseteq G \text{ and } G^+ \text{ is chordal}\}$$



The Cops-and-Robber Game

Treewidth is at most t if and only if $t+1$ cops can always catch the robber in G in a monotone game if the robber is *visible* (to the cop player)



$\text{tw}(K_n) = n - 1$ $\qquad$ $\text{tw}(P_n \times P_m) = \min(m, n)$ $\qquad$ $\text{tw}(T) = 1$

Treewidth

Many NP-hard problems are FPT parameterized by treewidth via dynamic programming on the tree decomposition.

For a given signature τ, monadic second order logic has

- element-variables $(x, y, z, \dots)$ and set-variables $(X, Y, Z, \dots)$
- relations $=$ (equation) and $x \in X$ (membership), as well as relations from τ
- quantifiers $\exists$ and $\forall$, as well as operators $\wedge$, $\vee$, $\neg$

If φ is a sentence, we write $G \models \varphi$ to indicate that φ holds on G (i.e., G is a model of φ)

Theorem [Courcelle'90]

For a $\texttt{MSO}_1$ sentence φ and graph G one can decide whether $G \models \varphi$ in time $f(\text{tw}(G), |\varphi|)n$ for some function f.

Conditional Lower Bounds

Exponential Time Hypothesis (ETH) [Impagliazzo, Paturi, 1990]

Roughly, 3-SAT on n variables cannot be solved in time $2^{o(n)}$.

Conditional lower bounds for tw are usually $2^{o(\mathrm{tw})}$, $2^{o(\mathrm{tw}\log \mathrm{tw})}$ or $2^{o(\mathrm{poly}(\mathrm{tw}))}$.

Rarer results: Unless the ETH fails,

- QSAT WITH k ALTERNATIONS admits a lower bound of a tower of exponents of height k in the treewidth of the primal graph PSPACE-complete [Fichte, Hecher, Pfandler, 2020]
- k-CHOOSABILITY and k-CHOOSABILITY DELETION admit double- and triple-exponential lower bounds in treewidth, respectively Π_2^p-complete and Σ_3^p-complete [Marx, Mitsou, 2016]
- $\exists\forall$-CSP admits a double-exponential lower bound in the vertex cover number Σ_2^p-complete [Lampis, Mitsou, 2017]

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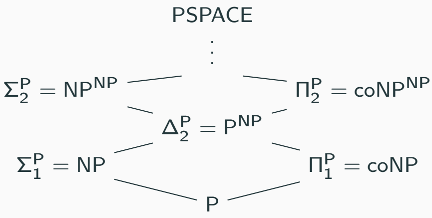
Conditional Lower Bounds

Question.
Does any NP-complete problem require at least double-exponential running time?



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Part 2.

Metric Graph Problem(s)

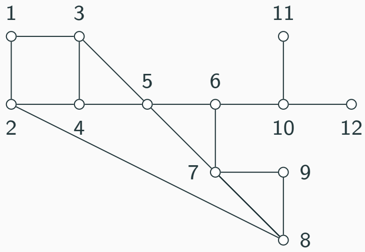
Metric Dimension

[Slater '75, Harary, Melter '76]

Def. A resolving set is a $S \subseteq V(G)$ such that $\forall u, v \in V, \exists z \in S$ with

$$d(z, u) \neq d(z, v).$$

Def. The minimum size of a resolving set of G is the metric dimension of G.



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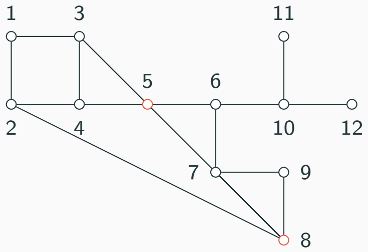

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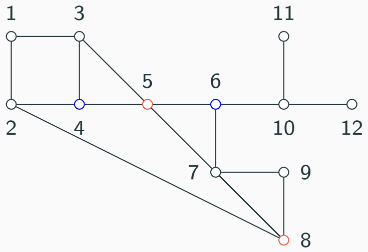

Vertices 4 and 6 are **not** resolved by 5 nor 8.

Metric Dimension

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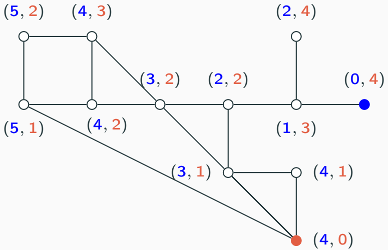


Observation. For any twins $u, v \in V(G)$ and any resolving set S of G, $S \cap \{u, v\} \neq \emptyset$.

