Vienna, Austria, 18-20 September 2024

On the Complexity of Recognizing k^+-real face Graphs

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Acknowledgments

Dagstuhl seminar
Beyond-Planar Graphs,
February 4-9, 2024


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example of a 6^+-real face picture

....

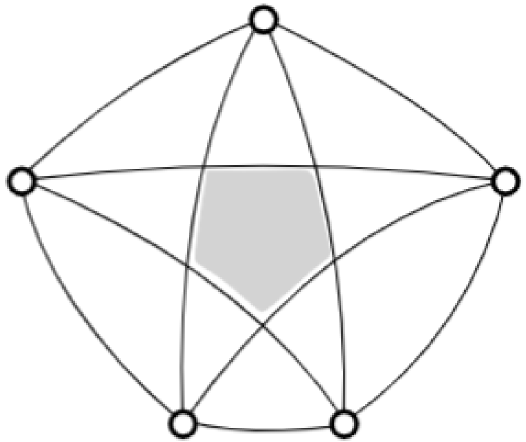
... but k^+-real face graphs are different objects

Outline

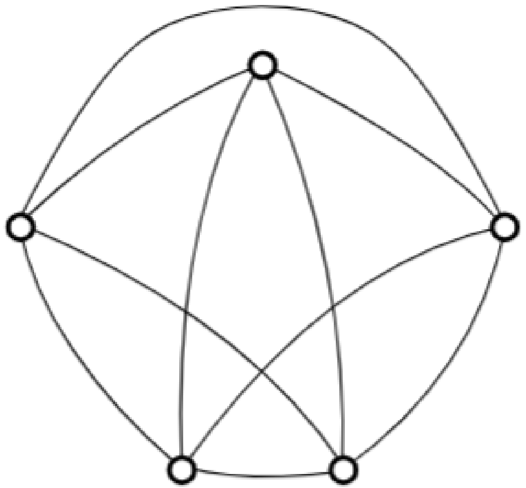
- k^+-real face graphs
  - Definition
  - State of the art
  - Recognition problem
- Contribution
- Open problems

Definition

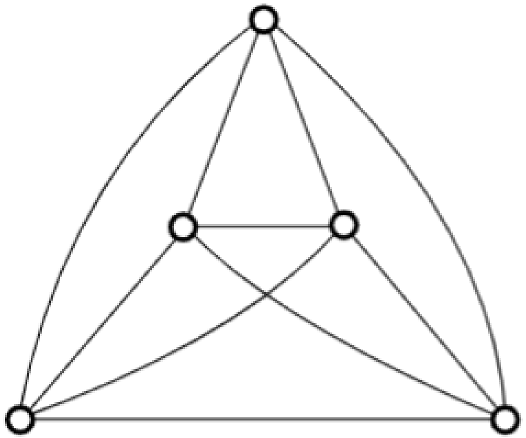
k⁺-real face drawing = <u>at least</u> k vertices per face (cell)



not k⁺-real face drawing



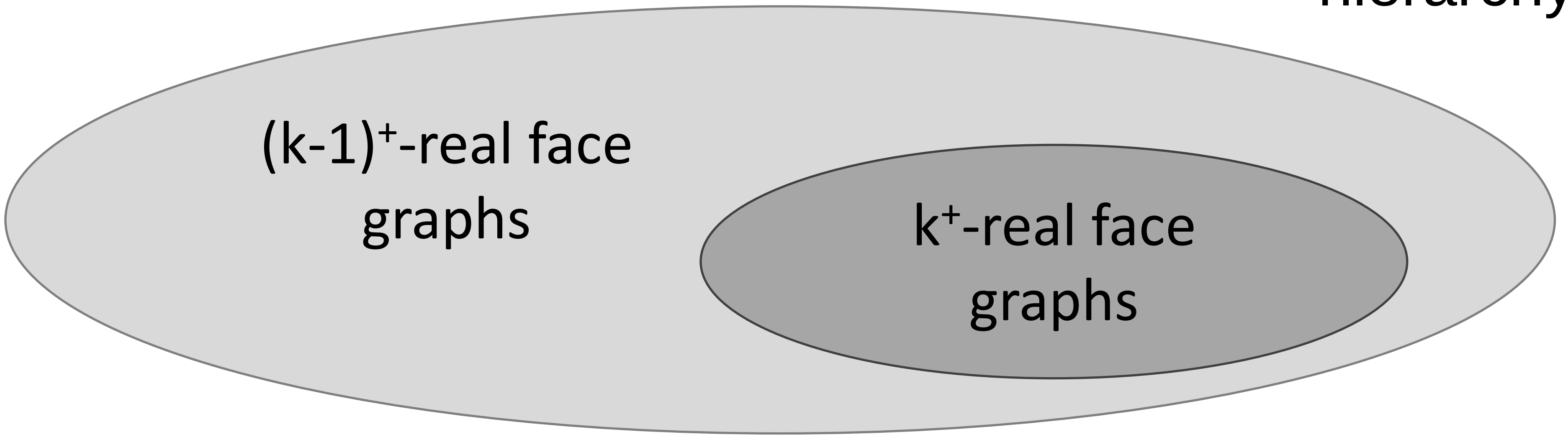
1⁺-real face drawing



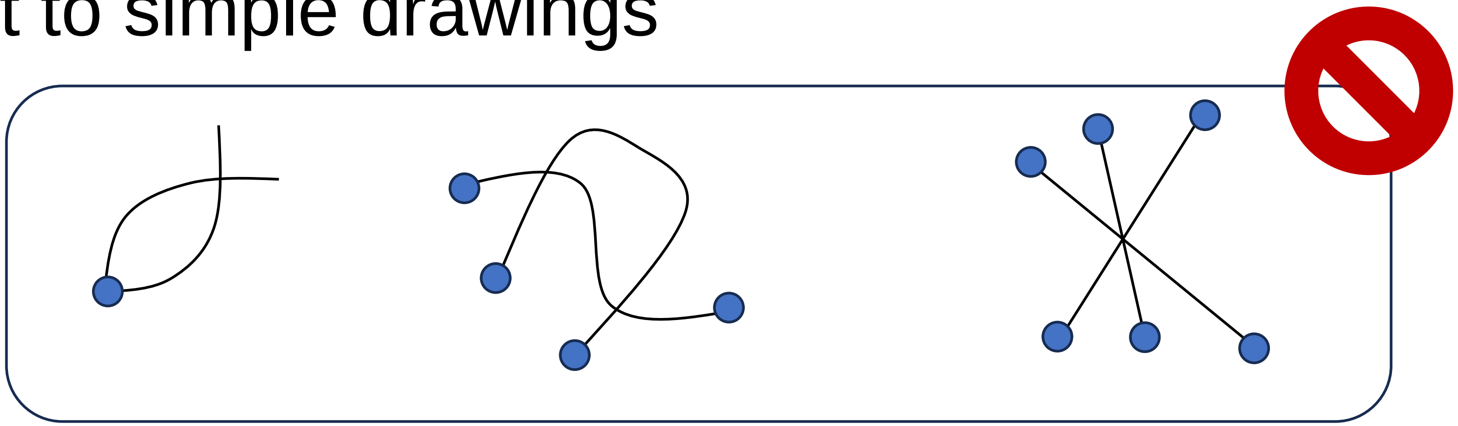
2⁺-real face drawing

Observations

hierarchy



we restrict to simple drawings



State of the art

- Binucci, Di Battista, D., Hong, Kaufmann, Liotta, Morin, Tappini - WG 2023
- Binucci, Di Battista, D., Dujmovic, Hong, Kaufmann, Liotta, Morin, Tappini – IEEE Access 2024

edge density

Graph Family	Crossings ($\chi \leq$)	Edges ($m \leq$)
k^+-real face graphs ($k \geq 3$)	$\frac{2-k}{k} \cdot m + n - 2$	$\frac{k}{k-2}(n-2)$
2^+-real face graphs	$n - 2$	$4n - 8$
1^+-real face graphs	$m + n - 2$	$5n - 10$
outer k^+-real face graphs ($k \geq 3$)	$\frac{2-k}{k} \cdot m + \frac{k-1}{k} \cdot n - 1$	$\frac{k-1}{k-2} \cdot n - \frac{k}{k-2}$
outer 2^+-real face graphs	$\frac{1}{2}n - 1$	$2.5n - 4$
outer 1^+-real face graphs	$m - 1$	$3n - 6$

