

The Erdős-Pósa property for circle graphs as vertex-minors

Sang-il Oum

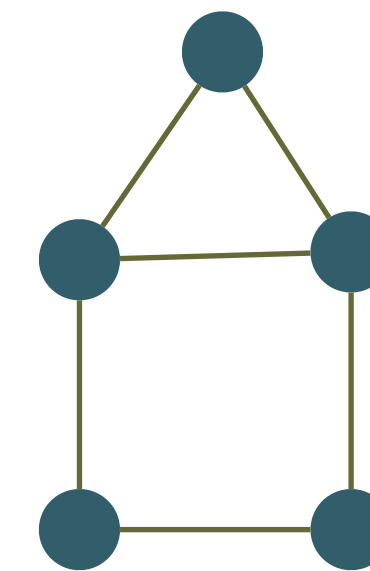
IBS Discrete Mathematics Group (Daejeon, Korea)

Joint work with

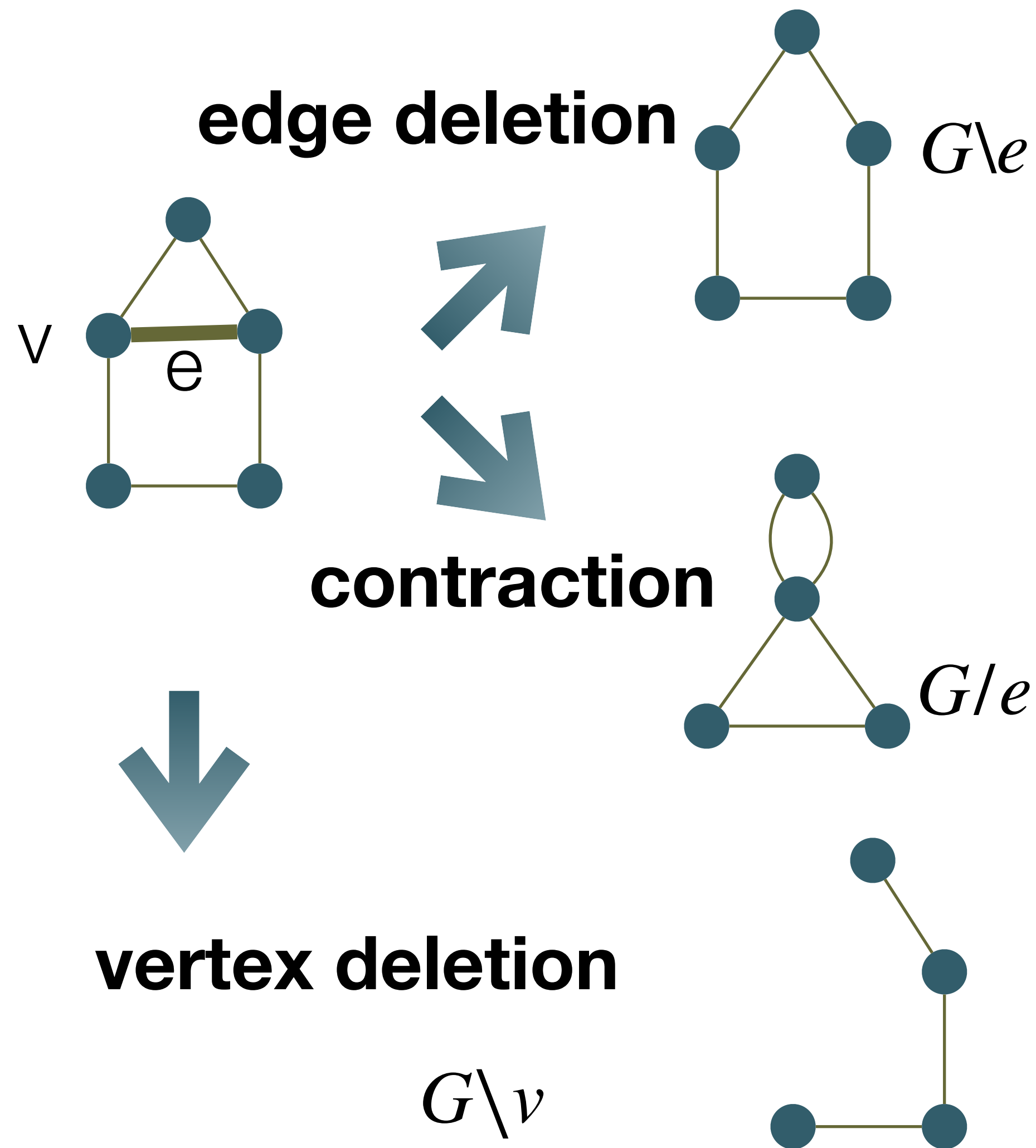
- Rutger Campbell (IBS → KAIST)**
- J. Pascal Gollin (IBS → Primorska)**
- Meike Hatzel (IBS → TU Darmstadt)**
- O-joung Kwon (Hanyang & IBS)**
- Rose McCarty (Georgia Tech)**
- Sebastian Wiederrecht (IBS → KAIST)**

Planar graphs and minors

A graph G is **planar** if it can be drawn on the plane with no edge crossing.



Operations that **preserve planarity**



H is a **minor** of G if
 H is obtained from G by a sequence of
deleting edges/vertices and
contracting edges

Easy Observation:
 If H is a **minor** of a planar graph G ,
 then H is planar.

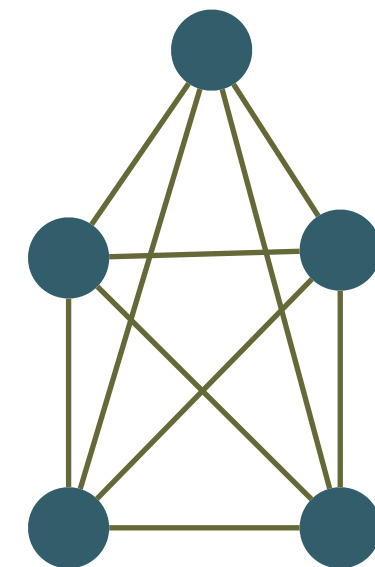
Kuratowski's theorem on planar graphs (1930)

Theorem (Kuratowski 1930 / Wagner 1937)

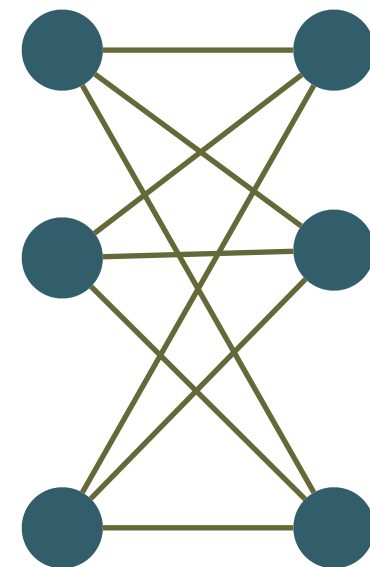
G is **planar**

if and only if

G has **no minor** isomorphic to **K_5 or $K_{3,3}$**



K_5



$K_{3,3}$

Two minor-minimal non-planar graphs

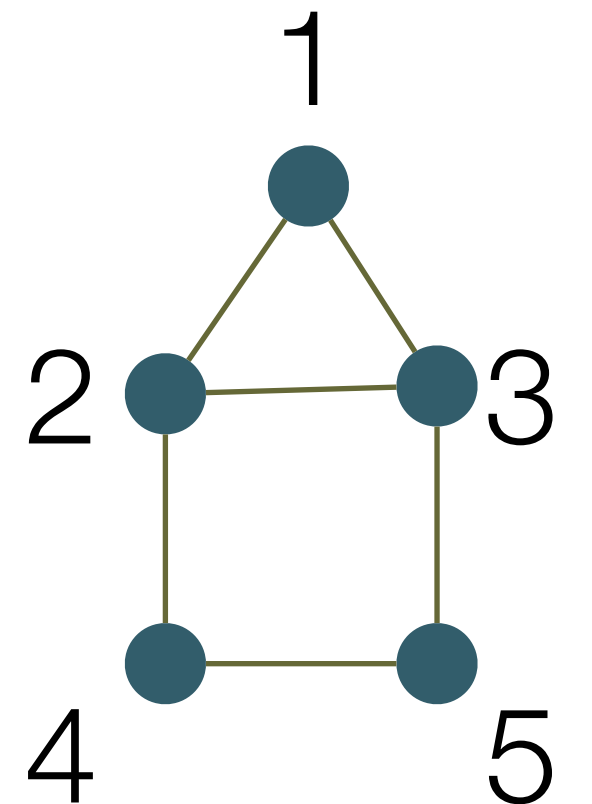
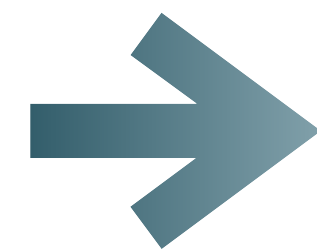
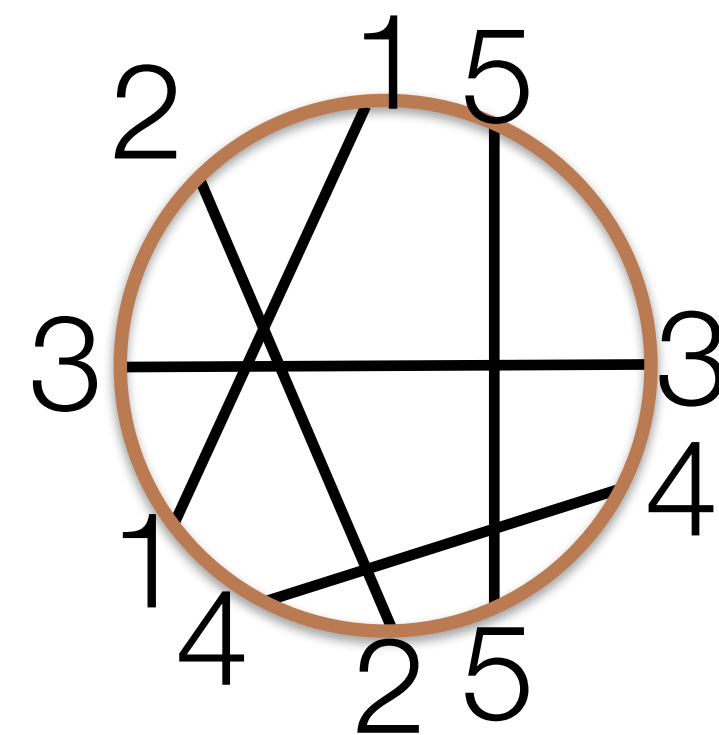
Circle graphs and **vertex-minors** / **pivot-minors**