Metric Dimension (MDim)

> **METRIC DIMENSION**
> **Input:** An undirected simple graph G and a positive integer k
> **Question:** Is $\mathrm{md}(G) \leq k$?

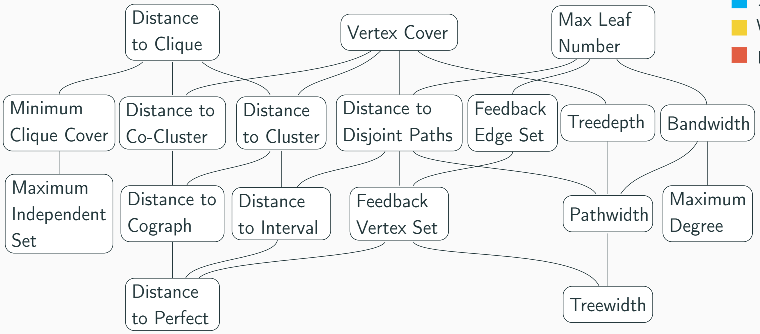
Polynomial-time

- Trees [Slater'75]
- Cographs [Epstein et al'15]
- Outerplanar [Diaz et al'17]

NP-complete

- Arbitrary [Garey, Johnson'79]
- Split [Epstein et al'15]
- Bipartite [Epstein et al'15]
- Co-bipartite [Epstein et al'15]
- Planar [Diaz et al'17]
- Interval [Foucaud et al'17]

Parameterized complexity of Metric Dimension



- FPT ($f(k) \cdot n^{O(1)}$-time algorithm)
- XP ($n^{f(k)}$-time algorithm)
- W[1]-hard (not FPT unless FPT = W[1])
- para-NP-hard (not XP unless P = NP)

n: size of input
k: size of parameter

A lower parameter is upper bounded by a function of the higher one

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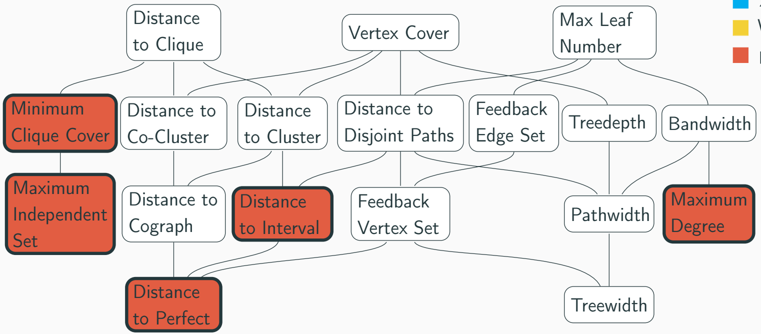

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From NP-hardness results on previous slide

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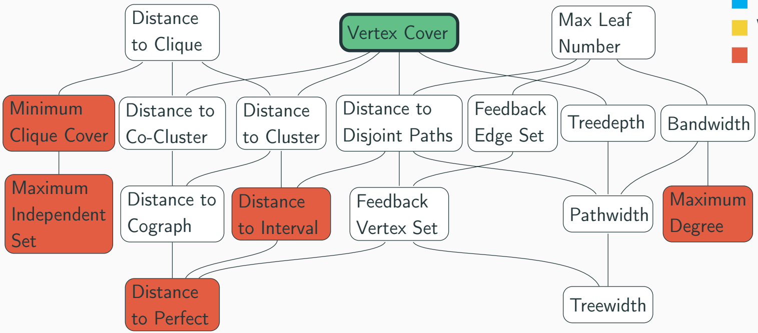

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W[2]-hard parameterised by solution size [Hartung, Nichterlein '13]

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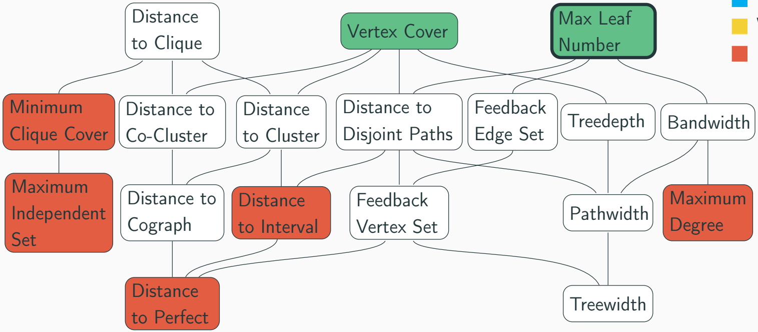

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[Eppstein '15]

