Harborth's conjecture for 4-regular planar graphs

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Fáry’s theorem

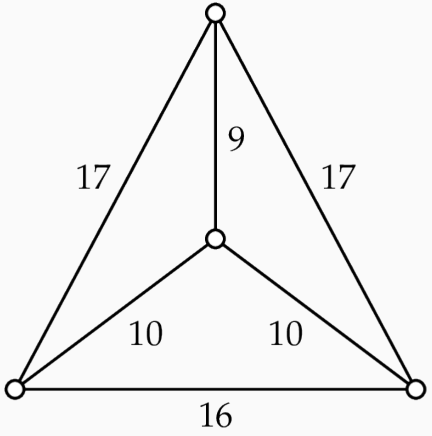
Theorem (Wagner 1936, Fáry 1948, Stein 1951)

Every planar graph has a straight-line embedding.

Harborth's conjecture

Conjecture (Harborth, Kemnitz, Möller, Süssenbach 1987)

Every planar graph has a straight-line embedding where every edge has integer (**equivalently, rational**) length.



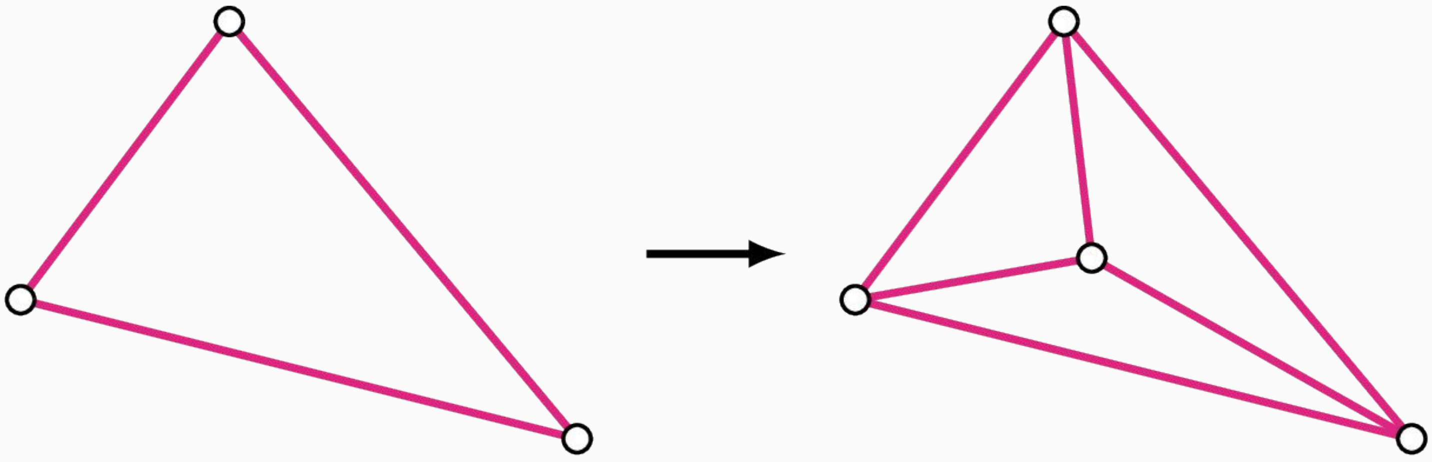
A possible approach

Proof idea for maximal planar graphs (Kemnitz, Harborth 2001):

- delete a vertex of minimum degree
- triangulate the resulting polygon and recurse
- add back the deleted vertex inside the polygon

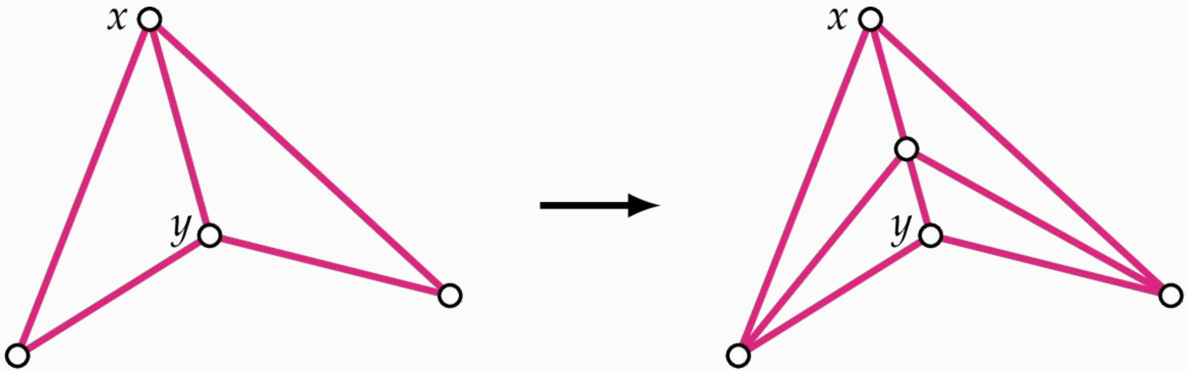
Vertices of degree 3

Theorem (Almering 1963): The set of points at rational distance to the vertices of a rational triangle are dense in the plane.



Vertices of degree 4

Theorem (Kemnitz and Harborth 2001): There is a point on the line passing through ***x*** and ***y*** at rational distance to each of the four vertices.