Circle graph = **intersection** graph of **chords** of a circle

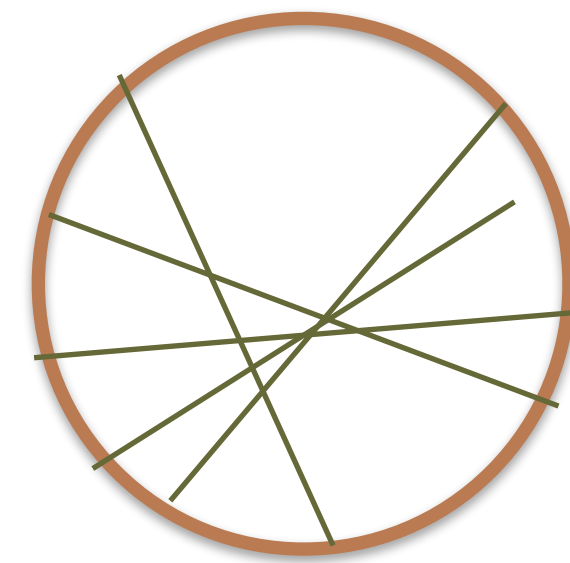
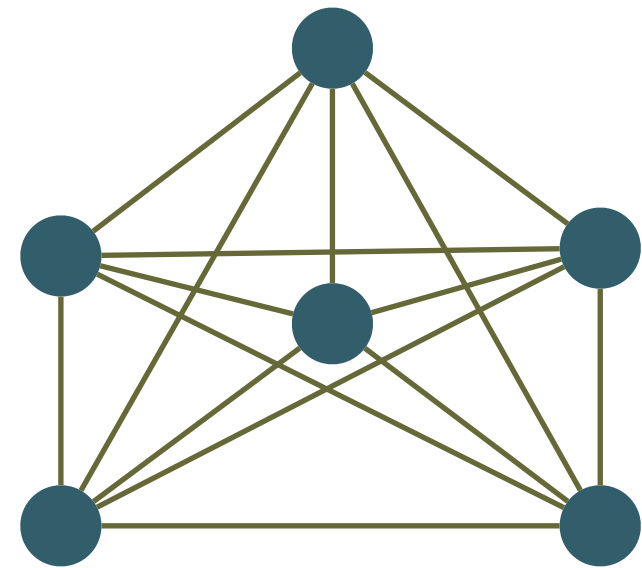
chords
intersect



vertices
edges

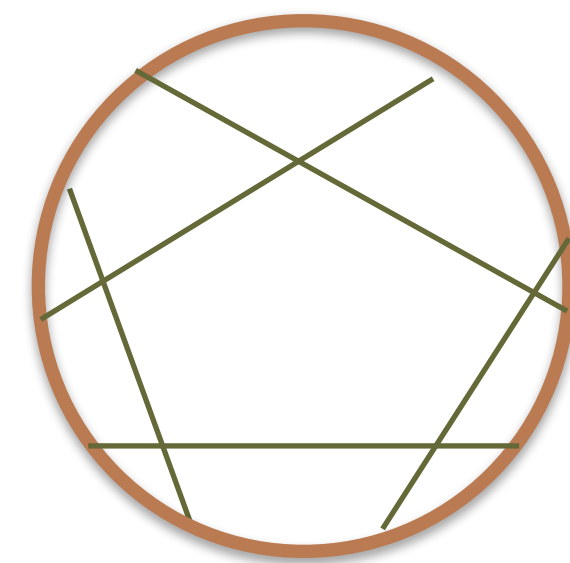
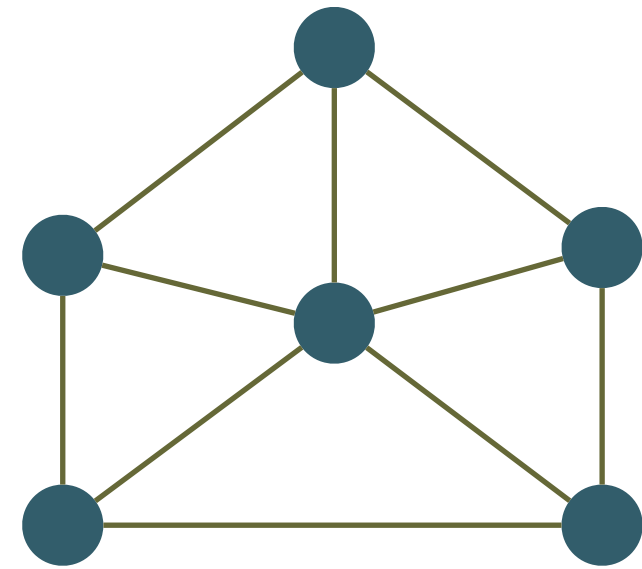


A minor of a circle graph is **not** necessarily a circle graph.