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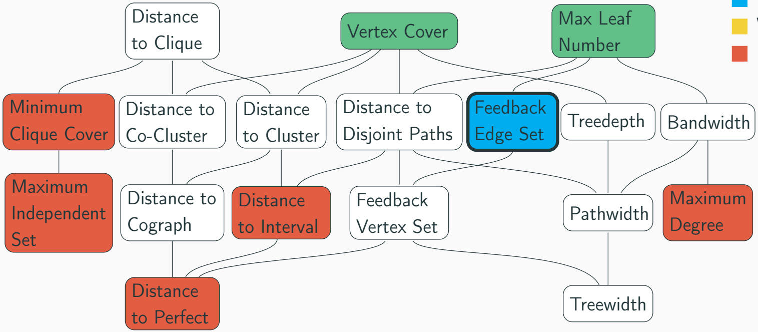

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[Epstein et al '15]

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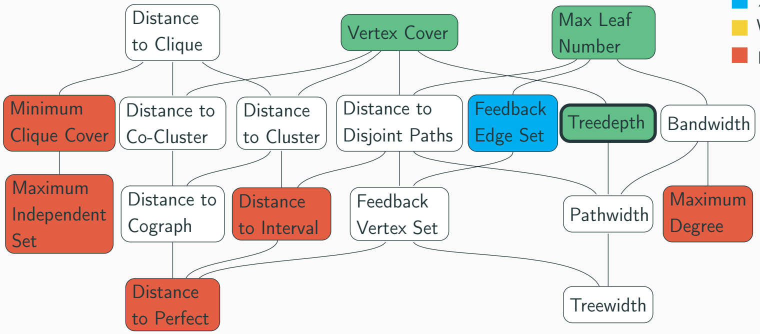

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[Gima et al '21]

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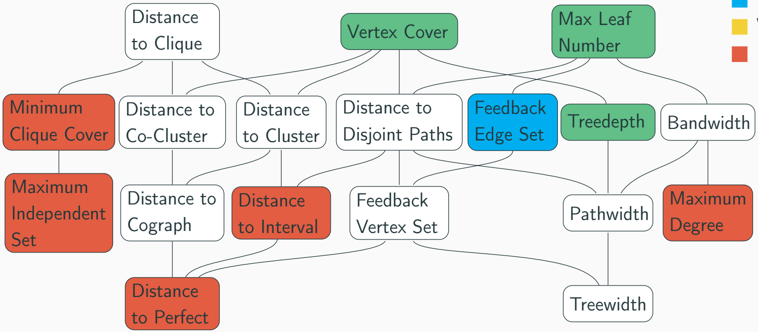

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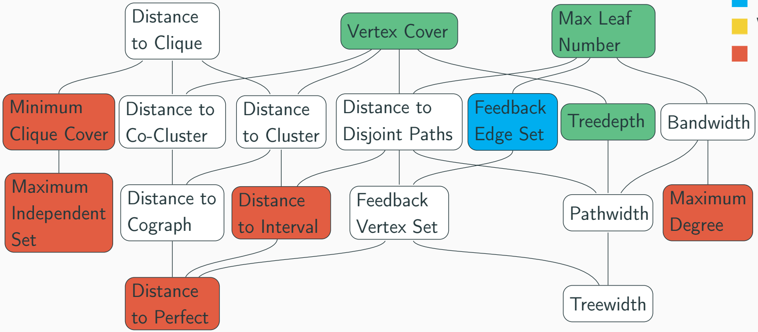
FPT parameterised by treelength + max degree [Belmonte et al '17]
and clique-width + diameter [Gima et al '21]

Parameterized complexity of Metric Dimension



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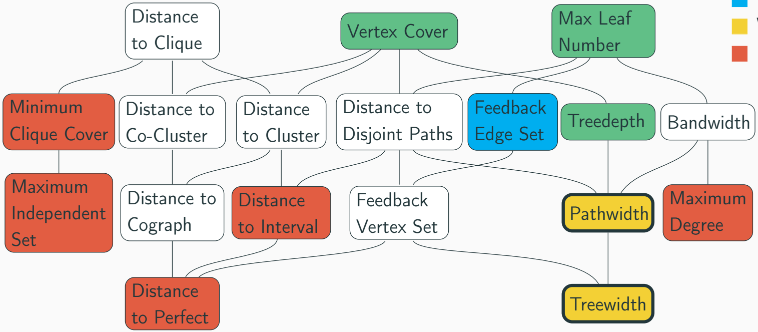
Q1: Complexity parameterised by **Feedback Vertex Set**? [Hartung, Nichterlein '13]