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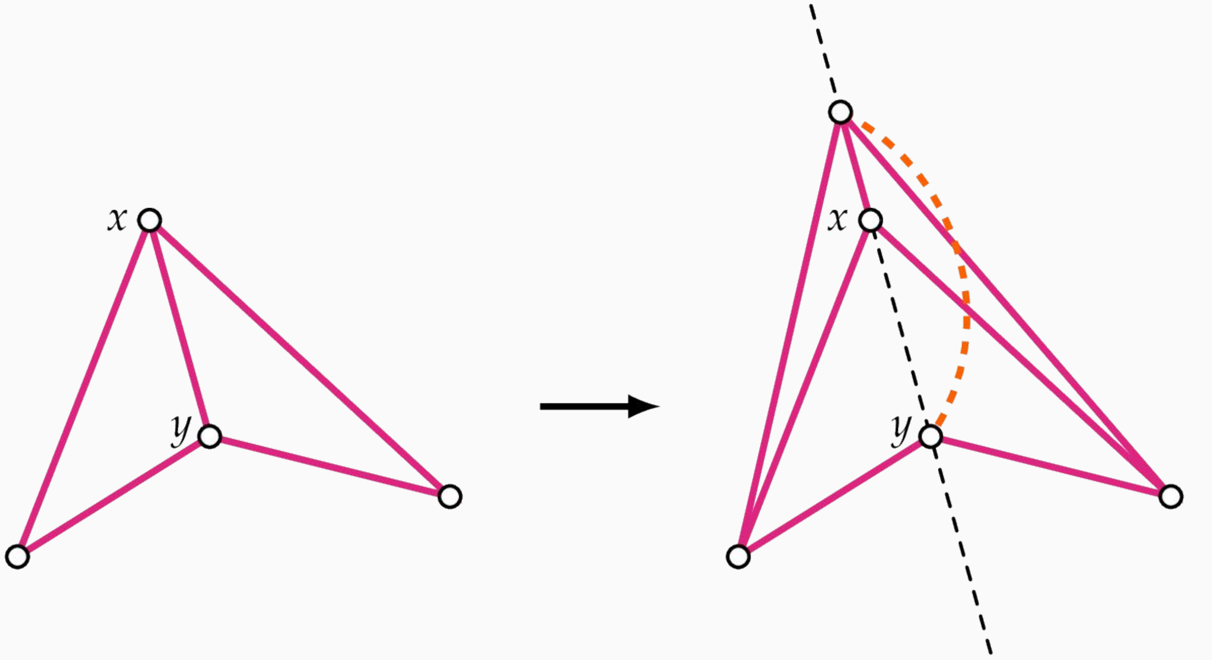
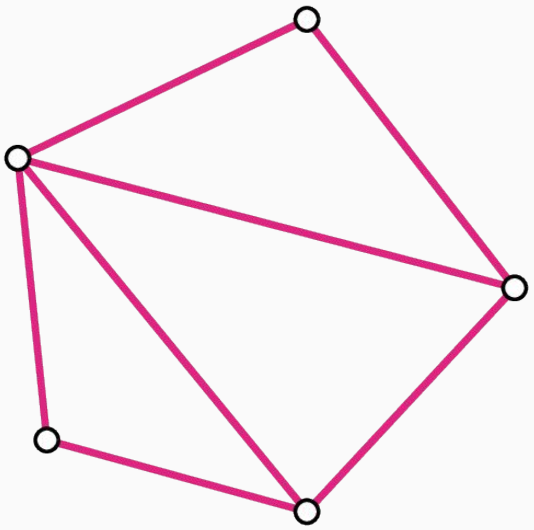


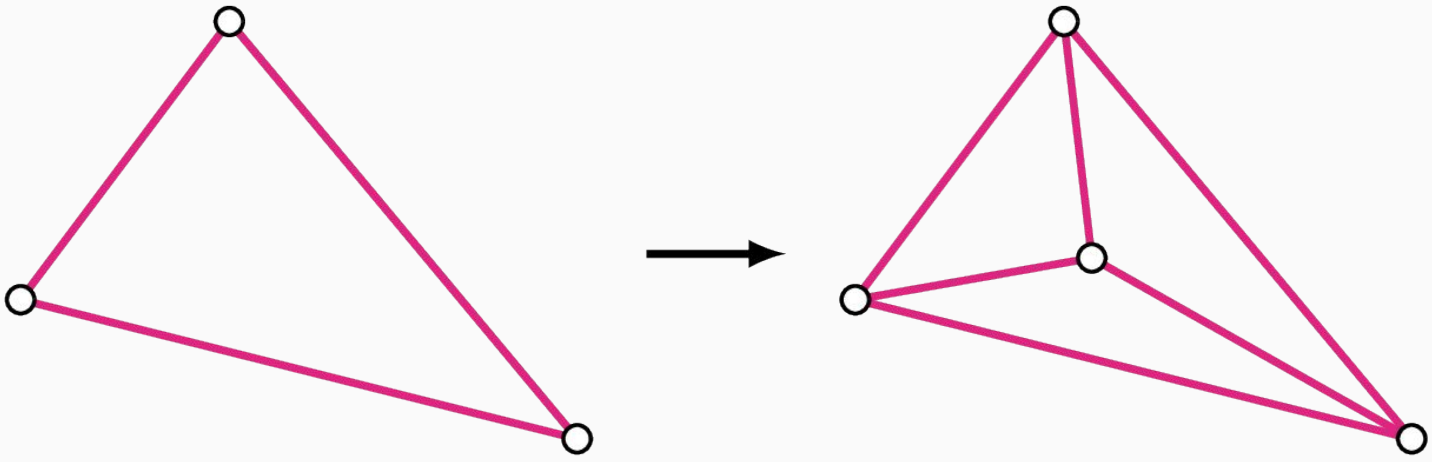
Vertices of degree 5

No idea...