State of the art

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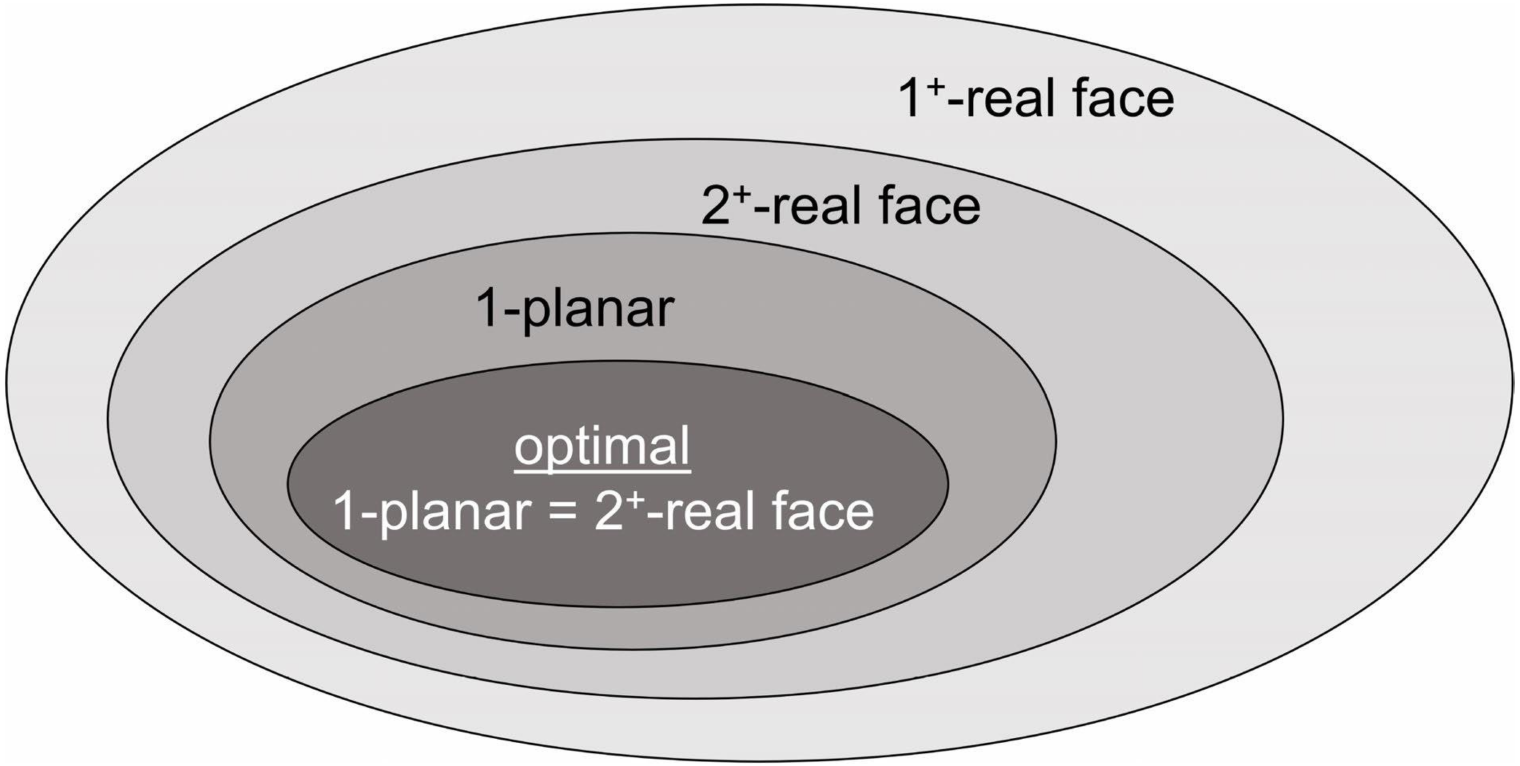
some relationships

open:
are there 2-planar graphs that are not 1^+-real face?



The recognition problem

- **Problem**. Given a graph G and a positive integer k, decide whether G is k^+-real face
  - recognition algorithms exist only for complete graphs and complete bipartite graphs, based on edge density and crossing number [Binucci Di Battista, D., Dujmovic, Hong, Kaufmann, Liotta, Morin, Tappini – IEEE Access 2024]

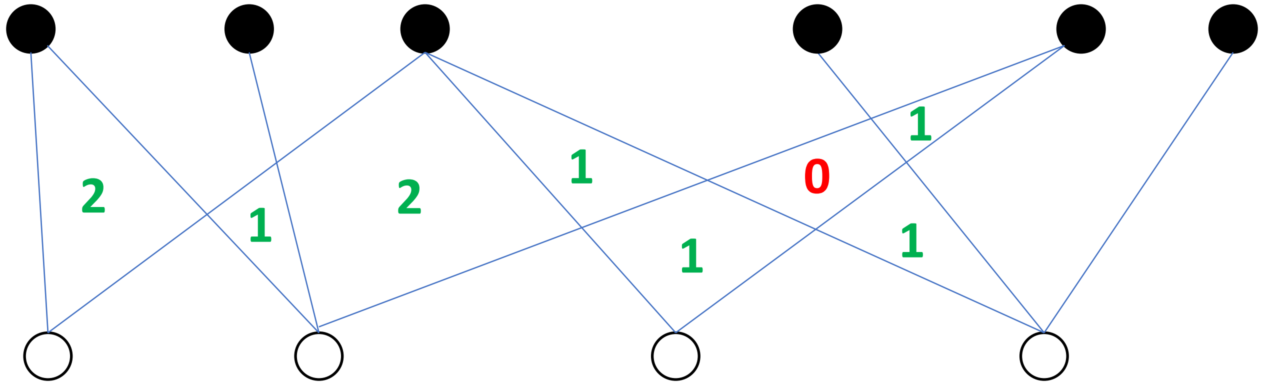
Contribution

- **General setting**
  - the recognition problem is NP-hard; namely, deciding whether a graph is k^+-real face (for k=1 or k=2) is NP-complete, even for biconnected graphs – reduction from 3-PARTITION
- **2-layer setting**
  - edge density results for each positive integer k
  - linear-time recognition algorithms for *optimal* k^+-real face graphs when $k \geq 1$
  - linear-time recognition algorithms for k^+-real face graphs when $k \geq 2$