Q2: Complexity parameterised by **treewidth**? [Eppstein '15], [Belmonte et al '17], [Díaz et al '17]

Parameterized complexity of Metric Dimension



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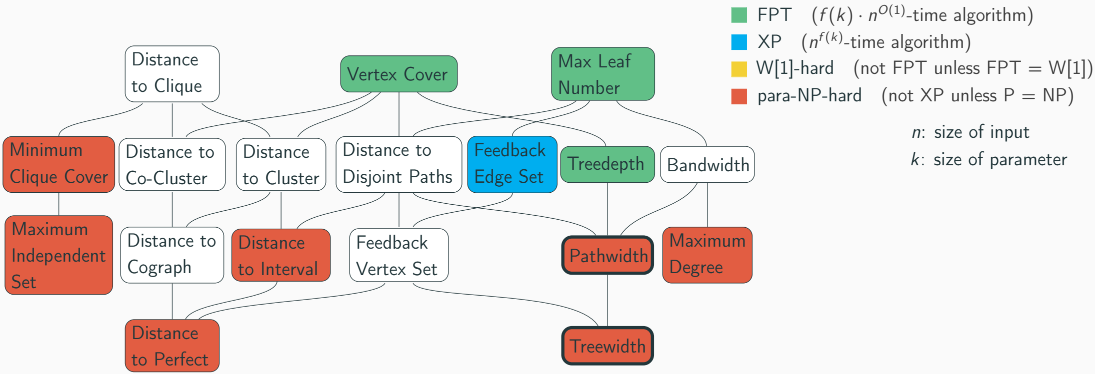
Q2 answered first by [Bonnet, Purohit '21].

Parameterized complexity of Metric Dimension



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Q2: Complexity parameterised by **treewidth**? [Eppstein '15], [Belmonte et al '17], [Díaz et al '17]

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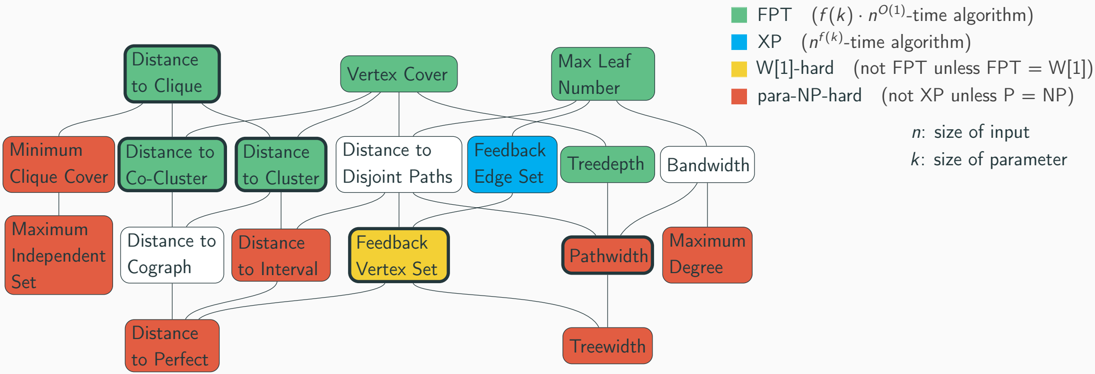
Parameterized complexity of Metric Dimension



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Q2: Complexity parameterised by **treewidth**? [Eppstein '15], [Belmonte et al '17], [Díaz et al '17]

Q1 answered for the combined parameter Feedback Vertex Set + Pathwidth
[Galby, Khazaliya, Mc Inerney, Sharma, Tale '23]

Part 3.

Our Technique and MDim

Results

Theorem [FGKLMST, 2024]

Metric Dimension and Geodetic Set

- can be solved in $2^{\mathrm{diam}^{\mathcal{O}(\mathrm{tw})}} \cdot n^{\mathcal{O}(1)}$ time
- no $2^{f(\mathrm{diam})^{o(\mathrm{tw})}} \cdot n^{O(1)}$ time algorithm assuming ETH

Theorem [FGKLMST, 2024]

Strong Metric Dimension

- can be solved in $2^{2^{\mathcal{O}(\mathrm{vc})}} \cdot n^{\mathcal{O}(1)}$ time, admits $2^{\mathcal{O}(\mathrm{vc})}$ kernel
- no $2^{2^{o(\mathrm{vc})}} \cdot n^{O(1)}$ time algorithm, or $2^{o(vc)}$ kernel, assuming ETH

A way to go

Theorem [FGKLMST, 2024]

METRIC DIMENSION and GEODETIC SET

- can be solved in $2^{\text{diam}^{\mathcal{O}(\text{tw})}} \cdot n^{\mathcal{O}(1)}$ time
- no $2^{f(\text{diam})^{o(\text{tw})}} \cdot n^{O(1)}$ time algorithm assuming ETH

Reduction.