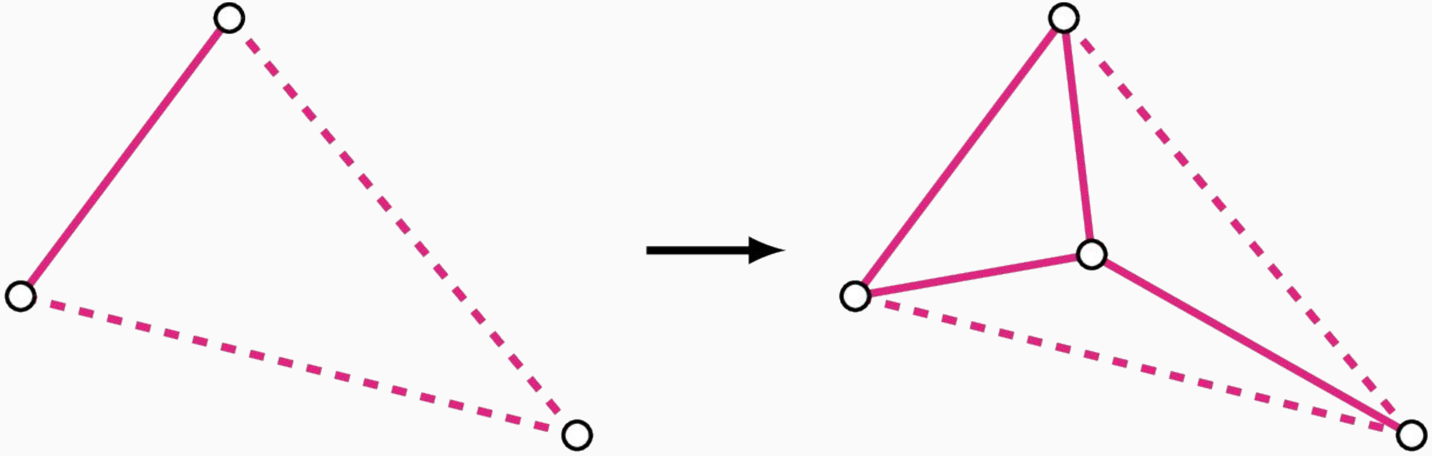
Almering revisited

Theorem (Almering 1963): The set of points at rational distance to the vertices of a rational triangle are dense in the plane.



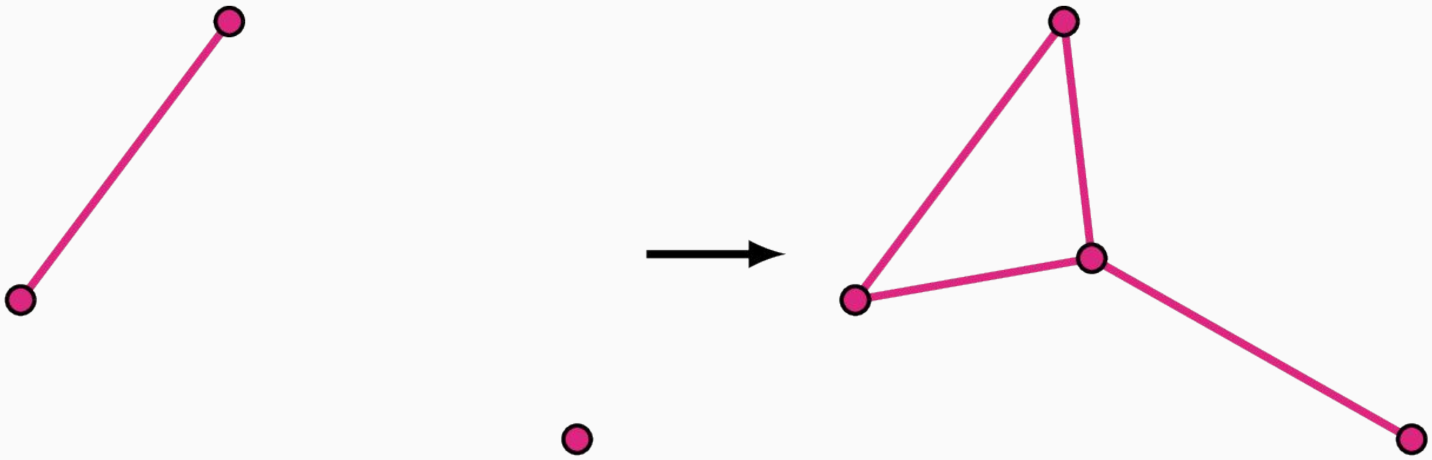
Almering revisited

Theorem (Berry 1992): Almering's theorem still holds even if, for two sides, only their squares are rational.



Almering revisited

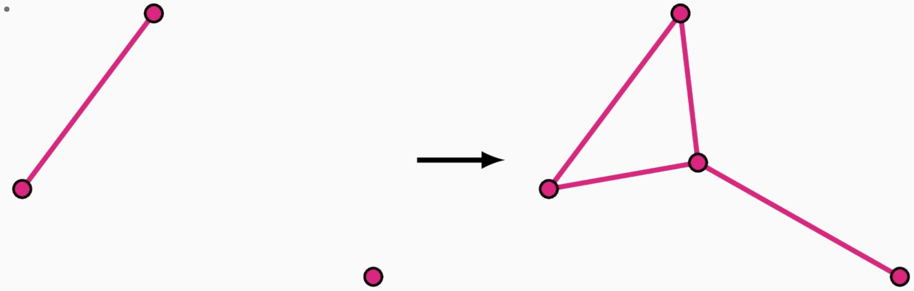
Theorem (Geelen, Guo, McKinnon 2008): A point at rational distance to three rational points is also rational.