Contribution

- **General setting**
  - the recognition problem is NP-hard; namely, deciding whether a graph is k^+-real face (for k=1 or k=2) is NP-complete, even for biconnected graphs – reduction from 3-PARTITION
- **2-layer setting** in this talk
  - edge density results for each positive integer k
  - linear-time recognition algorithms for *optimal* k^+-real face graphs when $k \geq 1$
  - linear-time recognition algorithms for k^+-real face graphs when $k \geq 2$

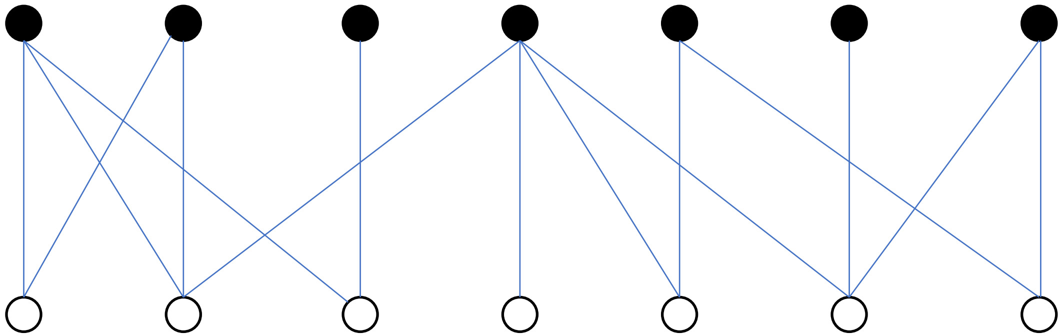
2-layer drawings: observations



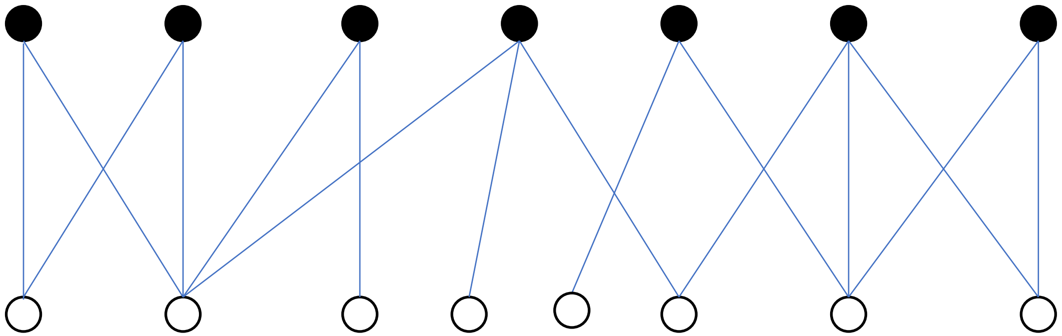
- each internal face has at most 2 vertices
- the external face contains all the vertices
- we can restrict our attention to connected graphs

2-layer drawings – Density results for k = 1,2

1$^+$-real face drawing

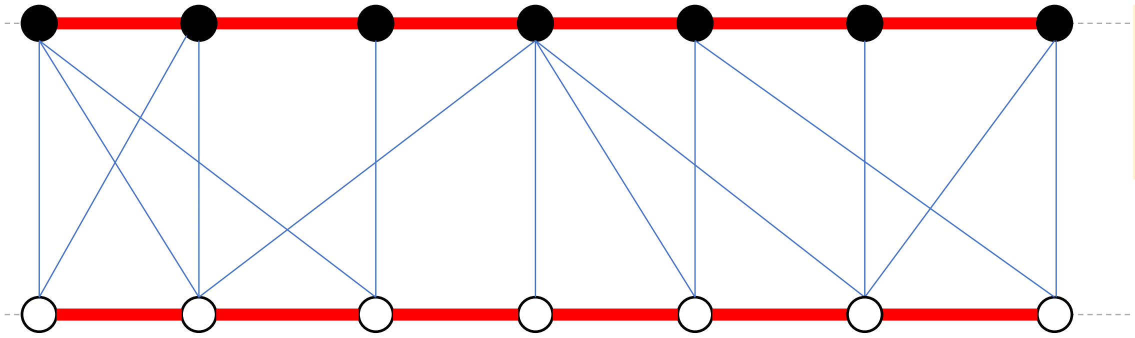


2$^+$-real face drawing



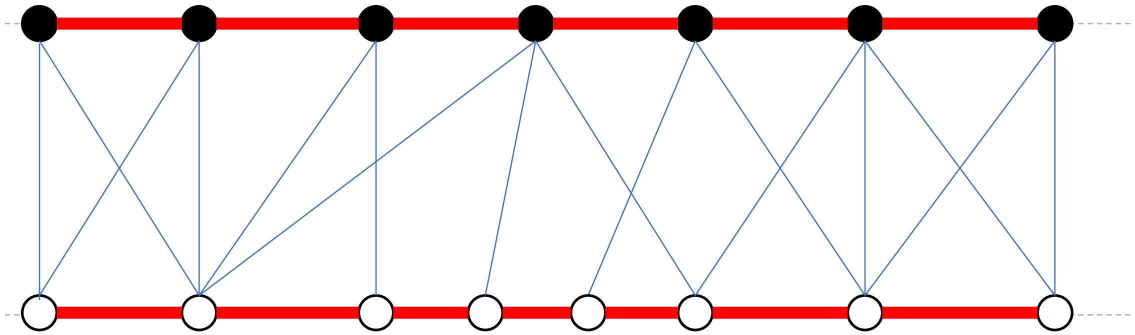
2-layer drawings – Density results for k = 1,2

1^+-real face drawing



adding **n-2** edges the drawing remains outer 1^+-real face

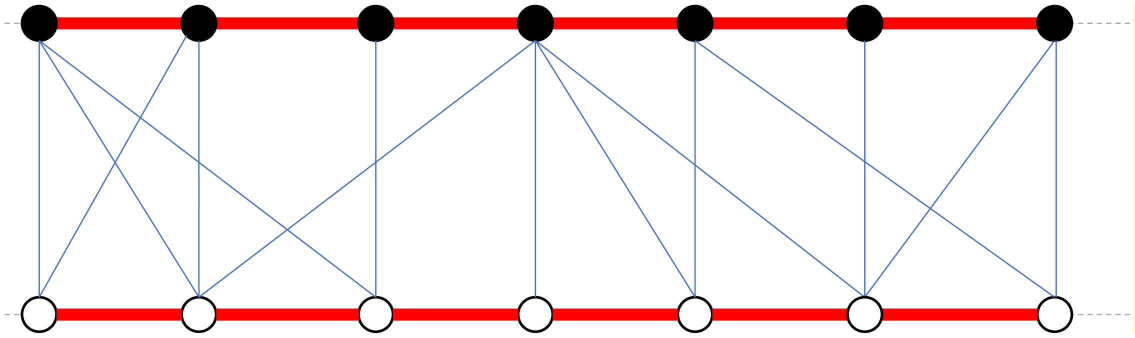
2^+-real face drawing



adding **n-2** edges the drawing remains outer 2^+-real face

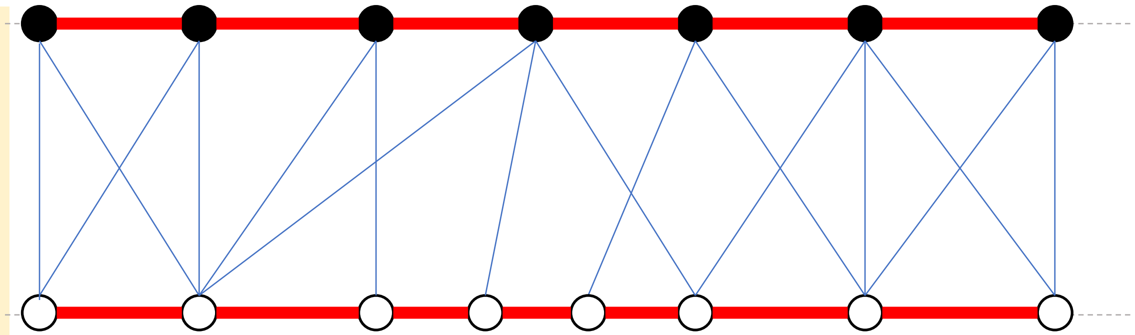
2-layer drawings – Density results for k = 1,2