3-PARTITIONED 3-SAT: φ $\rightarrow$ METRIC DIMENSION: (G, k)

$\text{tw}(G) = \log(n)$

$\text{diam}(G) = \text{const}$

3-Partitioned 3-SAT

3-Partitioned 3-SAT [Lampis, Melissinos, Vasilakis, 2023]

Input: 3-CNF formula φ with a partition of its variables into 3 disjoint sets X^α, X^β, and X^γ such that $|X^\alpha| = |X^\beta| = |X^\gamma| = n$ and each clause contains at most one variable from each of X^α, X^β, and X^γ

Question: Is ϕ satisfiable?

Theorem [Lampis, Melissinos, Vasilakis, 2023]

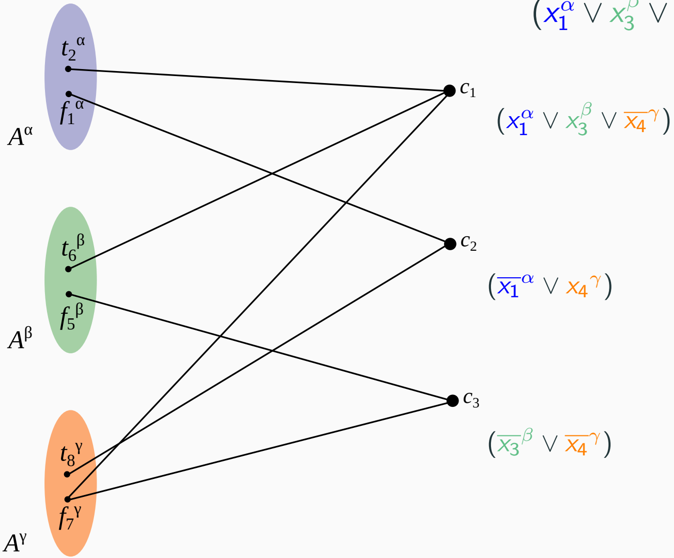
3-Partitioned 3-SAT: no $2^{o(n)}$ time algorithm assuming ETH

Encode SAT with small separator

$$(x_1^\alpha \lor x_3^\beta \lor \overline{x_4}^\gamma) \land (\overline{x_1}^\alpha \lor x_4^\gamma) \land (\overline{x_3}^\beta \lor \overline{x_4}^\gamma)$$

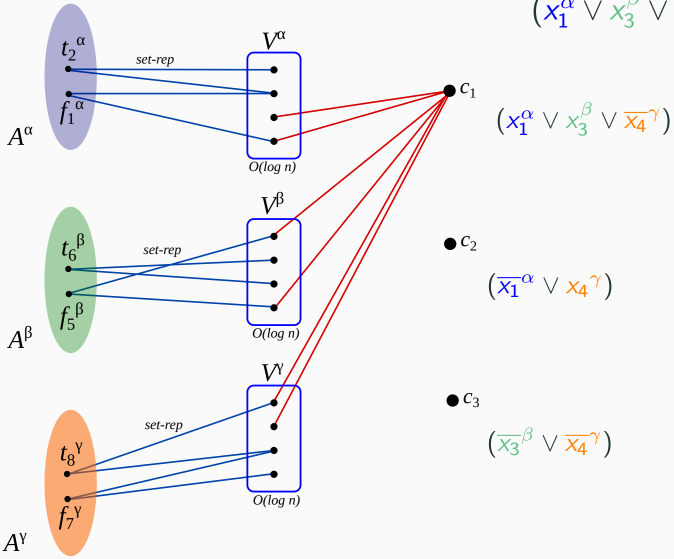
t_{2i}^α represents x_i^α

f_{2i-1}^α represents $\overline{x}_i^\alpha$



Set-Representation Gadget

$$\left(x_1^{\alpha} \vee x_3^{\beta} \vee \overline{x_4}^{\gamma}\right) \wedge \left(\overline{x_1}^{\alpha} \vee x_4^{\gamma}\right) \wedge \left(\overline{x_3}^{\beta} \vee \overline{x_4}^{\gamma}\right)$$