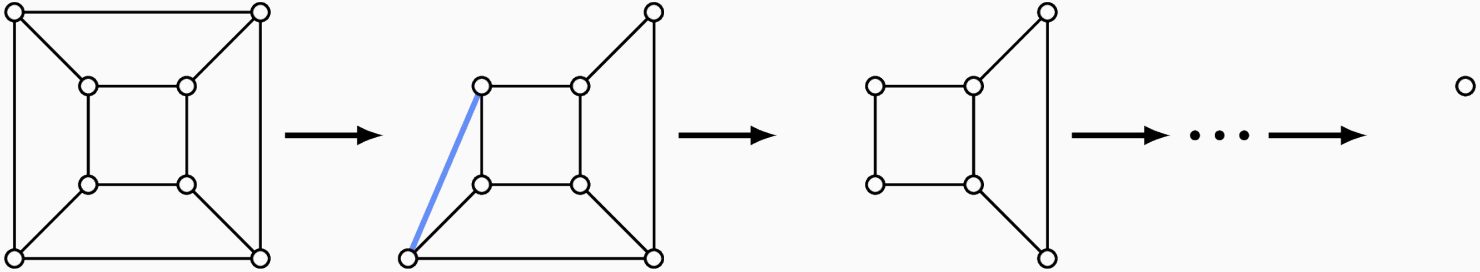
Generating graph drawings using Berry

Given a graph G, a ***3-elimination order*** (Biedl 2011) $v_1, v_2, \ldots, v_n$ satisfies:

- $n = 1$, or
- v_1 has degree at most 2, and $v_2, \ldots, v_n$ is a 3-elimination order for $G - v_1$, or
- v_1 has degree 3, and there are neighbors x and y such that $v_2, \ldots, v_n$ is a 3-elimination order for $(G - v_1) \cup (xy)$.



A 3-elimination order for the cube



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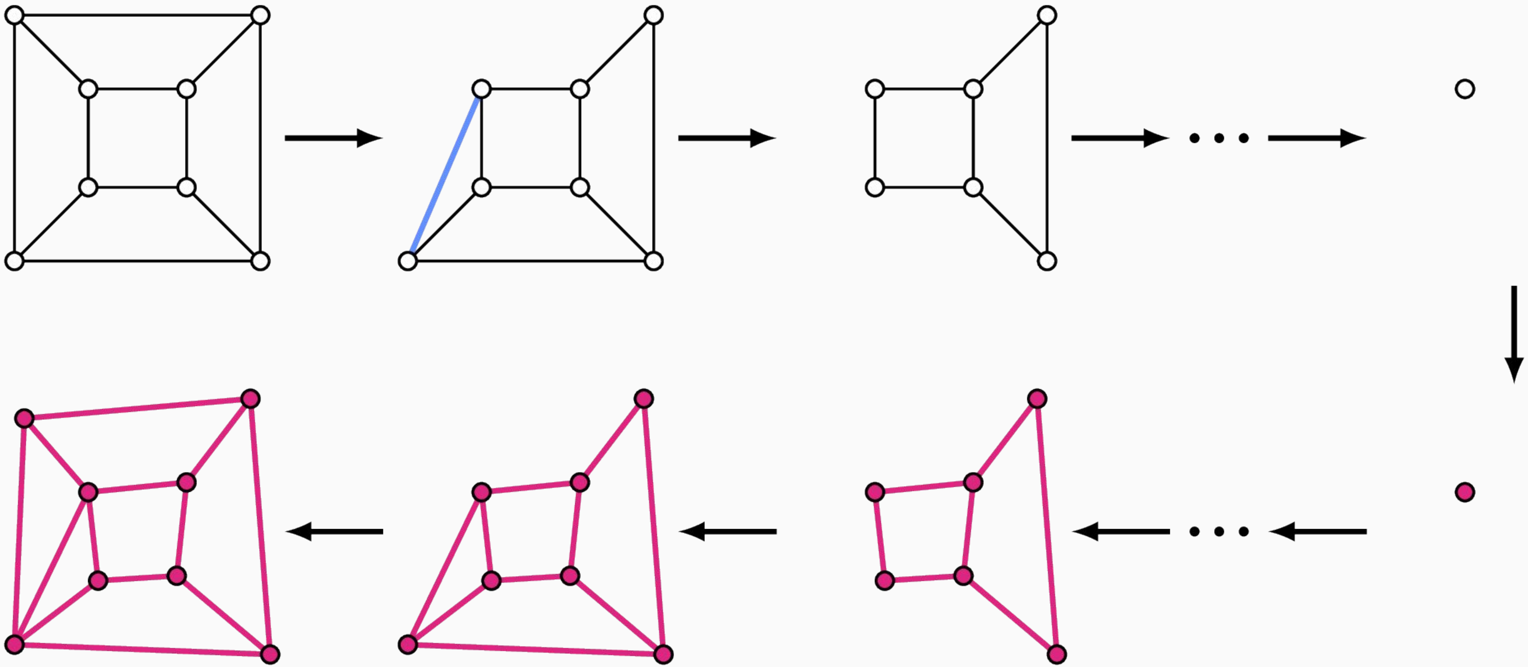
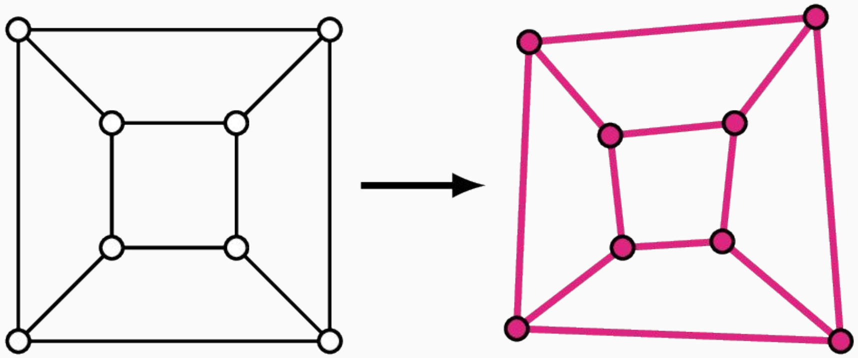


Drawing with 3-elimination orders

Theorem (Geelen et al. 2008, Biedl 2011): Any straight-line drawing of a planar graph with a 3-elimination order can be "approximated" by a rational drawing with rational vertex coordinates.