K_6 is a circle graph

minor ↓

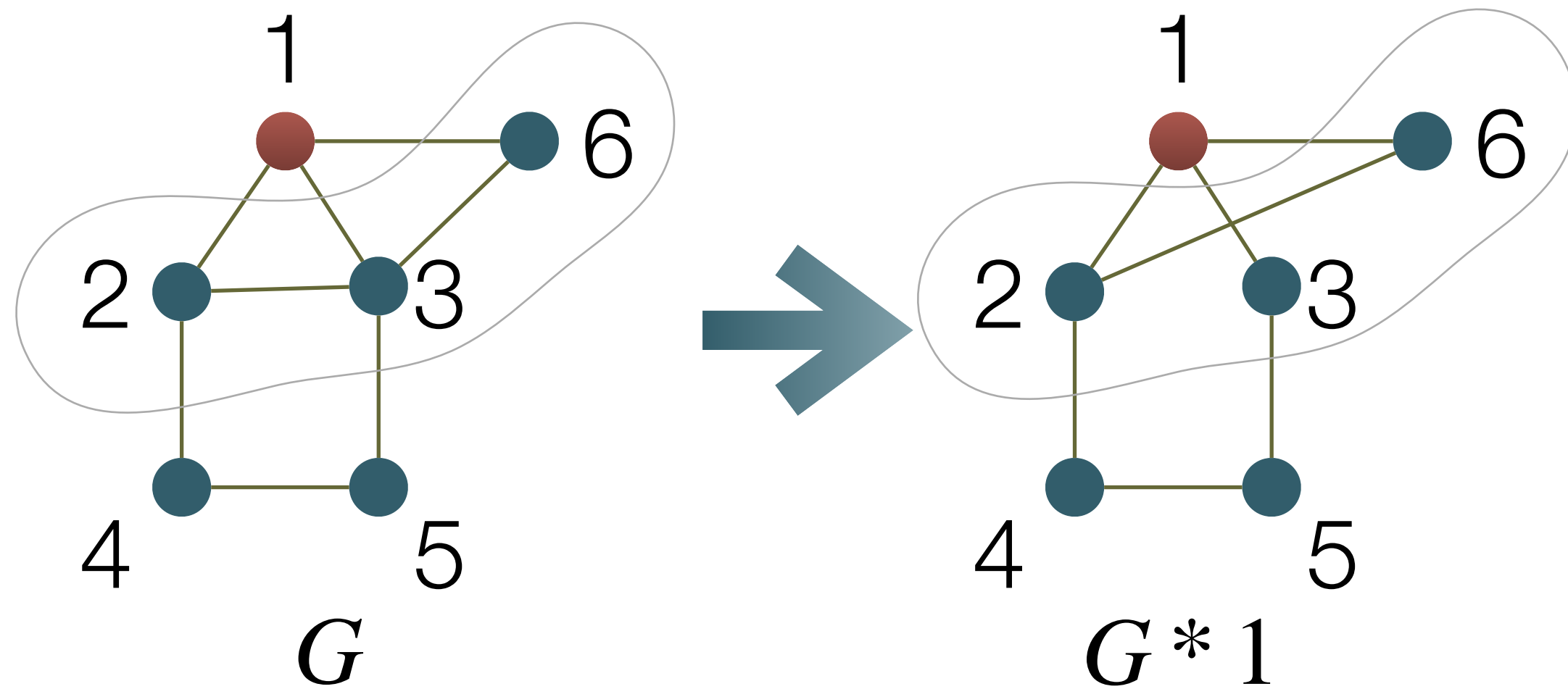


W_5 is **not** a circle graph



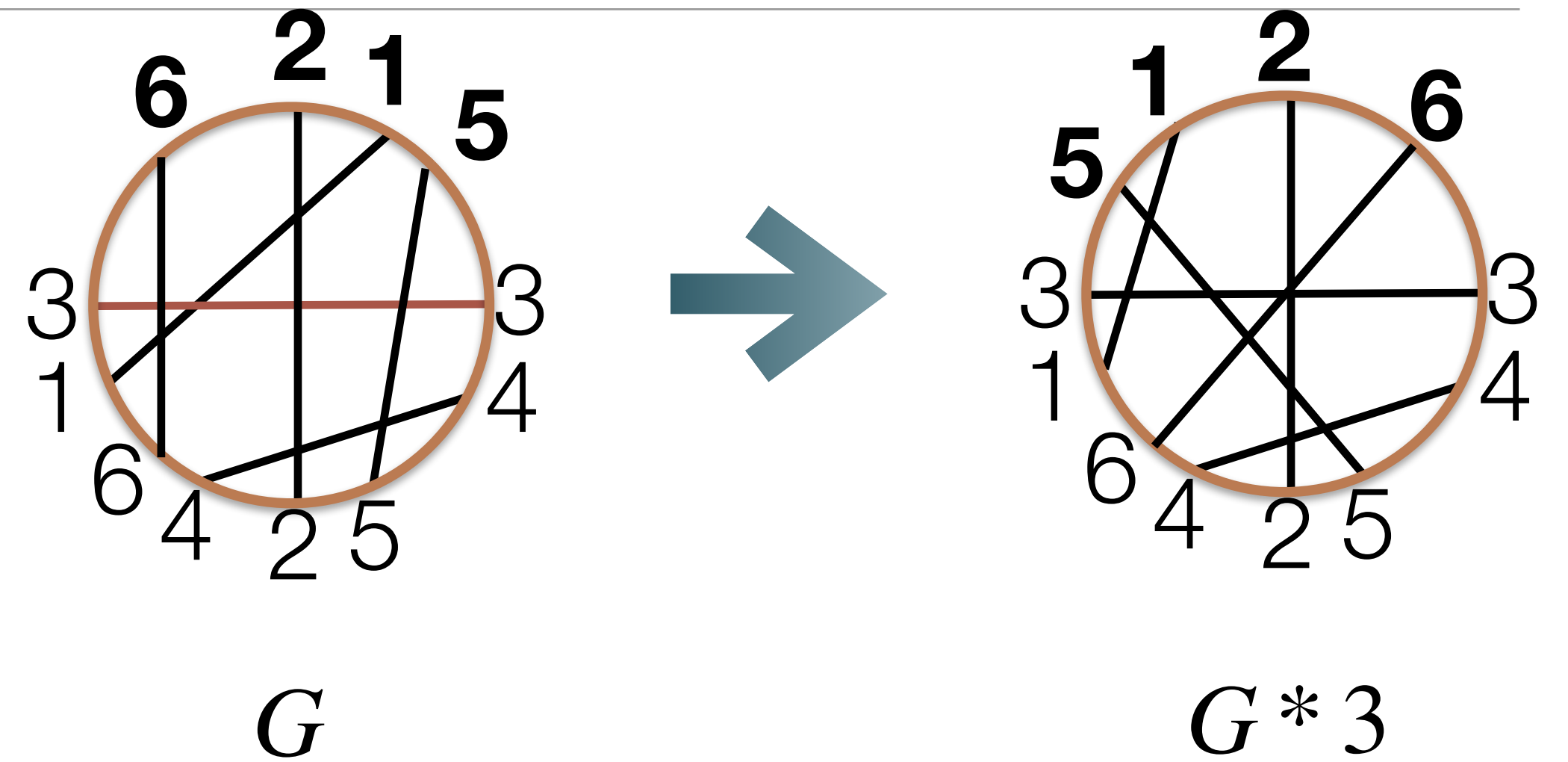
Impossible to
characterize circle graphs
in terms of forbidden
minors

Local complementation and vertex-minors



"local complementation at 1"

H is a **vertex-minor** of G if H can be obtained from G by a sequence of **local complementations** and **vertex deletions**



If G is a circle graph,
then $G*v$ is a circle graph.

Easy Observation:

If H is a vertex-minor of G and
G is a circle graph,
then H is a circle graph.

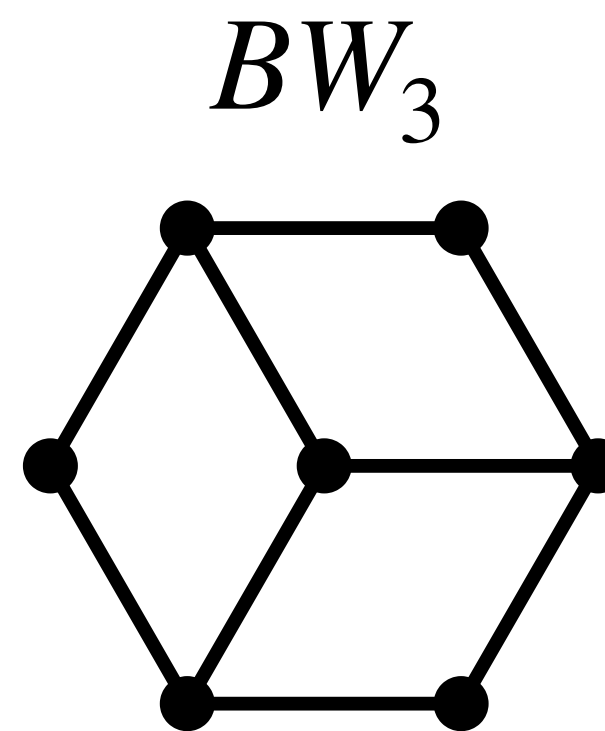
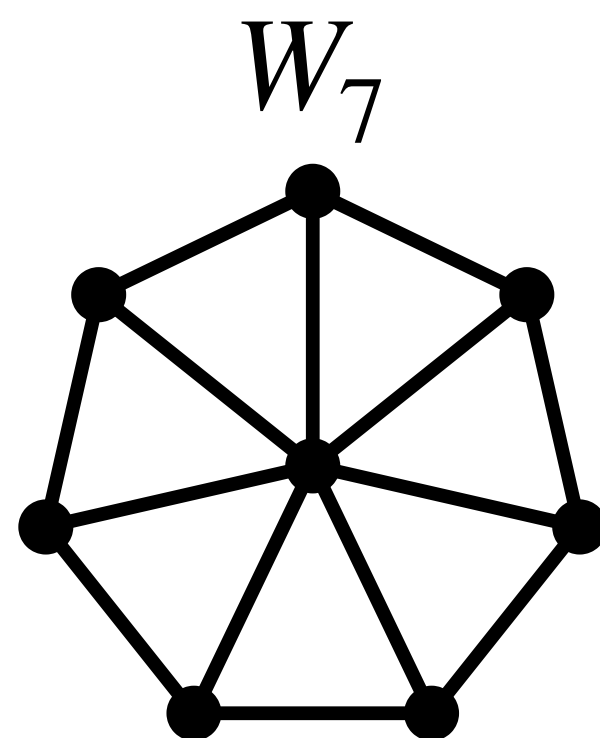
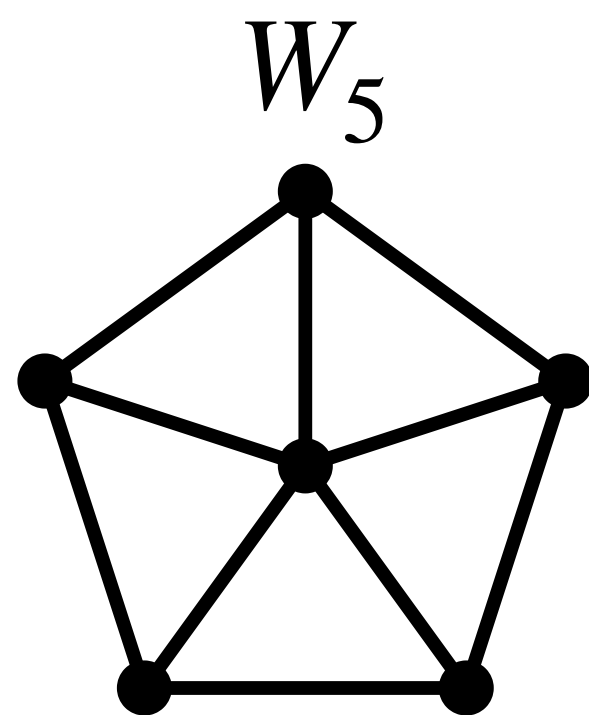
Forbidden **vertex-minors** for circle graphs

Theorem (Bouchet 1994)

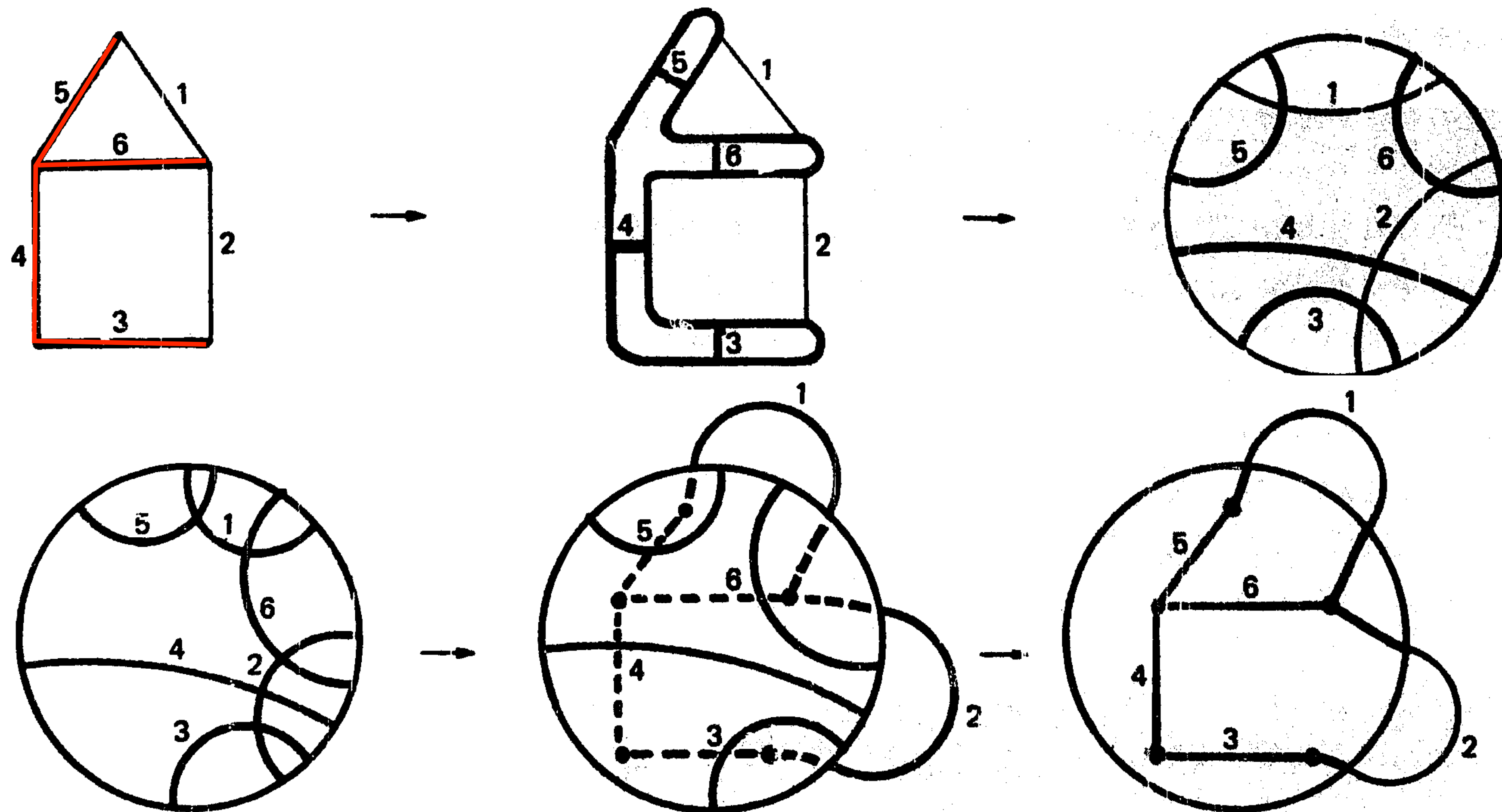
G is a **circle graph**

if and only if

G has no **vertex-minor** isomorphic to



Three vertex-minor minimal non-circle graphs



Theorem (de Fraysseix 1981)

A bipartite graph is a circle graph

if and only if it is a **fundamental graph** of a planar graph.