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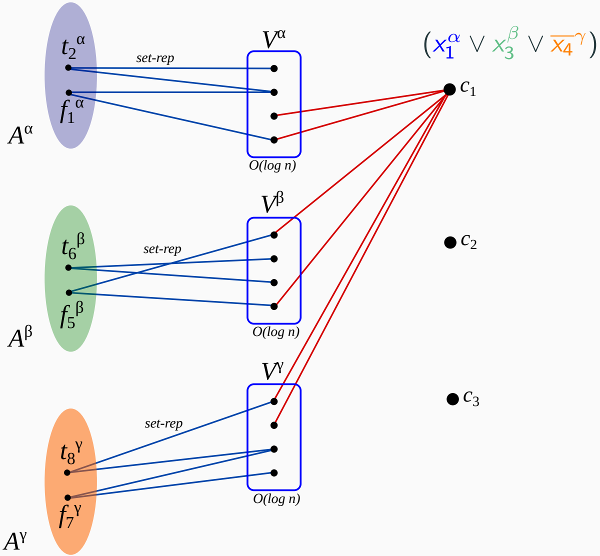


Let F_p be the collection of subsets of $\{1, \ldots, 2p\}$ that contain exactly p integers.

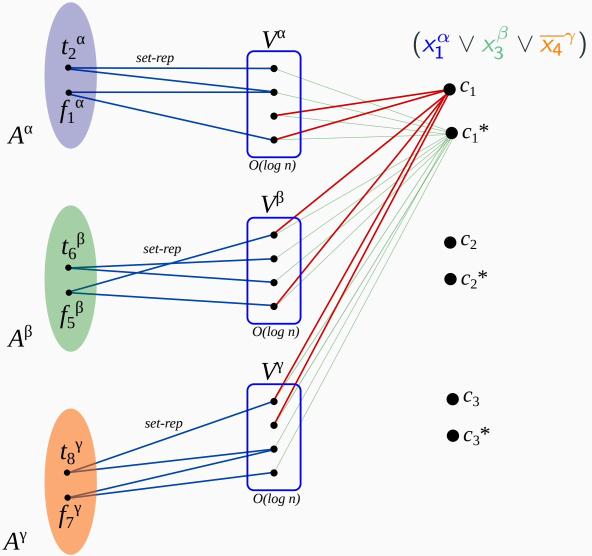
No set in F_p is contained in another set in F_p (Sperner family).

There exists $p = O(\log n)$ s.t. $\binom{2p}{p} \geq 2n$.
We define a 1-to-1 function

$$\text{set-rep} : \{1, \ldots, 2n\} \to F_p.$$

t_2^{α} is the **only** vertex in A^{α} that **does not** share a **common neighbour** with $c_1 = (x_1^{\alpha} \vee x_3^{\beta} \vee \overline{x_4}^{\gamma})$

Set-Representation Gadget



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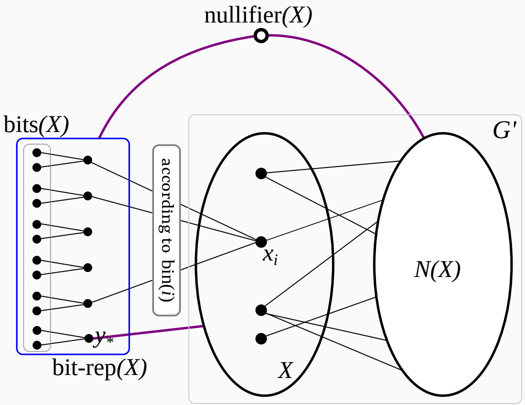
Bit-representation Gadget

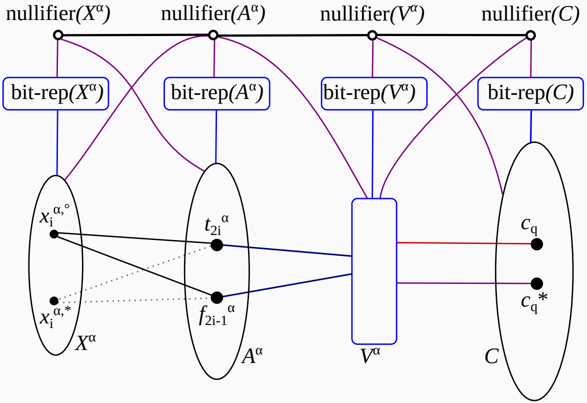
Observation. For any twins $u, v \in V(G)$ and any resolving set S of G, $S \cap \{u, v\} \neq \emptyset$.