Graphs with 3-elimination orders

- Cubic graphs (also see S. 2011 or Biedl 2011)
- At most 4 vertices of degree > 3 (Dubickas 2012)
- (2, 1)-sparse graphs (Biedl 2011)
    - Planar bipartite
    - Series-parallel
    - Arboricity 2
    - **Connected *subquartic* (maximum degree 4, but not 4-regular)** (Benediktovich 2013)

One more edge

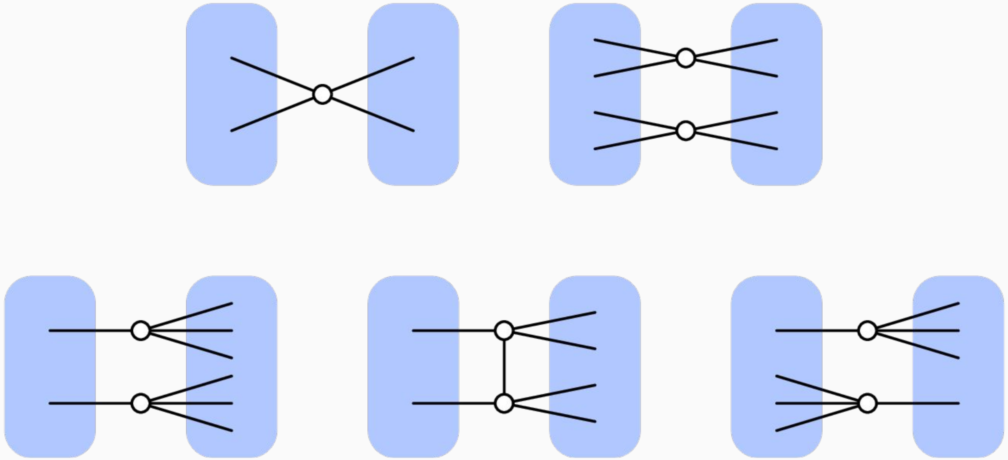
Our result: 4-regular graphs have drawings with all rational edge lengths.

Proof split into two parts:

- Vertex connectivity 1 or 2
- 3-vertex-connected

Low vertex connectivity

Proposition (S. 2013): Every 4-regular planar graph of ***edge*** connectivity 2 has a rational drawing.



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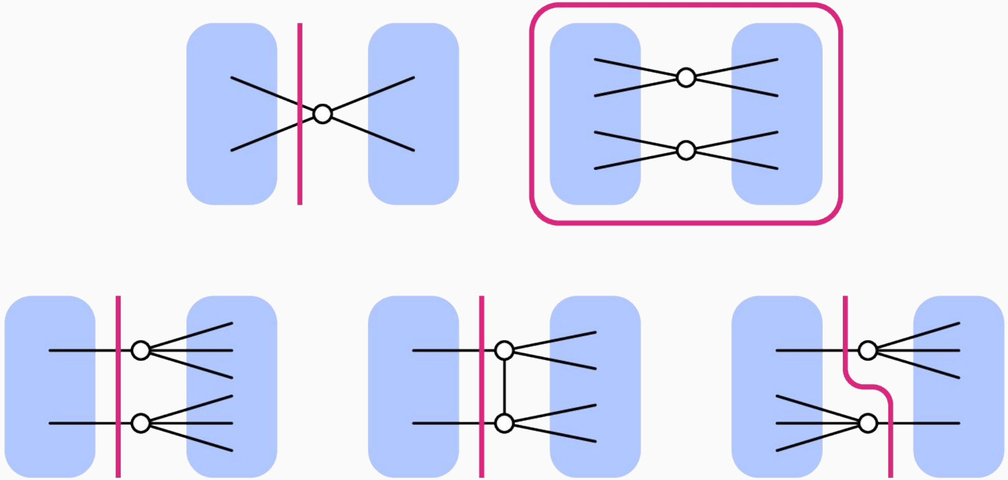
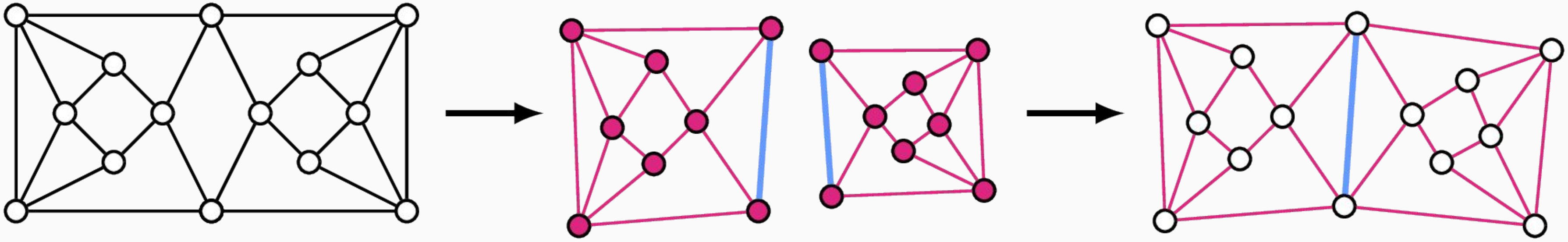