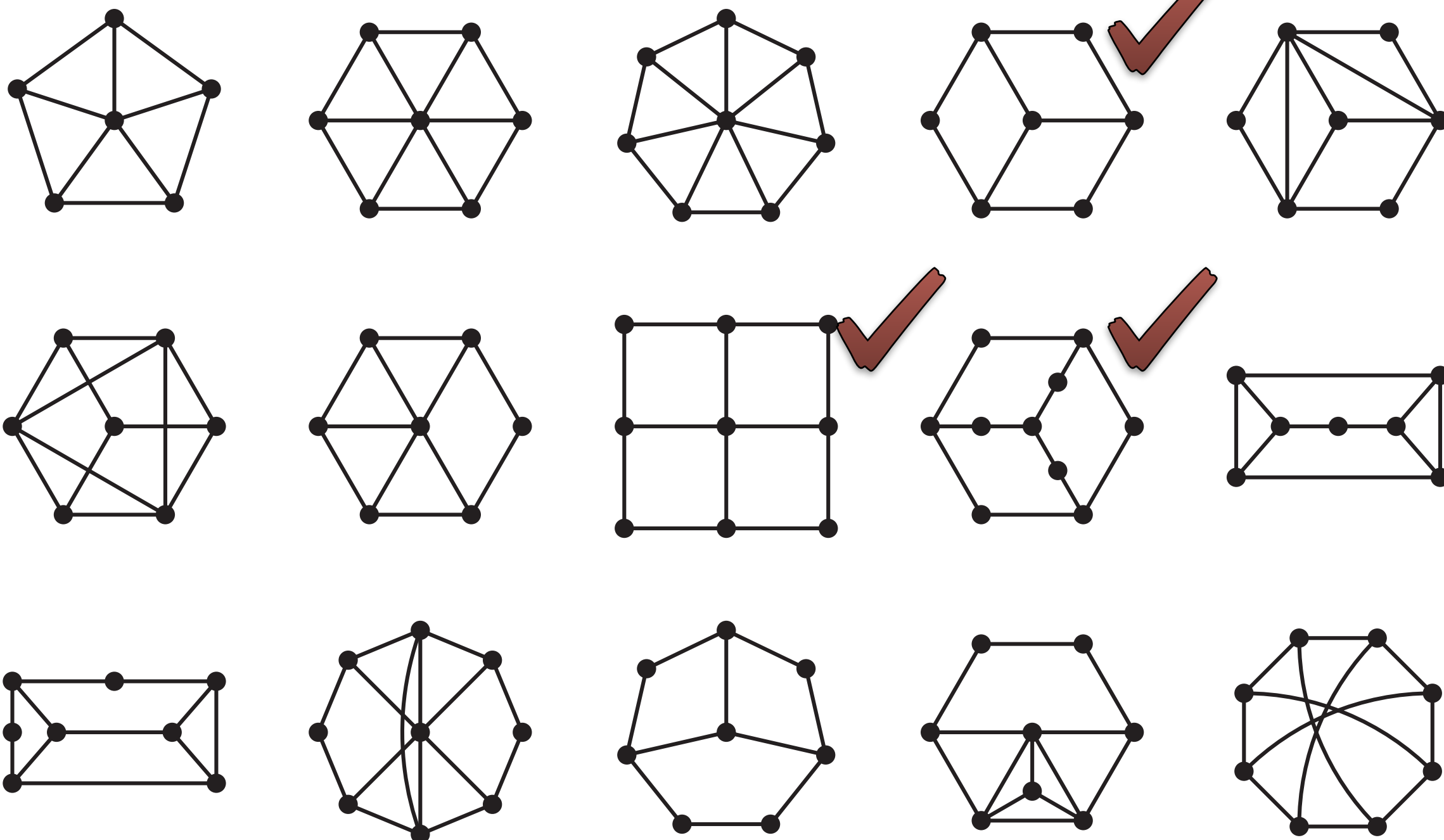
Forbidden **pivot-minors** for circle graphs

Theorem (Geelen, O. 2009)

G is a **circle graph**

if and only if

G has no **pivot-minor** isomorphic to

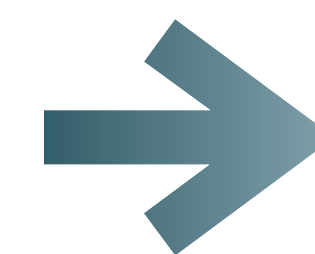


✓ 3 bipartite graphs

Fundamental graphs of
the Fano matroid

$M(K_5)$

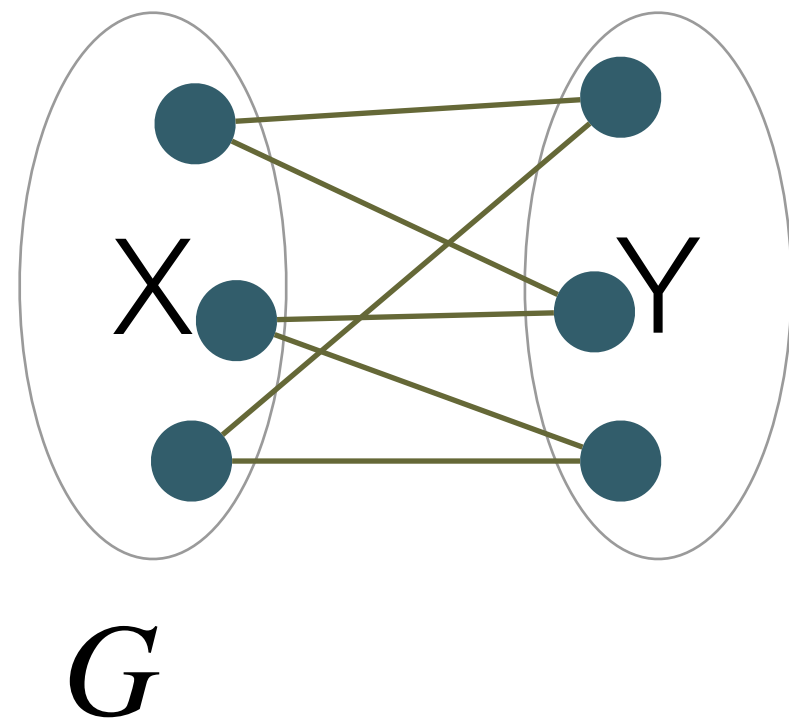
$M(K_{3,3})$



Implies
Kuratowski's theorem

Rank-width : *dense analogue of tree-width*

O., Seymour (2006)



$$\rho_G(X) = \text{rank} \begin{matrix} & \xleftarrow{\quad Y \quad} & \\ \uparrow X \uparrow & \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} & \\ & \xrightarrow{\quad Y \quad} & \end{matrix} = 2$$

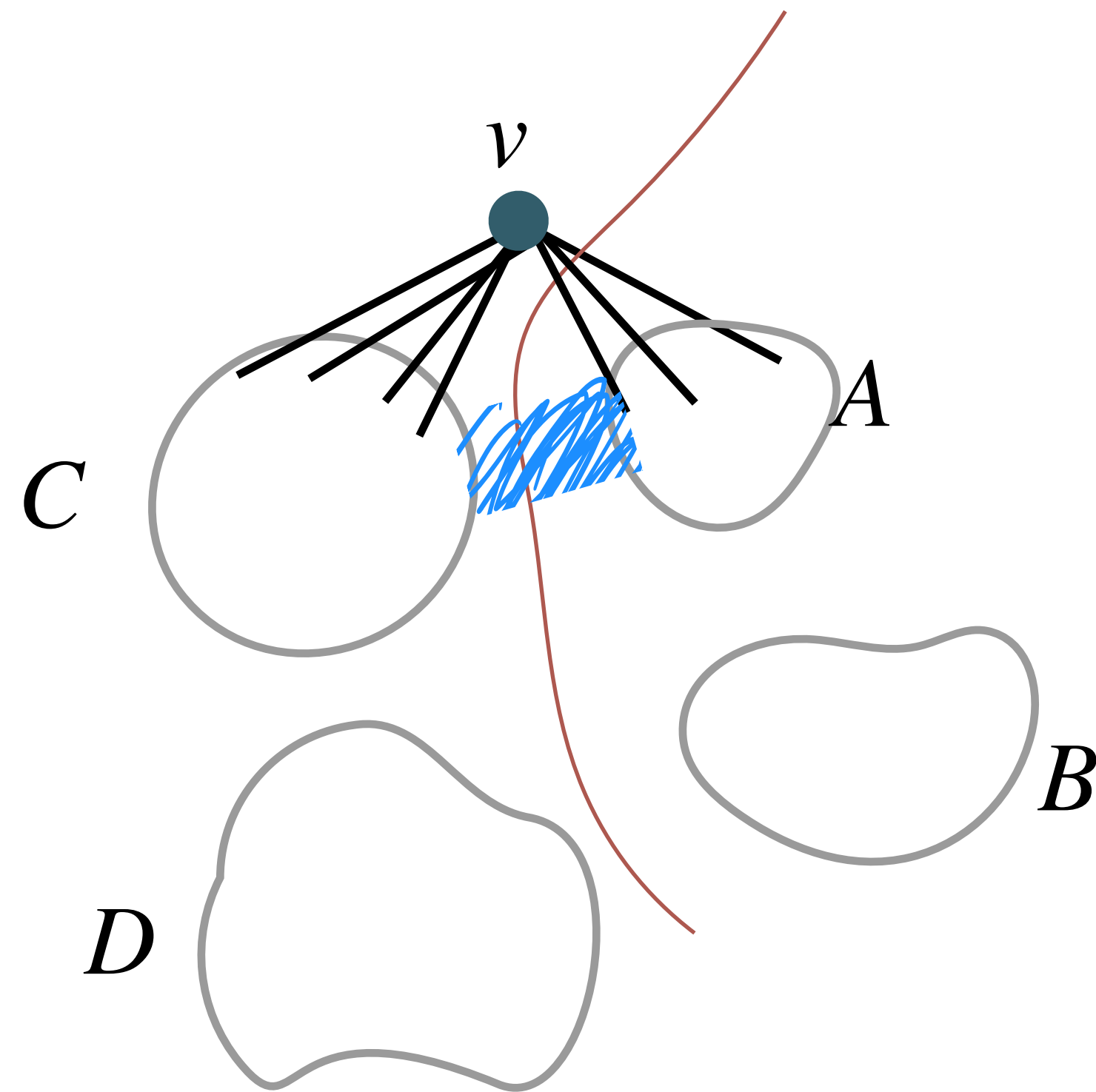
Cut-rank function of G:
rank of a $X^*(V-X)$ 0-1 matrix
over the binary field