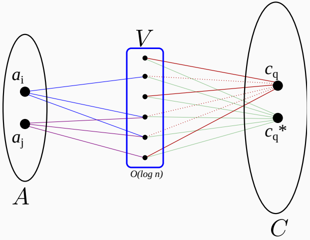
Purple edges represent all possible edges

- For any resolving set S, $|S \cap \text{bits}(X)| \geq \log(|X|) + 1$
- $|S \cap \text{bits}(X)|$ distinguishes each vertex in $X \cup \text{bit-rep}(X)$ from every other vertex in G
- nullifier(X) guarantees that the rest part of $V(G)$ does not affected by the gadget

Lower bound for Metric Dimension parameterized by tw



Purple — all possible edges
Blue — set-rep
Red — complementary to blue

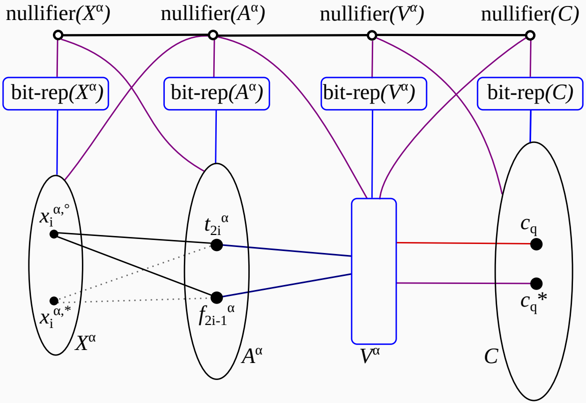


Note: $\text{tw}(G) = \log(n)$
$\text{diam}(G) = \text{const}$

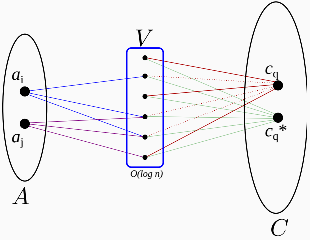
Theorem [FGKLMST, 2024]

Metric Dimension: no $2^{f(\text{diam})^{o(\text{tw})}} \cdot n^{O(1)}$ time algorithm assuming ETH

Lower bound for Metric Dimension parameterized by tw



Purple — all possible edges
Blue — set-rep
Red — complementary to blue

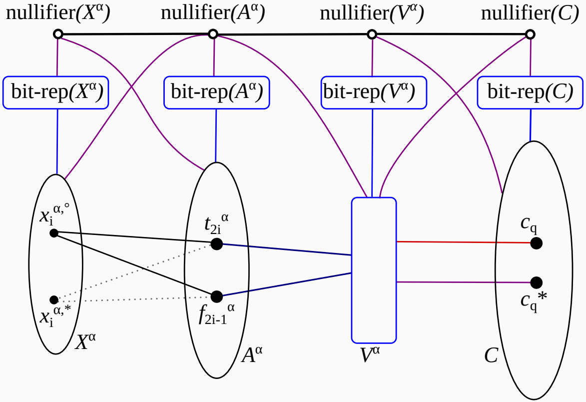


Note: $\mathrm{tw}(G) = \log(n)$
$\mathrm{diam}(G) = \mathrm{const}$

Theorem [FGKLMST, 2024]

METRIC DIMENSION: no $2^{f(\mathrm{diam})^{o(\mathrm{tw})}} \cdot n^{O(1)}$ time algorithm assuming ETH

Lower bound for Metric Dimension parameterized by tw



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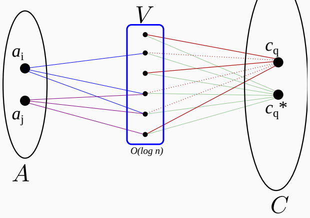


Note: $\text{tw}(G) = \log(n)$
$\text{diam}(G) = \text{const}$

Theorem **[FGKLMST, 2024]**

Metric Dimension: no $2^{f(\text{diam})^{o(\text{tw})}} \cdot n^{O(1)}$ time algorithm assuming ETH

Part 4.

Other Results and Applications

Geodetic Set and Strong MDim

GEODETIC SET

Input: An undirected simple graph G

Question: Does there exist $S \subseteq V(G)$ such that $|S| \leq k$ and, for any vertex $u \in V(G)$, there are two vertices $s_1, s_2 \in S$ such that a shortest path from s_1 to s_2 contains u?

Theorem **[FGKLMST, 2024]**

GEODETIC SET

- no $2^{f(\text{diam})^{o(\text{tw})}} \cdot n^{O(1)}$ time algorithm assuming ETH

Strong Metric Dimension

STRONG METRIC DIMENSION
Input: An undirected simple graph G
Question: Does there exist $S \subseteq V(G)$ such that $|S| \leq k$ and, for any pair of vertices $u, v \in V(G)$, there exists a vertex $w \in S$ such that either u lies on some shortest path between v and w, or v lies on some shortest path between u and w?