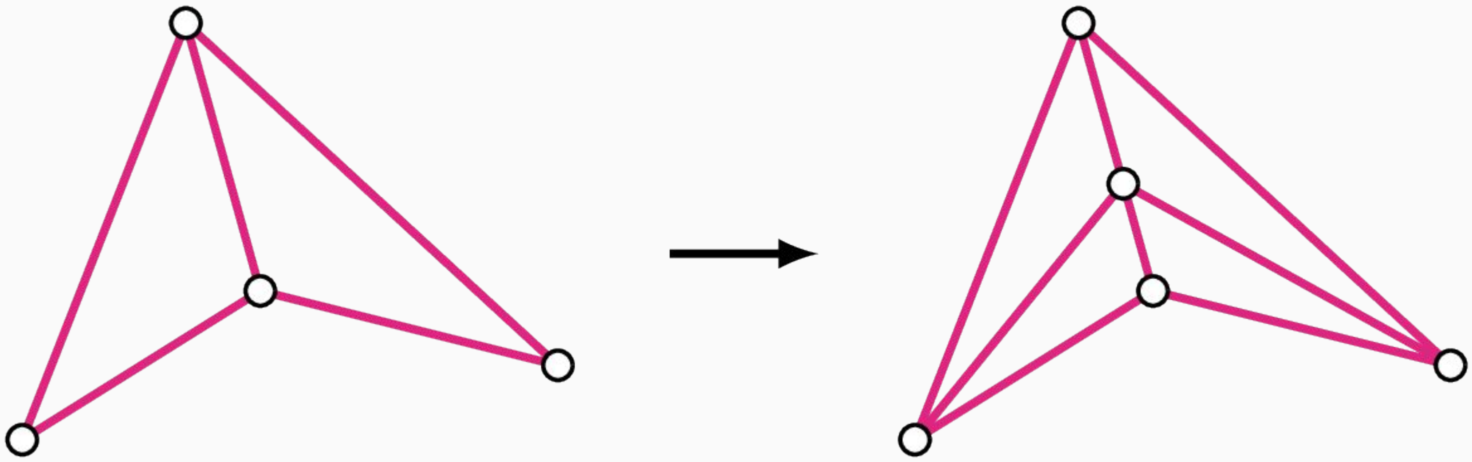
Low vertex connectivity

Glue together two subquartic graphs at an edge.



Adding vertices of degree 4

Revisiting Kemnitz and Harborth's incomplete solution for degree-4 vertices:



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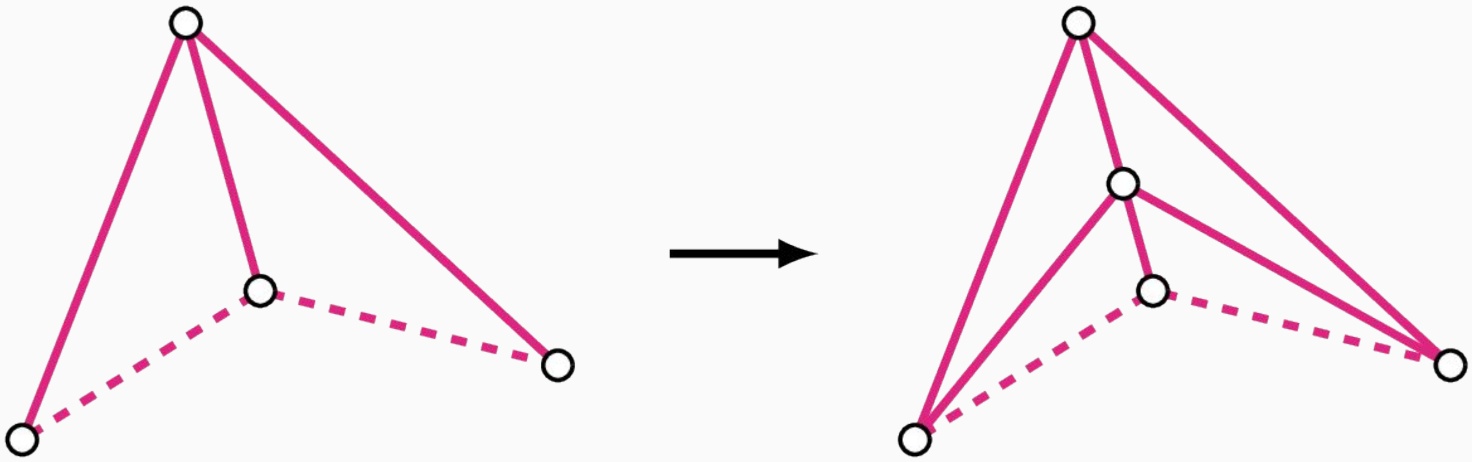