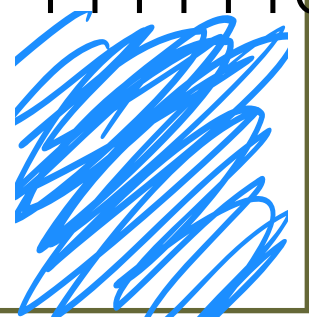
Rank-width of $G = \min k$ such that
 $V(G)$ can be recursively partitioned into
subsets of
"**cut-rank**" $\leq k$
until each set becomes a singleton

Clique-width and rank-width
are equivalent

Theorem (O. Seymour 2006)
 $\text{rw}(G) \leq \text{cw}(G) \leq 2^{\text{rw}(G)+1} - 1$

Cut-rank function and vertex-minors



	A	B
v	1111110	0000000
C		
D		

$$\rho_G(X) = \rho_{G*v}(X)$$

The cut-rank function is **invariant under** taking **local complementations**

If H is a vertex-minor of G , then for all X ,

$$\rho_H(X') \leq \rho_G(X)$$

for its corresponding X' .

Obstructions for bounded tree-width bounded rank-width

Theorem (Robertson and Seymour 1986)
Every graph of large tree-width contains a fixed planar graph as a minor.

Large tree-width
 $\Rightarrow$ large grid minor

Theorem (Geelen, Kwon, McCarty, Wollan 2020)
Every graph of large rank-width contains a fixed circle graph as a vertex-minor

Large rank-width
 $\Rightarrow$ large circle graph minor

Erdős-Pósa property

Every graph has
 $> k$ vertex-disjoint **cycles**
 or
 a vertex set X such that
 $|X| \leq f(k)$ and $G-X$ has no cycles

Dirac and Gallai: Does $f(k)$ exist?
 If so, then what is the best possible $f(k)$?

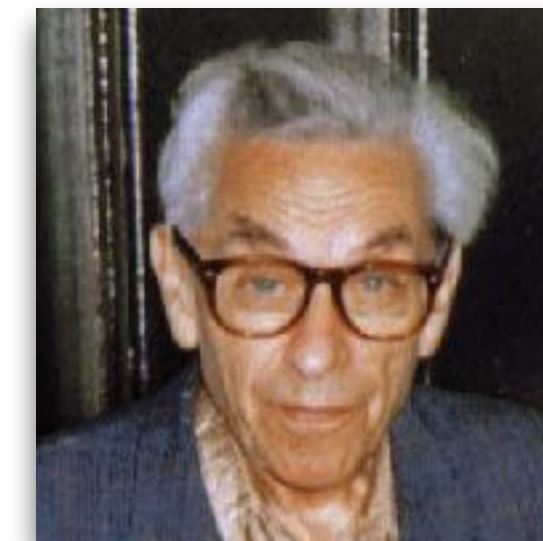
Lovász (1965): $f(1)=3$

tight K_5

Voss (1969): $f(2)=6$

Voss (1967): $f(3) \leq 12$

Erdős-Pósa (1965): $f(k) = \Omega(k \log k)$



We say {cycles} has the ***Erdős-Pósa property***

Many generalizations...

Characterizing the Erdős-Pósa property for minors

Theorem (Robertson, Seymour 1986)

Every graph has
 $> k$ vertex-disjoint **H-minors**
 or
 a vertex set X such that
 $|X| \leq f(k)$ and $G-X$ has no **H-minors**

$\Leftrightarrow$ H is planar.

Main ingredient for $\Leftarrow$: Grid Theorem (Robertson and Seymour)

If H is **planar**, graphs with no H minor have **tree-width** $\leq g(H)$.

Our goal:

vertex-minor analogue of the Erdős-Pósa theorem

Erdős-Pósa for vertex-minors?

having
 kH
as a minor

H : **planar**



$G-X$ has
no H minor

having
 kH
as a vertex-minor

H : **circle**



?

$G-X$ has no H
vertex-minor

Erdős-Pósa for vertex-minors?

having
 kH
as a minor

H : **planar**



$G-X$ has
no H minor

having
 kH
as a vertex-minor

H : **circle**



?

there is a graph G'
locally equivalent to
 G such that
 $G'-X$ has no H
vertex-minor

Deletion or (Local Complementations + Deletion) **Fails**

Lemma:

For every t , there exists a graph G such that

(1) no vertex-minor isomorphic to $2P_4$

(2) every graph G' locally equivalent G and every set X of t vertices, $G'-X$ has a vertex-minor isomorphic to P_4 .

Proof Sketch: Take $G=K_{\{t+2,t+2,\dots,t+2\}}$
(complete multipartite graph with $t+3$ parts)

Main Theorem: Erdős-Pósa for vertex-minors

Theorem:

For every circle graph H with ≥ 1 edge and $k > 0$,
there is an integer t such that

every graph G either

(1) has a vertex-minor isomorphic to kH

or

(2) has a **t -perturbation** G' having no vertex-minor isomorphic to H

Perturbation of a graph

Potential definitions of perturbations

- $\text{Adj}(G') = \text{Adj}(G) + M$ + diagonal matrix over the binary field for a matrix M of $\text{rank} \leq r$ (**rank- t perturbation**)
- $V(G') = V(G)$ and there is a graph H such that $|V(H)| = |V(G)| + t$ and both G' and G

These two concepts are equivalent up to the local equivalence.

(1) **Any t -perturbation is locally equivalent to a rank- $2t$ perturbation.**

(2) **Any rank- t perturbation is a t -perturbation.**

Lemma: The set of t -perturbations of G is closed under taking vertex-minors.

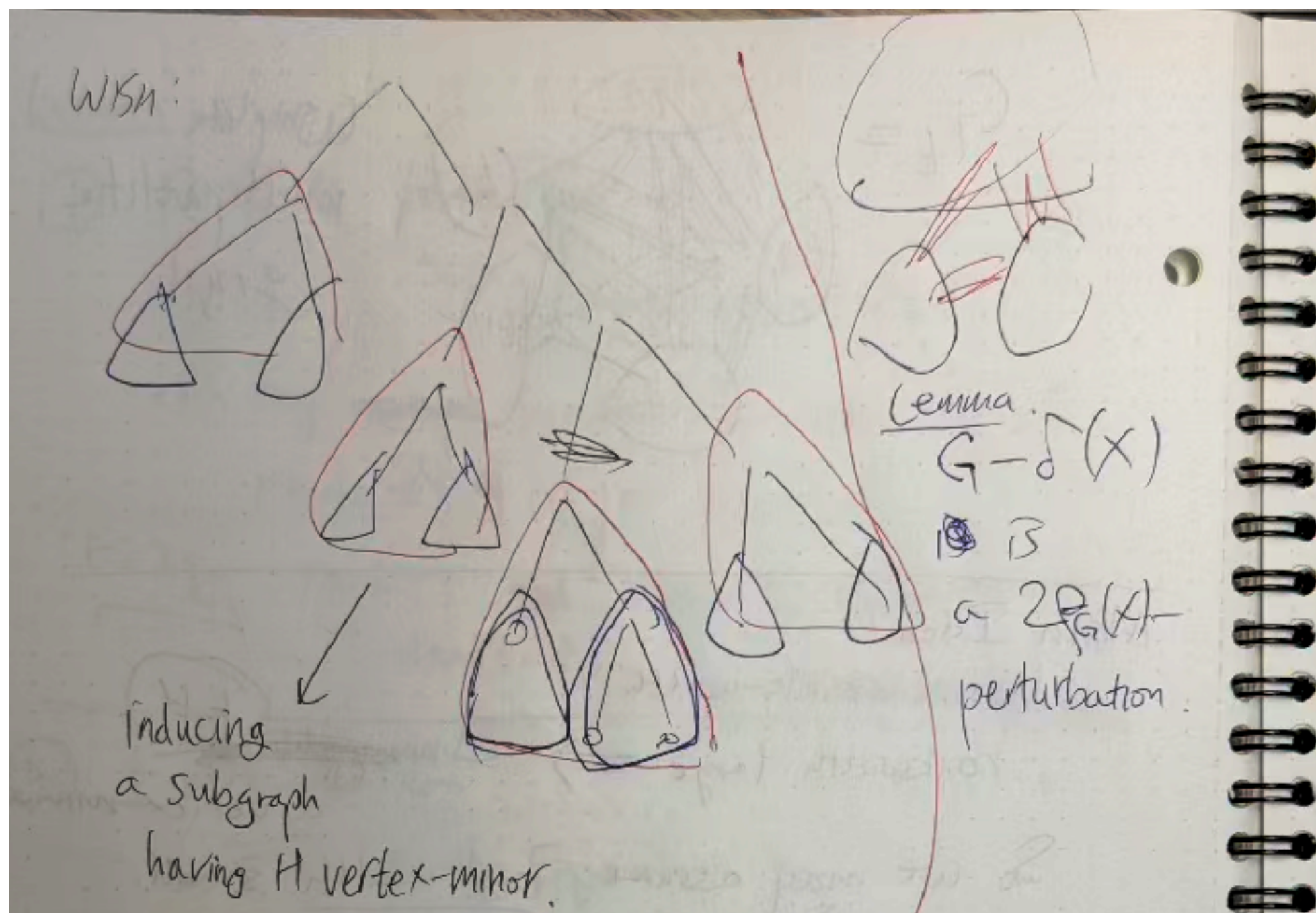
False for rank- t perturbations and their locally equivalent graphs

Proof Sketch

Let us assume H is connected for simplicity.