1^+-real face drawing



outer 1^+-real face graphs have at most 3n-6 edges ⇒ 2-layer 1^+-real face graphs have at most 3n-6-(n-2) = 2n-4 edges

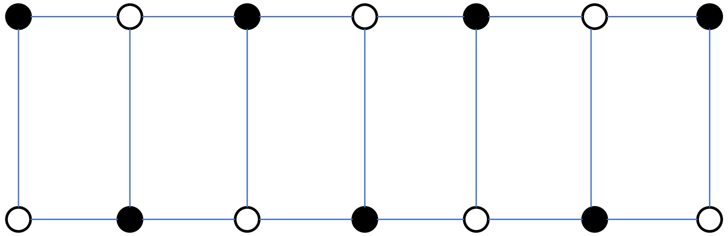
2^+-real face drawing



outer 2^+-real face graphs have at most 2.5n-4 edges ⇒ 2-layer 2^+-real face graphs have at most 2.5n-4-(n-2) = 1.5n-2 edges

2-layer drawings – Density results for k = 1,2

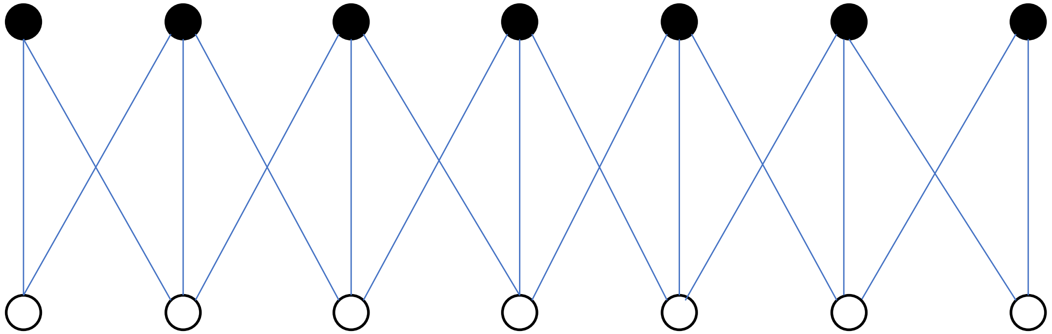
optimal 2^+-real face graph (1.5n-2 edges)



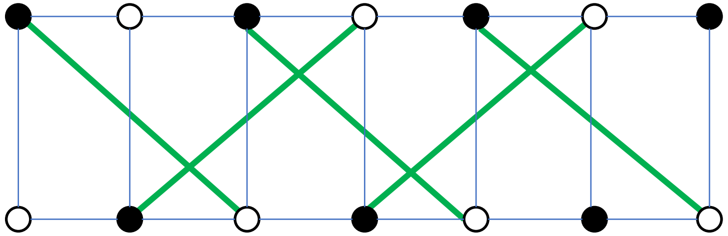
ladder

⇔

2^+-real face drawing



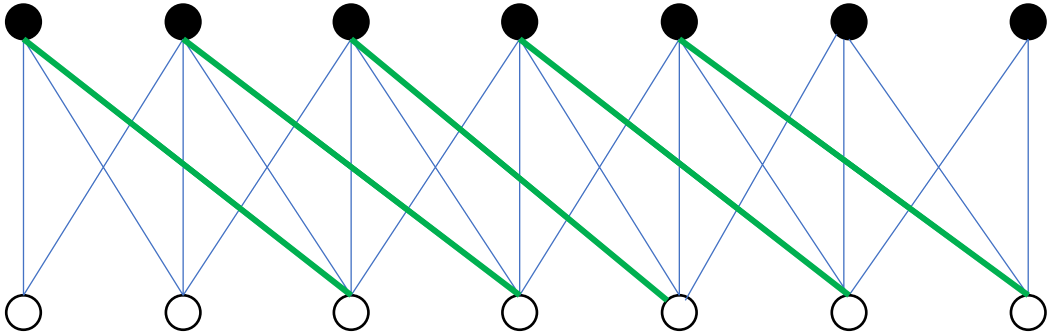
optimal 1^+-real face graph (2n-4 edges)



enriched ladder

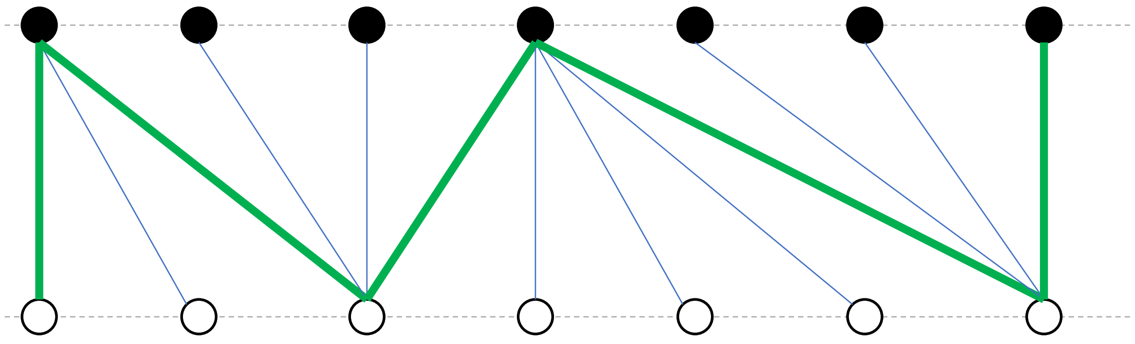
⇔

1^+-real face drawing



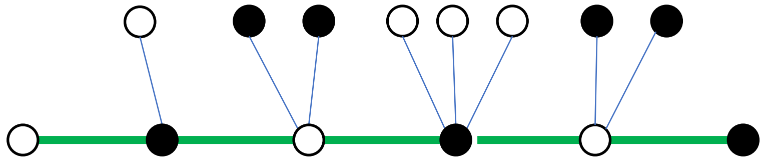
2-layer drawings – Density results for k ∈ [3,n]

k⁺-real face graph (n-1 edges)