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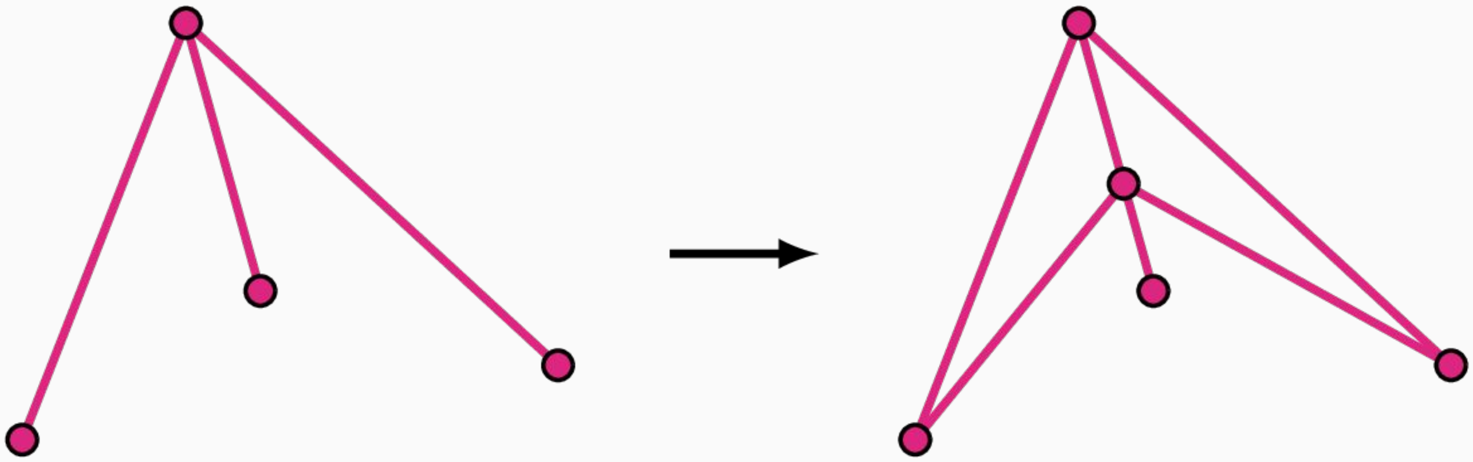
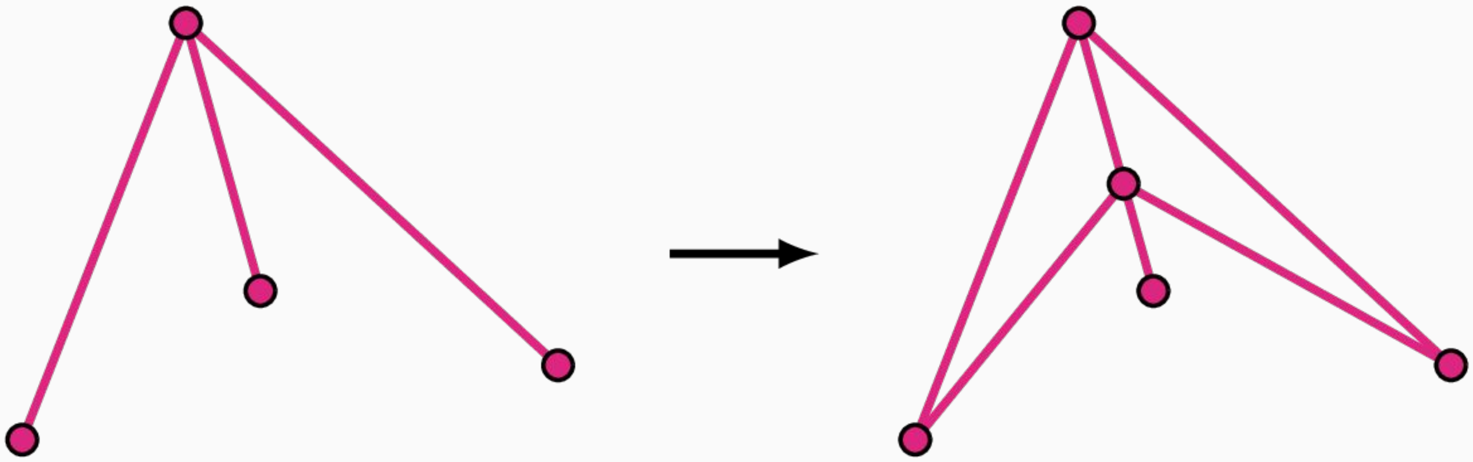
- Like Geelen et al., place vertices at rational coordinates



Adding vertices of degree 4

Goal: Find a configuration where

- Kemnitz and Harborth’s solution lands on the line segment
- The rest of the graph can be drawn around it



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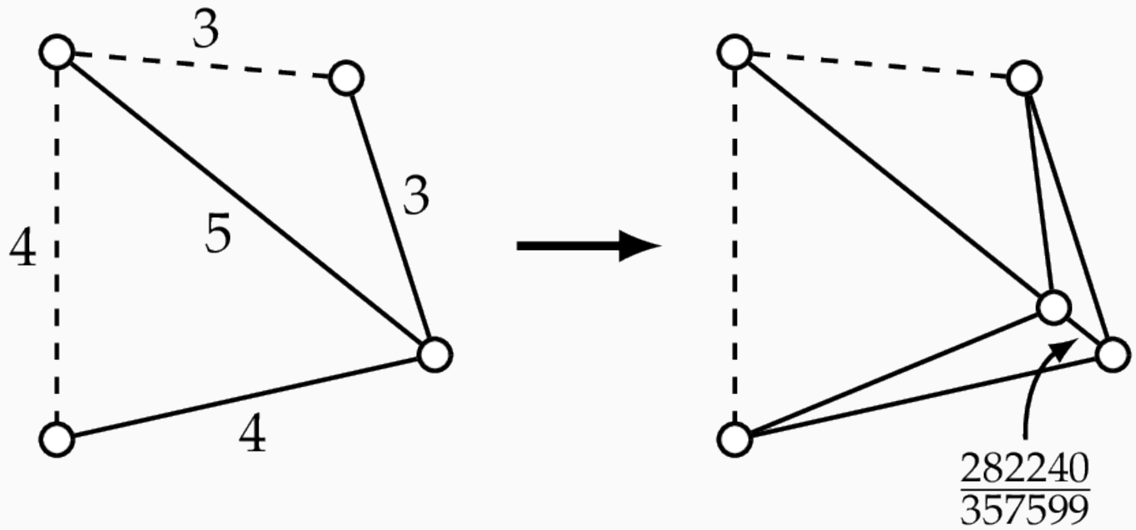