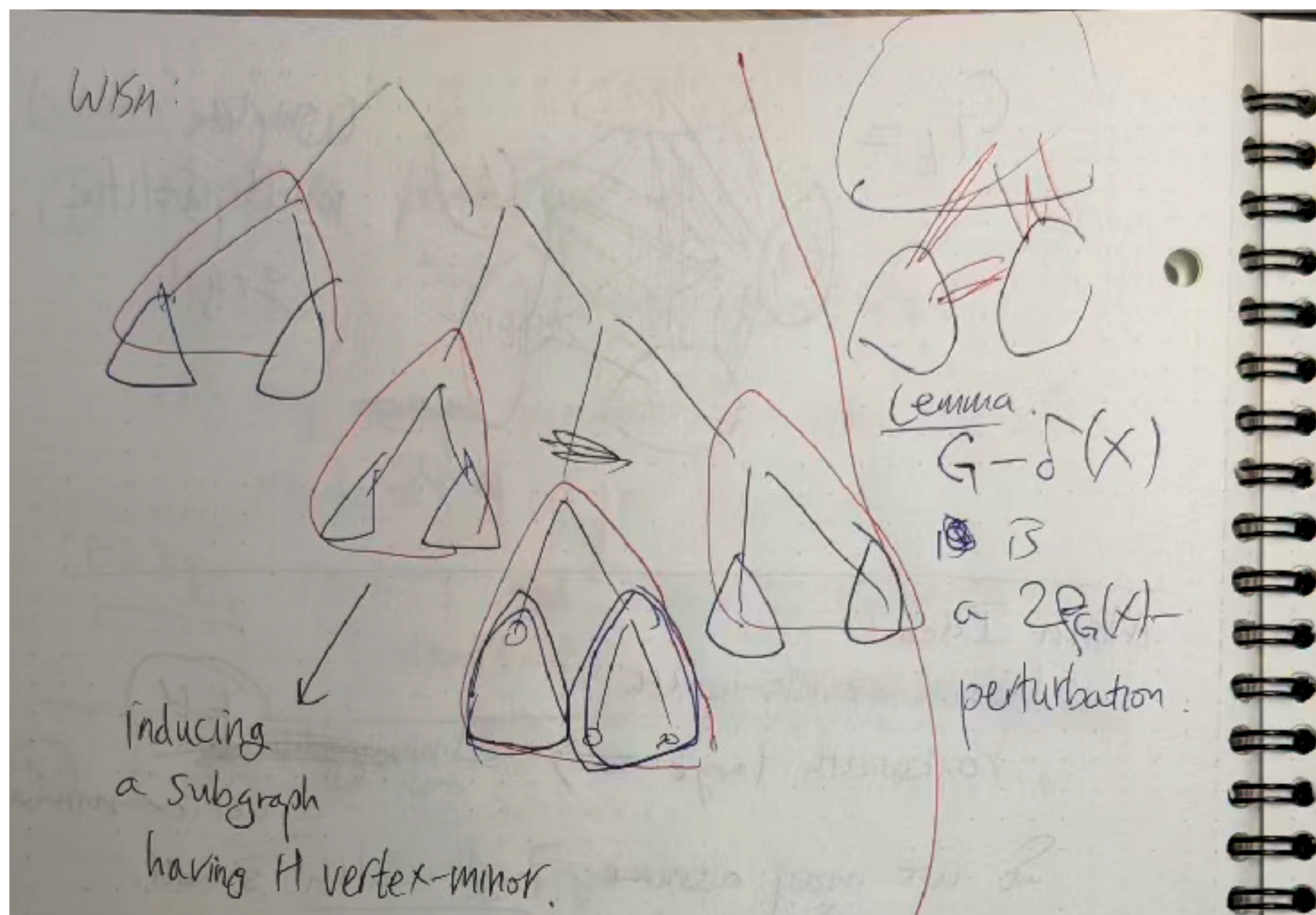
(The paper discusses the case when H is disconnected as well, based on the lemma of Robertson and Seymour.)

- We may assume that rank-width is small.
Otherwise, kH is a vertex-minor. (kH is a circle graph)
- Let (T, L) be a rank-decomposition of small width.
- **1st try:** Take disjoint subtrees of T , whose leaves induces a subgraph having H vertex-minors.



If we can't find many
disjoint such subtrees,
then
we have a low perturbation of G
having no H vertex-minors

(GOOD)



If we can't find many
disjoint such subtrees,
then
we have a low perturbation of G
having no H vertex-minors

(GOOD)

Problem: Suppose we found many disjoint such subtrees.
How do we get kH vertex-minors?

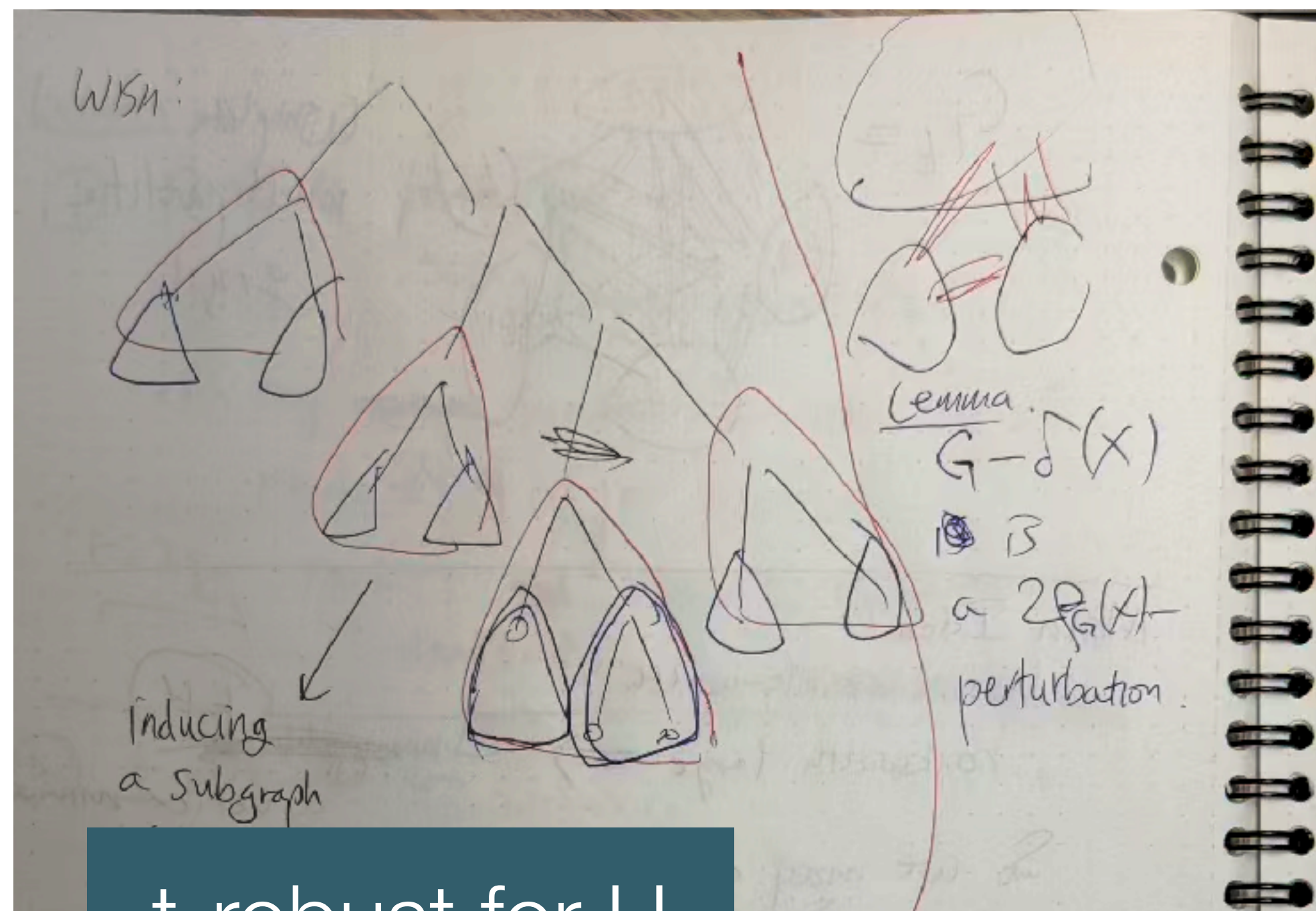
- Applying one local complementation in one part may destroy H vertex-minors in other parts.

2nd try: “Robustness”

- Aim: Find X such that $G[X]$ has H as a vertex-minor, **no matter what local complementations are made outside of X .**
- We say $G[X]$ is **t-robust for H** if any t -perturbation of $G[X]$ has H as a vertex-minor.