2-layer drawing

⇔

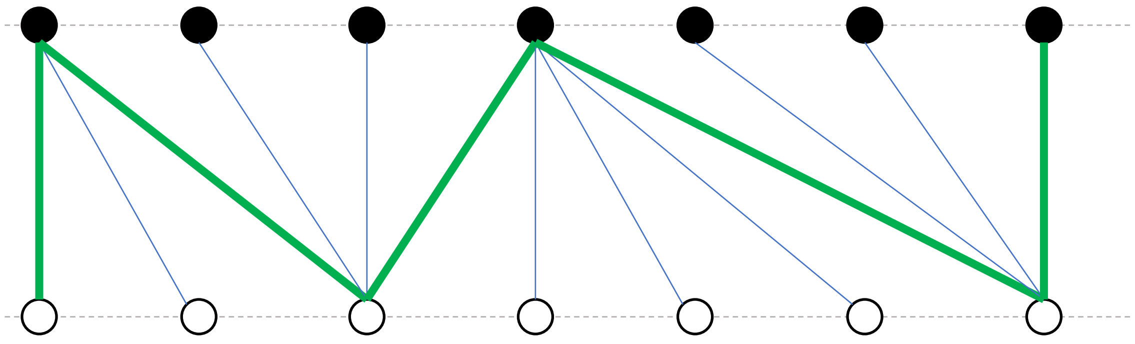


caterpillar

2-layer k⁺-real face drawing ⇔ planar ⇔ caterpillar [Eades, McKay, Wormald, ‘96]

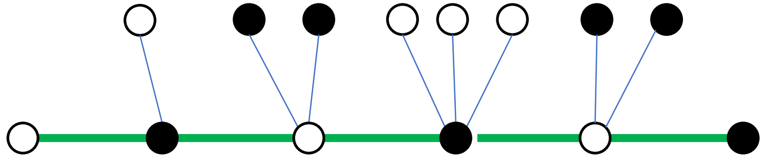
2-layer drawings – Density results for k $\in$ [3,n]

k^+-real face graph (n-1 edges)



2-layer drawing

$\Leftrightarrow$

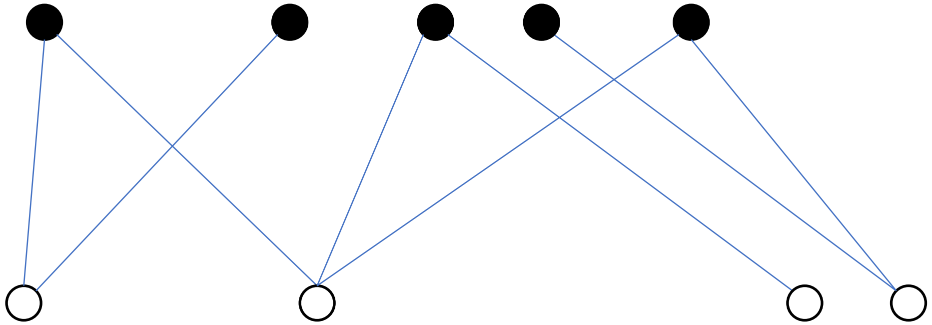


caterpillar

this immediately leads to an O(n)-time recognition algorithm for $k \geq 3$

Recognition of 2-layer 2^+-real face graphs

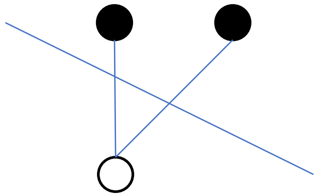
- There is a relationship between 2-layer 2^+-real face embeddings and 2-layer RAC embeddings
- **2-layer RAC drawing**



right angle crossings

a 2-layer embedding is RAC drawable ⇔ it has no fan crossings

[Di Giacomo, D., Eades, Liotta 2014]

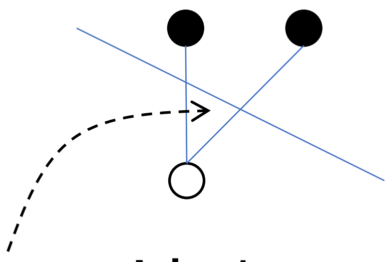


fan crossing

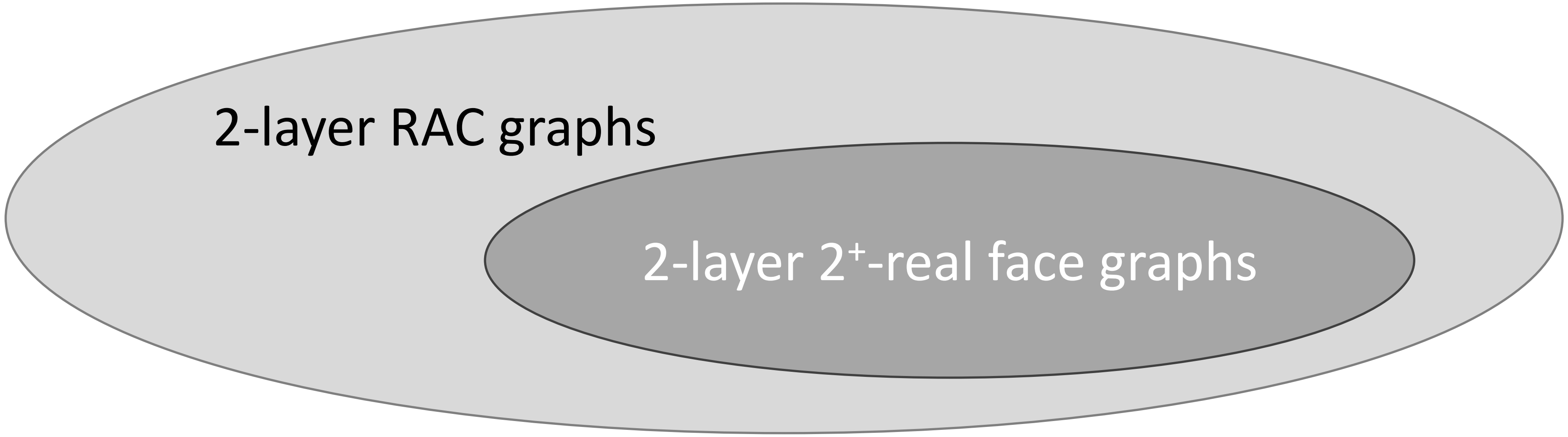
Recognition of 2-layer 2^+-real face graphs

- There is a relationship between 2-layer 2^+-real face embeddings and 2-layer RAC embeddings

any 2-layer 2^+-real face embedding is also RAC



internal face with 1 vertex only

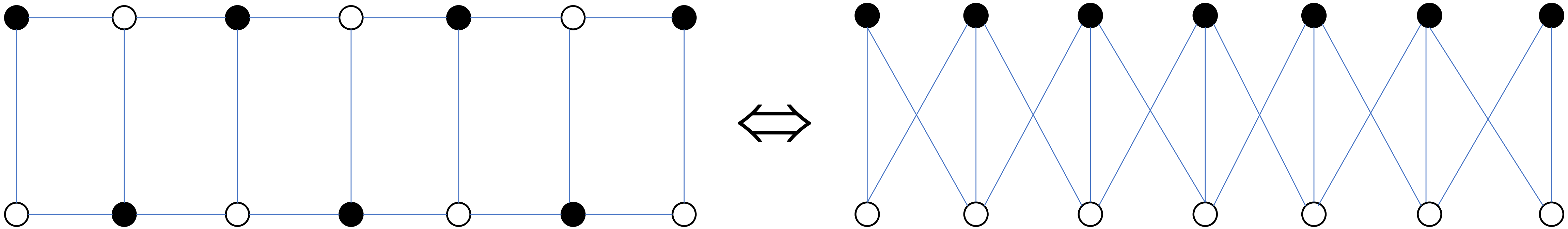


2^+-real face graphs – the biconnected case

Biconnected 2-layer 2^+-real face graph ⇔ biconnected 2-layer RAC graphs ⇔ spanning subgraph of a ladder