Theorem [FGKLMST, 2024]

STRONG METRIC DIMENSION

- no $2^{2^{o(\mathrm{vc})}} \cdot n^{O(1)}$ time algorithm, or $2^{o(vc)}$ kernel, assuming ETH

Match with the Algorithms

Theorem [FGKLMST, 2024]

Metric Dimension and Geodetic Set

- can be solved in $2^{\mathrm{diam}^{\mathcal{O}(\mathrm{tw})}} \cdot n^{\mathcal{O}(1)}$ time
- no $2^{f(\mathrm{diam})^{o(\mathrm{tw})}} \cdot n^{O(1)}$ time algorithm assuming ETH

Theorem [FGKLMST, 2024]

Strong Metric Dimension

- can be solved in $2^{2^{\mathcal{O}(\mathrm{vc})}} \cdot n^{\mathcal{O}(1)}$ time, admits $2^{\mathcal{O}(\mathrm{vc})}$ kernel
- no $2^{2^{o(\mathrm{vc})}} \cdot n^{O(1)}$ time algorithm, or $2^{o(vc)}$ kernel, assuming ETH

Applications of the Technique

Theorem [Chalopin, Chepoi, Mc Inerney, Ratel, COLT 2024]

POSITIVE NON-CLASHING TEACHING DIMENSION for Balls in Graphs

- no $2^{2^{o(\mathrm{vc})}} \cdot n^{O(1)}$ time algorithm, or $2^{o(\mathrm{vc})}$ kernel, assuming ETH

Theorem [Chakraborty, Foucaud, Majumdar, Tale, 2024]

LOCATING-DOMINATING SET (resp., TEST COVER)

- no $2^{2^{o(\mathrm{tw})}} \cdot n^{O(1)}$ (resp., $2^{2^{o(\mathrm{tw})}}(|U| + |\mathcal{F}|)^{O(1)}$) time algorithm assuming ETH

Part 5.

Open Problems

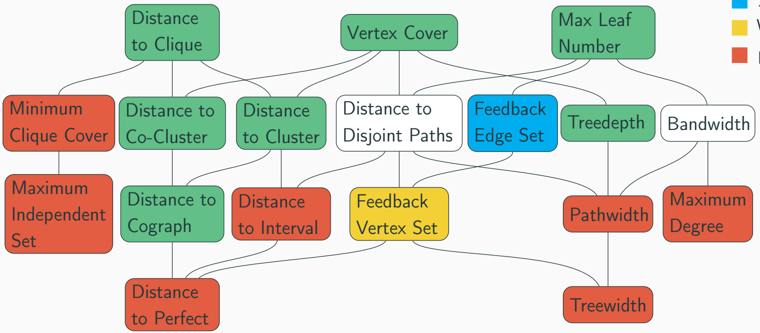
Open Questions

Q1: Are there certain properties shared by distance-based graph problems, that imply such running times? Is there a possible way to generalize our approach to a broader class of problems.

Q2: For which classic problems in NP are the best known FPT algorithms parameterized by tw, $\mathtt{vc}$ (or other parameters) double-exponential?

Q3: For which classic problems do the best known kernelization algorithms output a kernel with $2^{O(\mathtt{vc})}$ vertices?

… and for Metric Dimension



- FPT ($f(k) \cdot n^{O(1)}$-time algorithm)
- XP ($n^{f(k)}$-time algorithm)
- W[1]-hard (not FPT unless FPT = W[1])
- para-NP-hard (not XP unless P = NP)

n: size of input

k: size of parameter

Q4: XP or para-NP-hard parameterised by Feedback Vertex Set?

Q5: W[1]-hard or FPT parameterised by Feedback Edge Set?

Q6: Distance to Disjoint Paths? Bandwidth?

Thank you for your attention!

Further directions

Q1: Are there certain properties shared by distance-based graph problems, that imply such running times? Is there a possible way to generalize our approach to a broader class of problems.

Q2: For which classic problems in NP are the best known FPT algorithms parameterized by tw, $\mathtt{vc}$ (or other parameters) double-exponential?

Q3: For which classic problems do the best known kernelization algorithms output a kernel with $2^{O(\mathtt{vc})}$ vertices?

For Metric Dimension:

Q4: XP or para-NP-hard parameterised by Feedback Vertex Set?

Q5: W[1]-hard or FPT parameterised by Feedback Edge Set?

Q6: Distance to Disjoint Paths? Bandwidth?

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