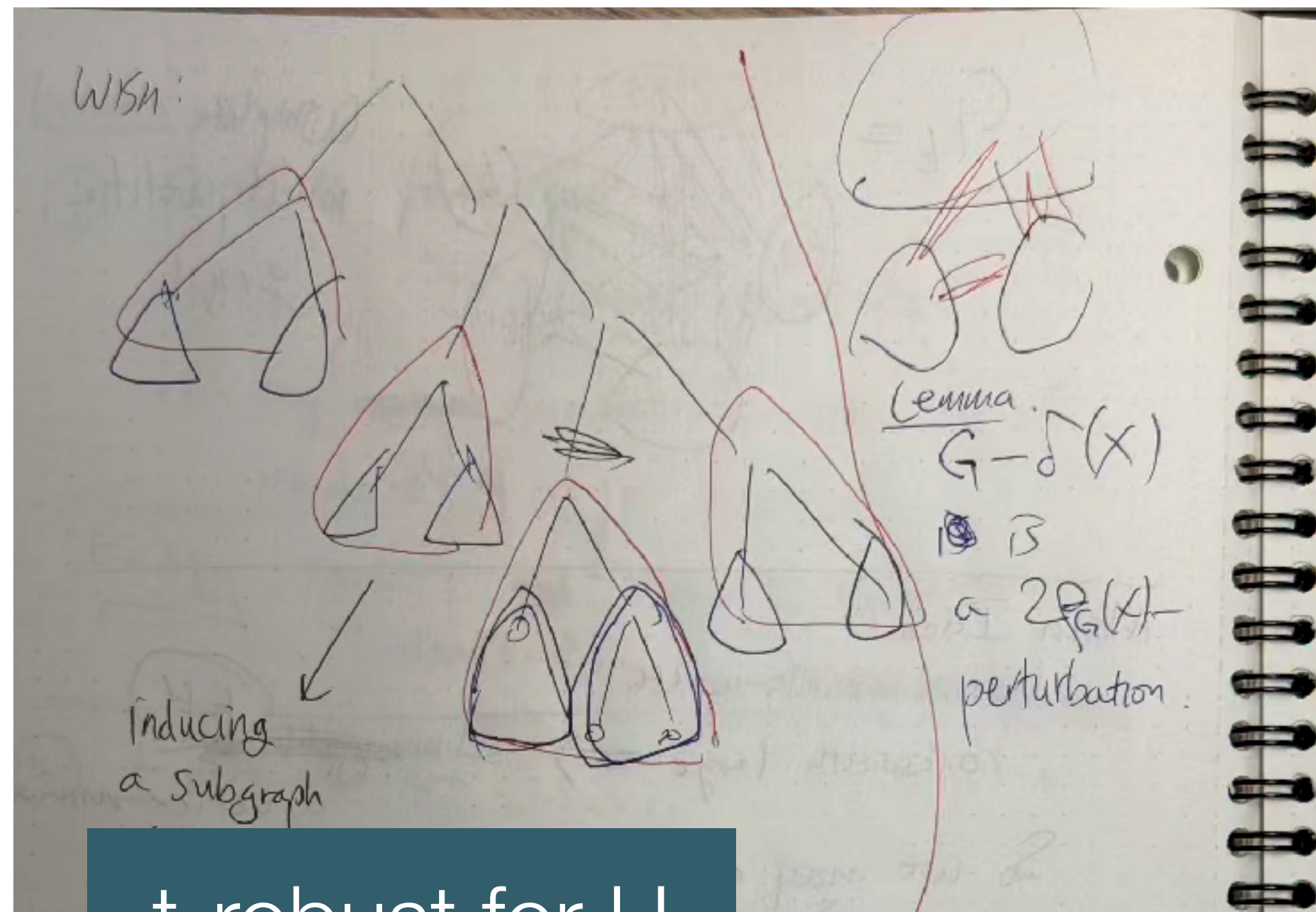
Lemma: If $\text{cut-rank}_G(X)=k$ and both G_1 and G_2 are vertex-minors of G on X , then G_1 is a 2^{k+1} -perturbation of G_2 .

- Goal: Find many disjoint subtrees, each inducing a subgraph t -robust for H with $t= 2^{(\text{rank-width}+1)}$



If we can't find many
disjoint such subtrees,
then
we have a low perturbation of G
having no H vertex-minors

(GOOD)



Assume we found many disjoint such subtrees.

Lemma: If $\text{cut-rank}_G(X)=k$ and both G_1 and G_2 are vertex-minors of G on X , then G_1 is a 2^{k+1} -perturbation of G_2 .

How to clean edges between distinct H 's to get kH ?

Another trick: **Well-quasi-ordering**

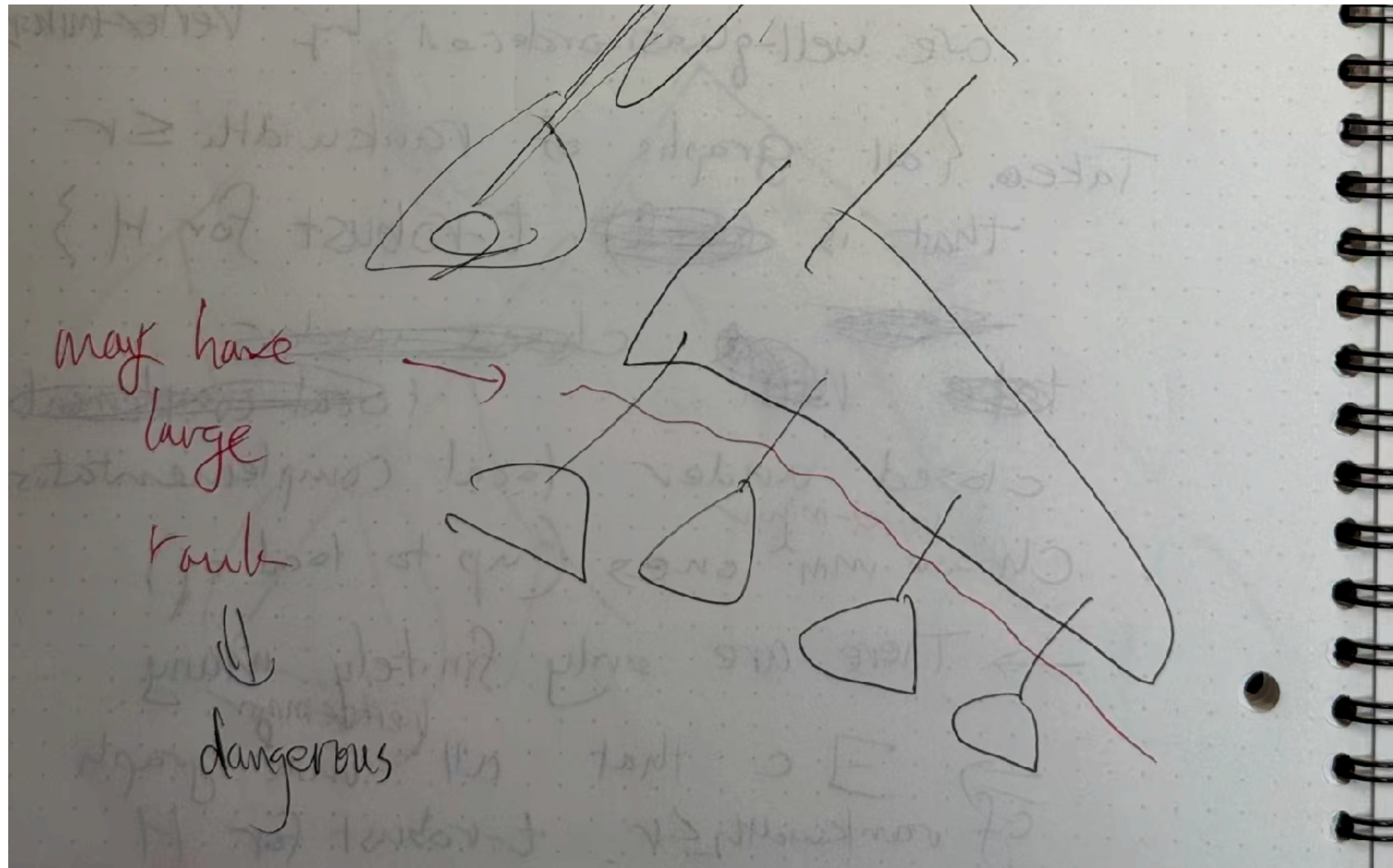
Graphs of bounded rank-width are well-quasi-ordered by vertex-minors. (O. 08)

{all vertex-minor-minimal graphs of rank-width $\leq r$, that is t -robust for H }
is finite

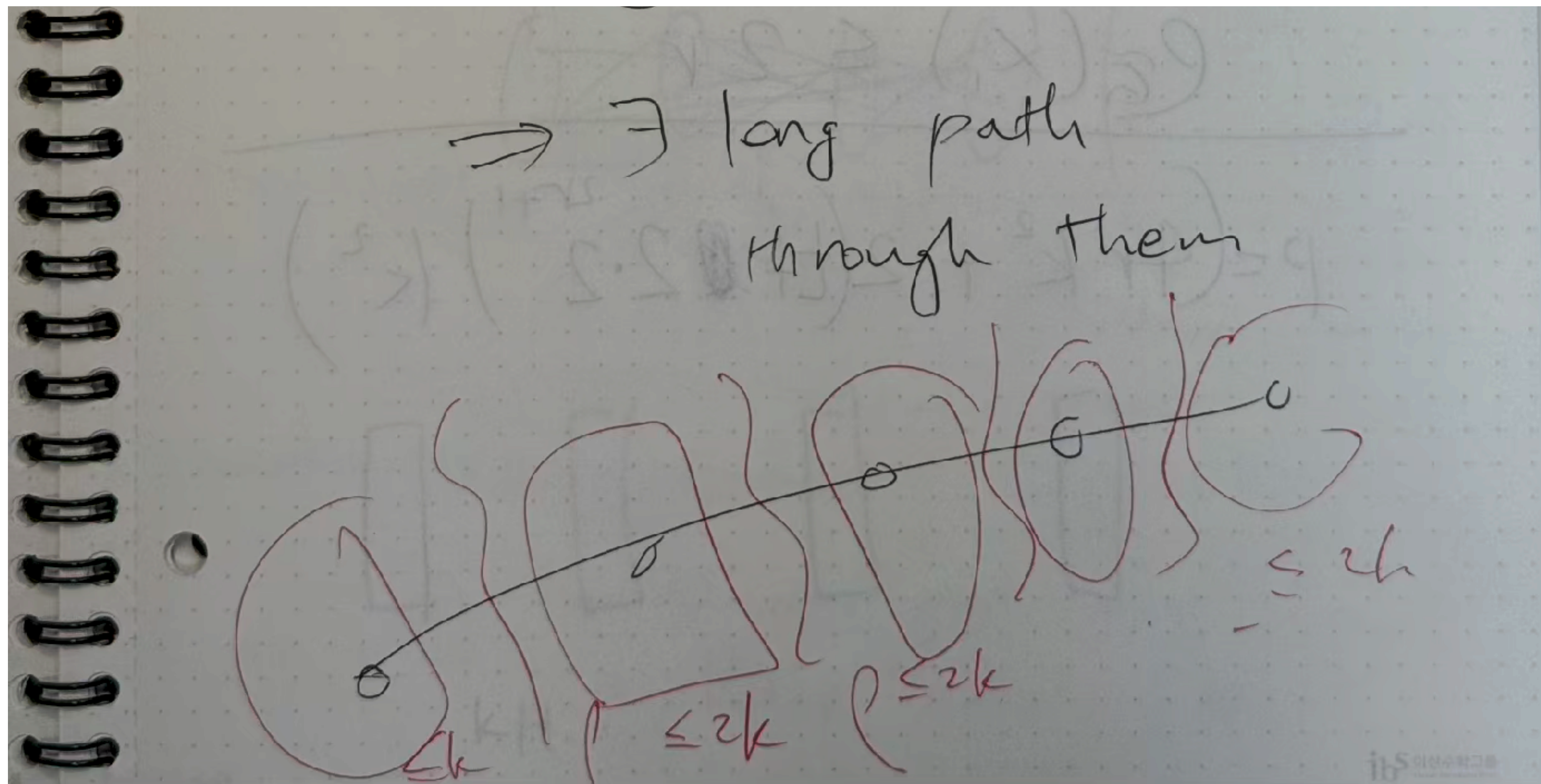
$\Rightarrow$ There are only finitely many vertex-minor-minimal graphs
of rank-width $\leq r$, t -robust for H

There is c such that
every induced subgraph t -robust for H
contains a vertex-minor on $\leq c$ vertices that is t -robust for H .