remember

2^+-real face drawing

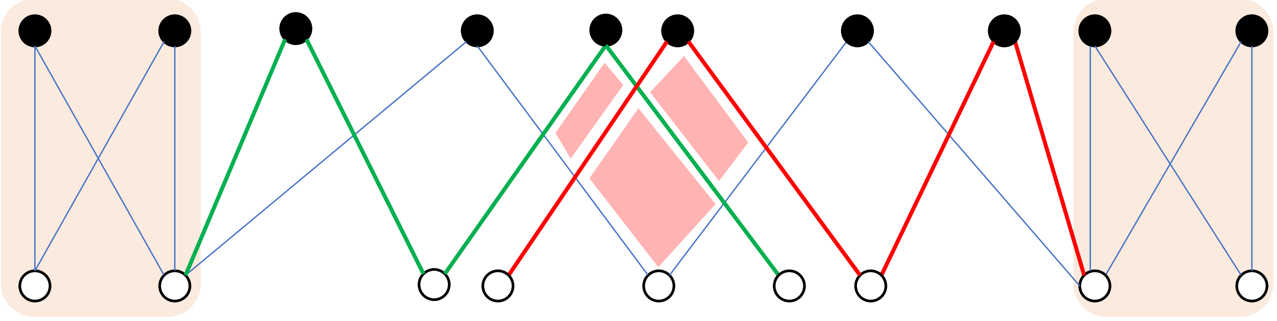


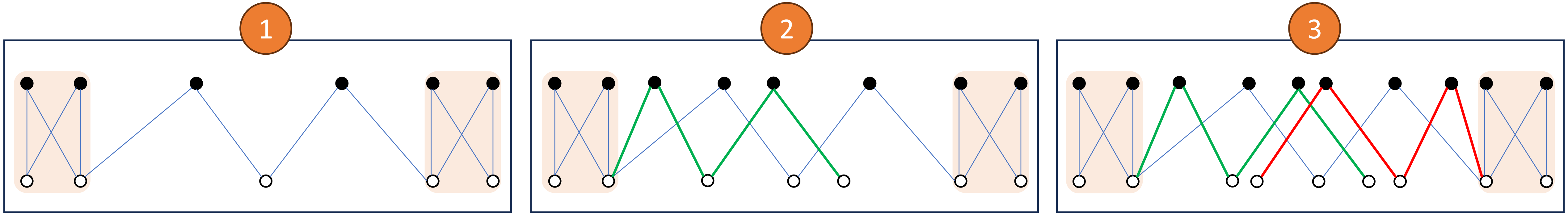
ladder

biconnected 2-layer 2^+-real face graphs can be recognized in O(n) time (using [Di Giacomo, D., Eades, Liotta 2014])

2^+-real face graphs – the 1-connected case

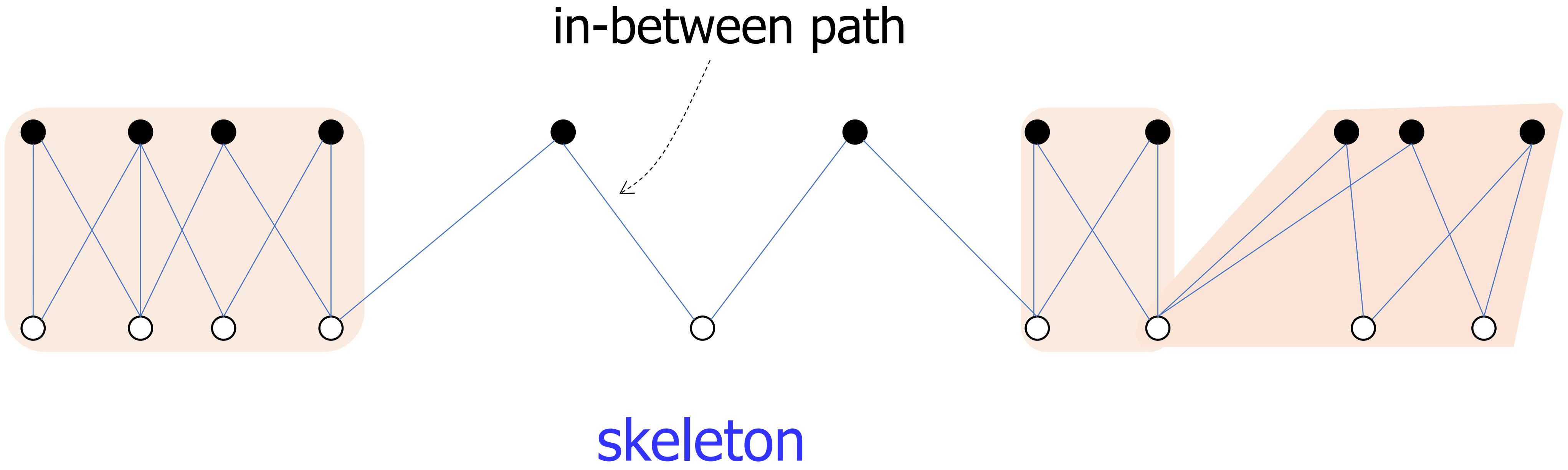
There exist infinitely many 2-layer RAC graphs that are not 2-layer 2^+-real face





2^+-real face graphs – the 1-connected case

- Structure of a 2-layer RAC embedding



2^+-real face graphs – the 1-connected case

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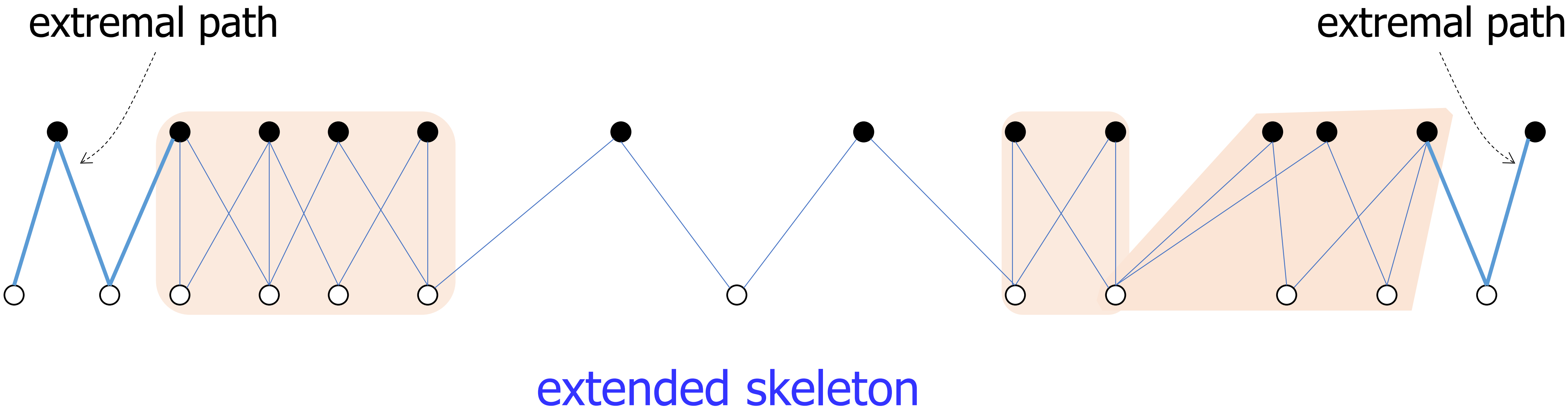

2^{+}-real face graphs – the 1-connected case

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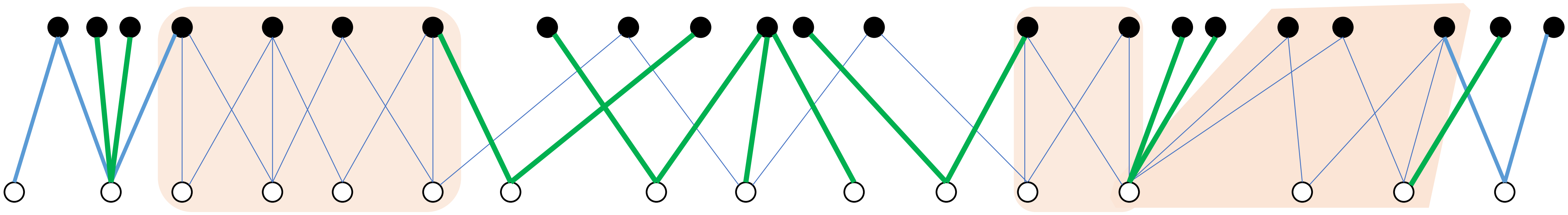
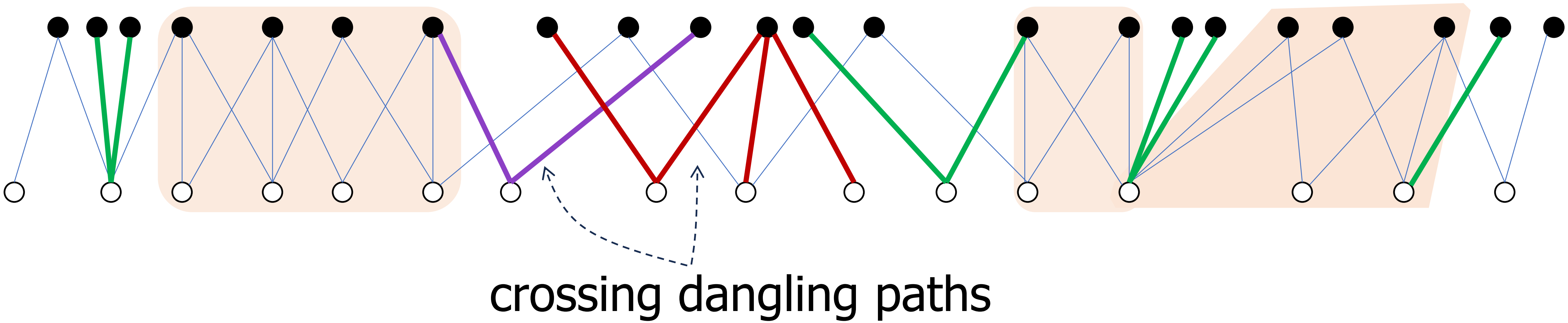


extended skeleton + dangling paths

2^+-real face graphs – the 1-connected case

- Structure of a 2-layer RAC embedding



2^+-real face graphs – the 1-connected case

A 2-layer RAC embedding is 2^+-real face $\Leftrightarrow$ it does not contain two crossing dangling paths

2^+-real face graphs – the 1-connected case

A 2-layer RAC embedding is 2^+-real face $\Leftrightarrow$ it does not contain two crossing dangling paths

- O(n)-time recognition algorithm

2^+-real face graphs – the 1-connected case

A 2-layer RAC embedding is 2^+-real face $\Leftrightarrow$ it does not contain two crossing dangling paths

- O(n)-time recognition algorithm
  - recursively remove all degree-1 vertices to recognize the skeleton (if any, it has a unique RAC embedding)

2^+-real face graphs – the 1-connected case

A 2-layer RAC embedding is 2^+-real face $\Leftrightarrow$ it does not contain two crossing dangling paths

- O(n)-time recognition algorithm
  - recursively remove all degree-1 vertices to recognize the skeleton (if any, it has a unique RAC embedding)
  - iteratively add dangling paths attached to in-between paths in a greedy fashion (if fan-crossings or crossings between dangling paths arise reject)

2^+-real face graphs – the 1-connected case

A 2-layer RAC embedding is 2^+-real face $\Leftrightarrow$ it does not contain two crossing dangling paths

- O(n)-time recognition algorithm
  - recursively remove all degree-1 vertices to recognize the skeleton (if any, it has a unique RAC embedding)
  - iteratively add dangling paths attached to in-between paths in a greedy fashion (if fan-crossings or crossings between dangling paths arise reject)
  - recognize and add extremal paths and the attached dangling paths (if possible)

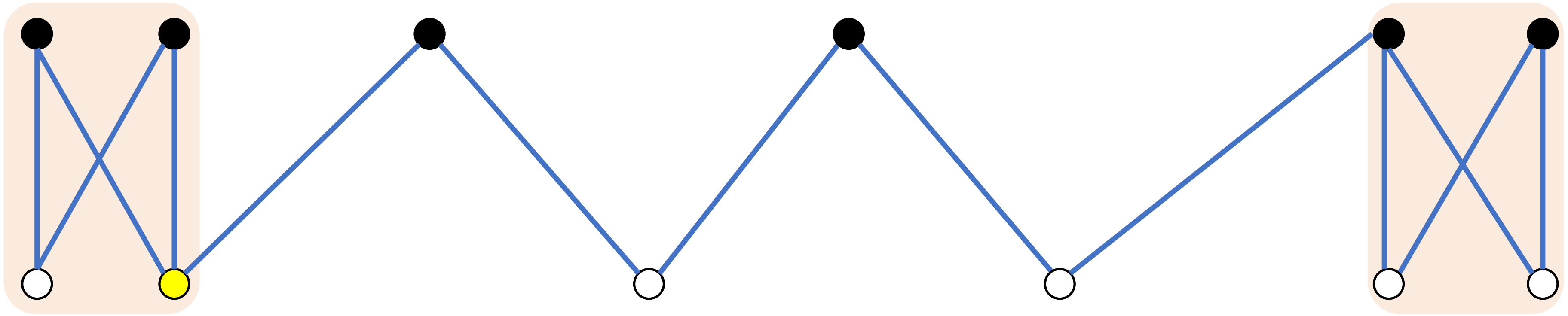
2^+-real face graphs – the 1-connected case

A 2-layer RAC embedding is 2^+-real face $\Leftrightarrow$ it does not contain two crossing dangling paths