One more trick: Pick subtrees arranged linearly from many subtrees



One more trick: Pick subtrees arranged linearly from many subtrees



Proposition: H : connected graph, G : rank-width $\leq r$

$\Rightarrow$ Either

(1) there is a p -perturbation of G with no H vertex-minor

or

(2) there is a vertex-minor G' of G

that is a union of disjoint sets X_i

$|X_i| \leq c$,

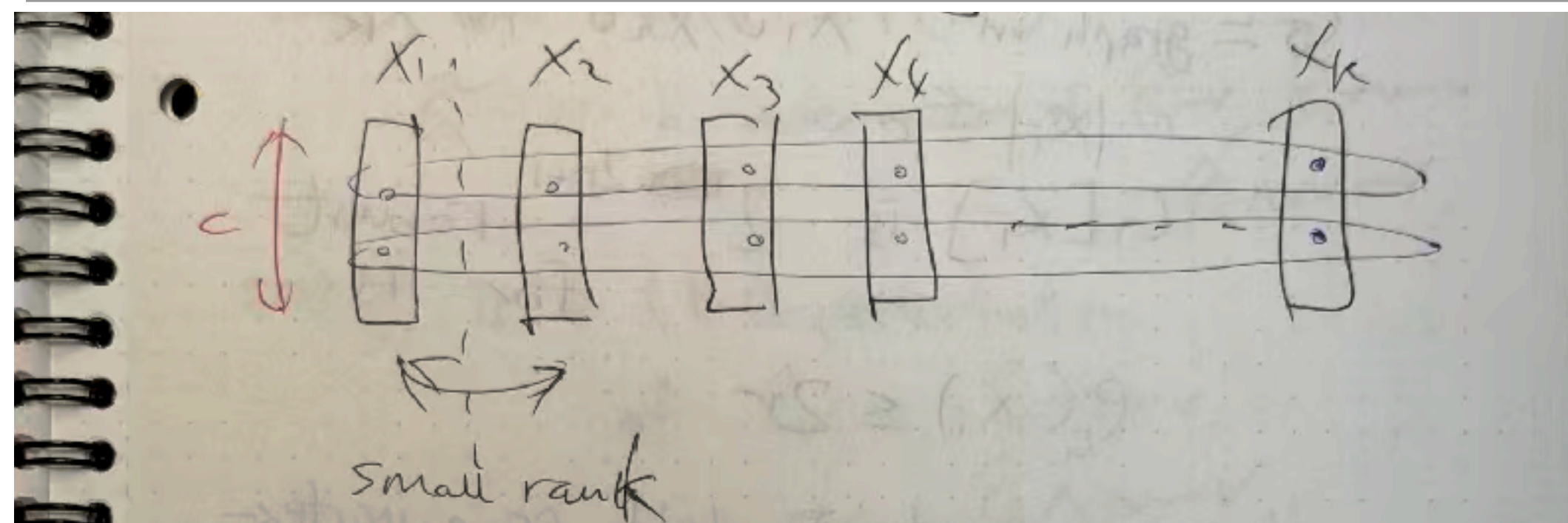
$G'[X_i]$ is t -robust for H ,

$\text{cut-rank}_{G'}(X_i) \leq 2r$.

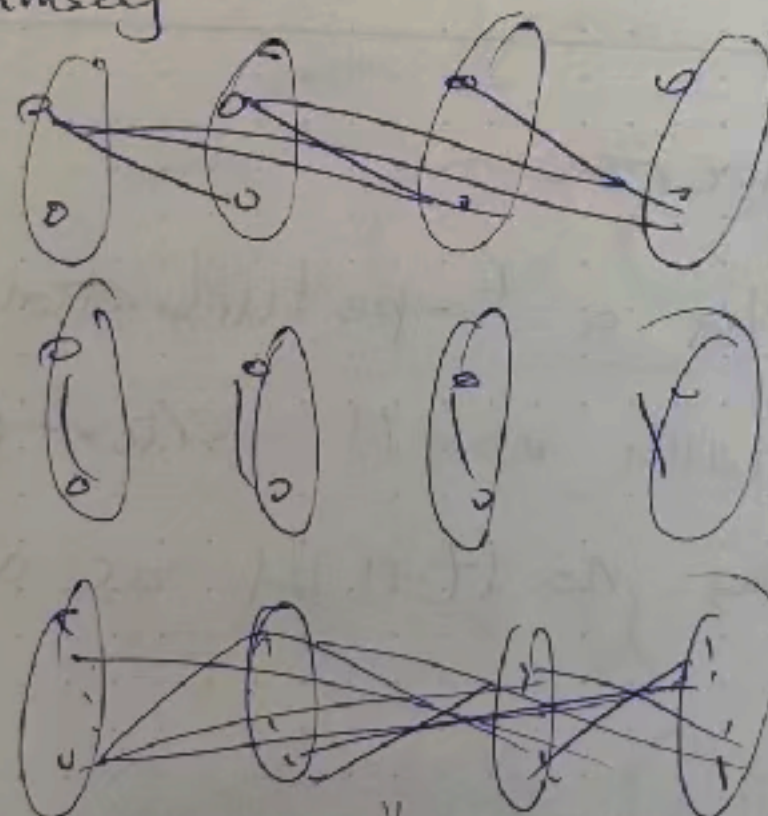
$$p = 4rk^2 + 2(t + 2 \cdot 2^{2r+1})k^2$$

How do we get kH ?

Ramsey



By Ramsey



Apply local complementations
to clean!

kH

Converse

Converse

- **Proposition:** If G has a t -perturbation G' having no H vertex-minor, then G has no $(t+1)H$ as a vertex-minor.
- Tool: Bouchet's theorem
If H is a vertex-minor of G and v is a vertex of G not in H , then at least one of $G-v$, G^*v-v , $G^*v^*w^*v-v$ (for a neighbor w of v) contains H as a vertex-minor.

Summary: Main Theorem

Theorem 1.1. *For any circle graph H with at least one edge and for any positive integer k , there exists an integer $t = t(k, H)$ so that every graph G either has a vertex-minor isomorphic to kH , or has a t -perturbation $\tilde{G}$ which has no vertex-minor isomorphic to H .*

Open Problems

- What happens if H is not a circle graph?
 - **Conjecture 1: The Erdős-Pósa property still holds.**
 - **Conjecture 2: The Erdős-Pósa property fails.**
- Is it true for pivot-minors?
 - True for bipartite graphs G ; Corollary for binary matroids.

Corollary 1.4. *For any planar multigraph H and for any positive integer k , there exists an integer $p = p(k, H)$ so that each binary matroid M either has a minor isomorphic to $M(kH)$, or has a rank- p perturbation $\widetilde{M}$ which has no minor isomorphic to $M(H)$.*

- Matroids: James Davies, Meike Hatzel, and Fernanda Rivera Omaña

Thank you

We are hiring!



<https://dimag.ibs.re.kr/hiring/>

THE IBS DISCRETE MATHEMATICS GROUP (DIMAG) RESEARCH FELLOWSHIP



DIMAG website: <https://dimag.ibs.re.kr>
Daejeon, Korea

Inquiry: dimag@ibs.re.kr

The IBS Discrete Mathematics Group (DIMAG) in Daejeon, Korea invites applications for **four** research fellowship positions (senior researcher positions).

DIMAG is a research group that was established on December 1, 2018 at the Institute for Basic Science (IBS), led by its Chief Investigator (CI) Sang-il Oum. DIMAG is located at the headquarters of IBS in Daejeon, South Korea, a city of 1.5 million people. Currently, DIMAG consists of researchers from various countries and the work is done in English. DIMAG is co-located with the IBS Extremal Combinatorics and Probability Group (ECOPRO).

Successful candidates for the research fellowship positions will be new or recent PhDs with outstanding research potential in all fields of **Discrete Mathematics** with emphasis on **Structural Graph Theory, Extremal Graph Theory, Combinatorial Optimization, Matroid Theory, and Algorithms**. This appointment is for about two years, and the starting salary is no less than KRW 63,000,000. The appointment is one-time renewable up to 5 years in total, contingent upon the outstanding performance of the researcher. The expected appointment date is September 1, 2026, and it can be adjusted to earlier or later, but no later than December 1, 2026. This is purely a research position and will have no teaching duties. However, if desired, successful candidates may have the opportunity to teach at KAIST.

A complete application packet should include

- ① AMS standard cover sheet or cover letter
- ② Curriculum Vitae including a list of publications
- ③ Research statement
- ④ Consent to Collection and Use of Personal Information (form available on the application page)
- ⑤ 3 recommendation letters

For full consideration, applicants should upload items 1, 2, 3, and 4 on the application website (<https://dimag.ibs.re.kr/apply>) and arrange their recommendation letters to be submitted directly by the letter writers through the link sent via email during the online application process by **December 3, 2025**, Wednesday (Anywhere on Earth). Recommendation letters submitted by the applicants themselves will not be considered. DIMAG encourages applications from individuals of diverse backgrounds.