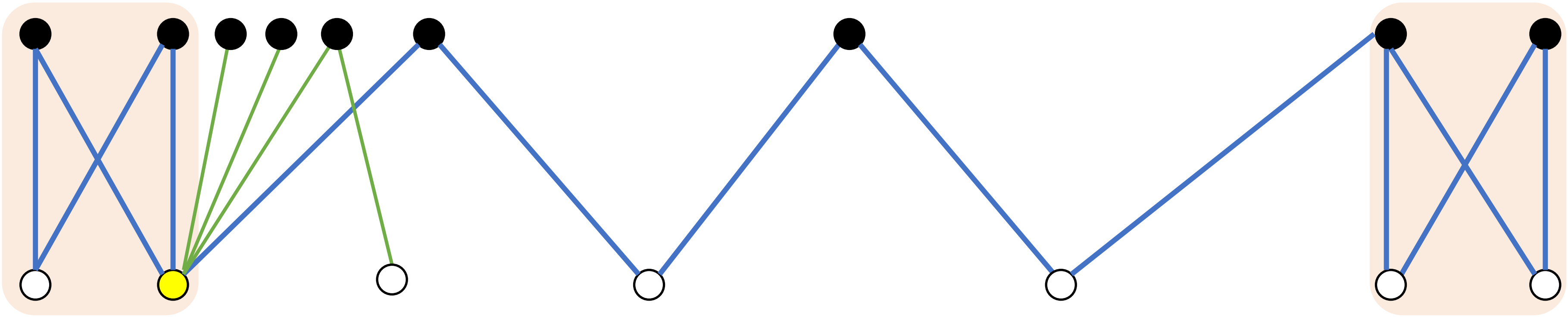
- O(n)-time recognition algorithm
  - recursively remove all degree-1 vertices to recognize the skeleton (if any, it has a unique RAC embedding)
  - iteratively add dangling paths attached to in-between paths in a greedy fashion (if fan-crossings or crossings between dangling paths arise reject)
  - recognize and add extremal paths and the attached dangling paths (if possible)

- The different steps exploit variants of linear-time algorithms in [Di Giacomo, D., Eades, Liotta 2014]

Example of the greedy strategy



Example of the greedy strategy



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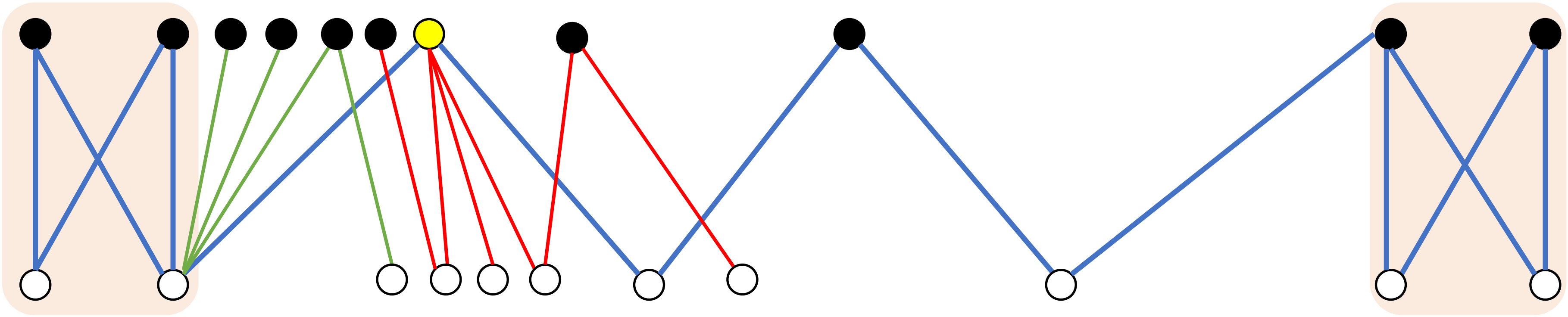

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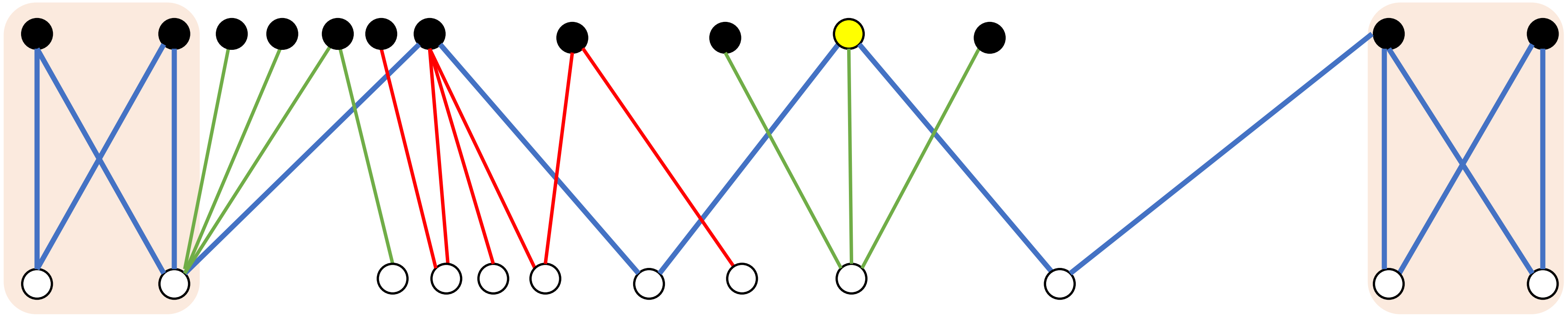
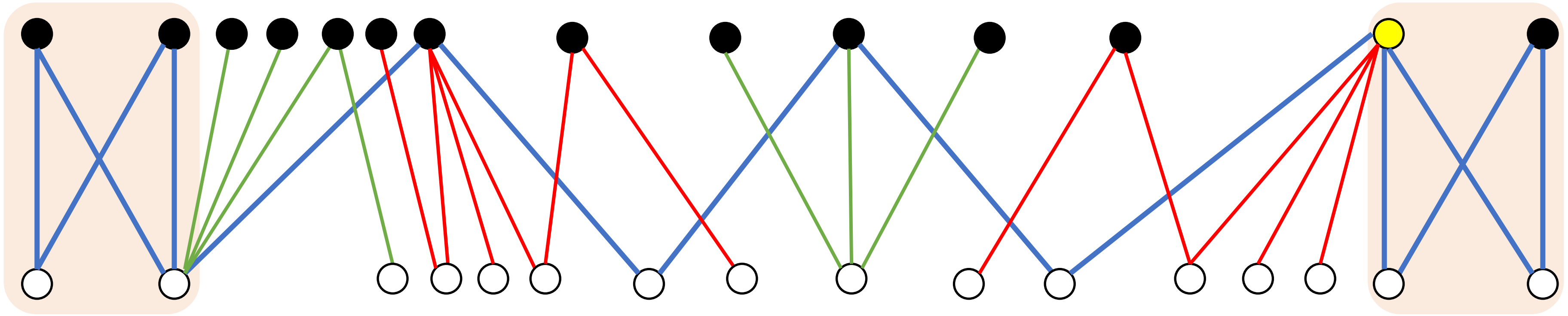


Example of the greedy strategy



Open Problems

- Can we efficiently recognize 2-layer 1^+-real face graphs?
  - We have preliminary results … we are optimistic
- What is the complexity of recognizing *outer* k^+-real face graphs?
- In the unconstrained scenario, are there subfamilies of k^+-real face graphs that can be recognized efficiently?
- What about parameterized algorithms for the unconstrained scenario?
  - note that the problem is neither FPT nor XP with parameter k

Thanks for your attention!

Motivations and Observations

- Theoretical motivations
  - extension of planar drawings with face sizes above a desired threshold
  - generalization to non-planar graphs of the classical guarding planar graph problem (vertices that cover faces)
- Practical motivation
  - faces consisting of crossing points only might be less readable