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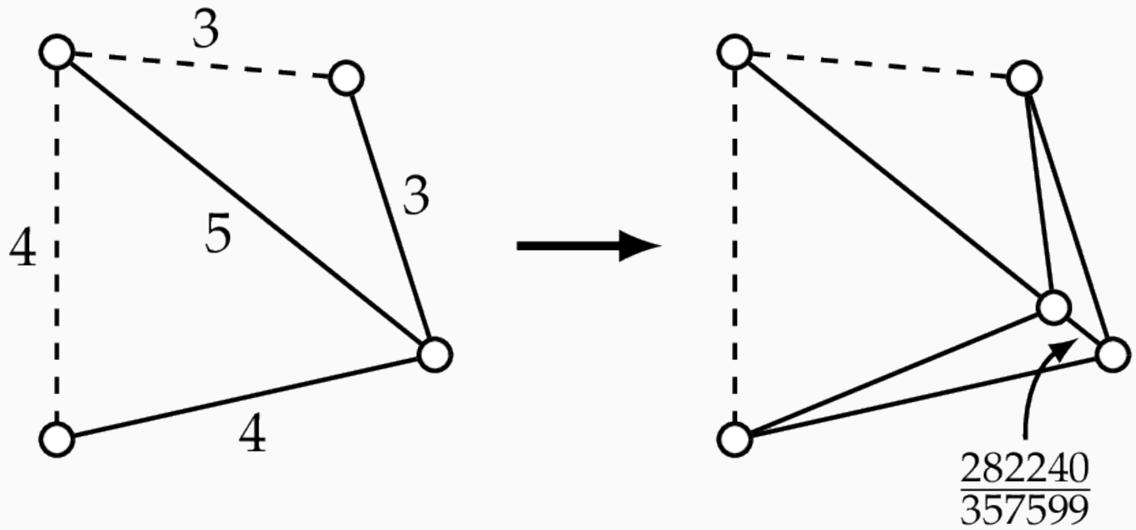
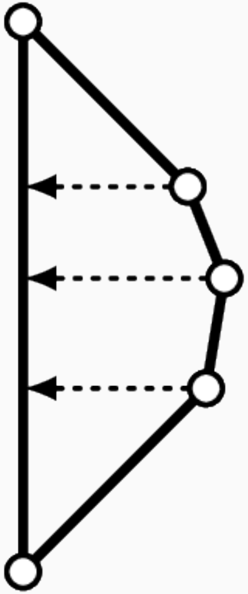


Adding vertices of degree 4

Theorem (Mchedlidze, Nöllenburg, Rutter GD 2013): In a maximal plane graph, a drawing of a cycle as a convex "one-sided" polygon can be extended to the whole graph if all its chords are internal.



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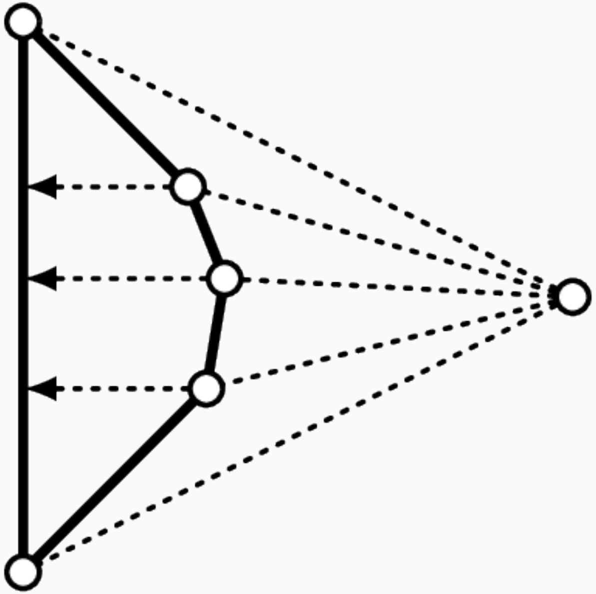

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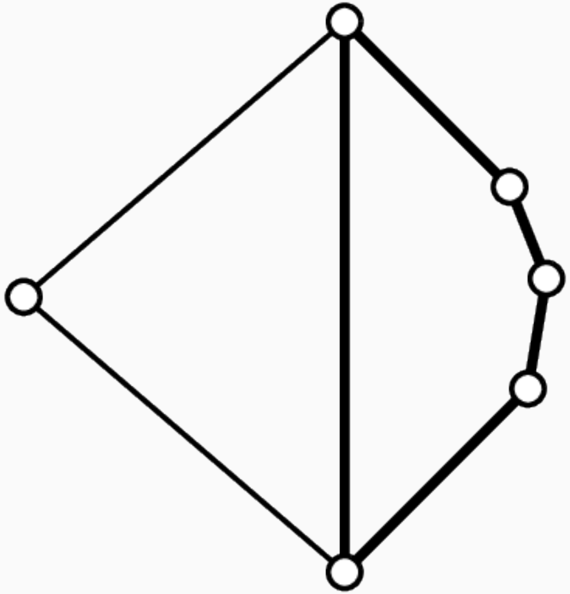

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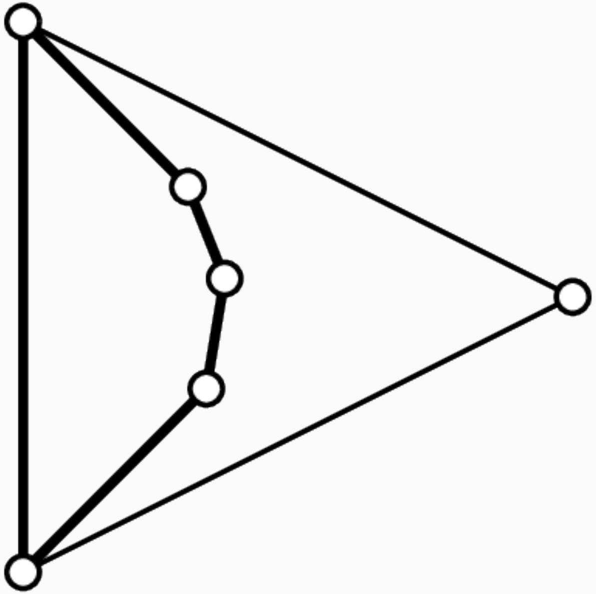

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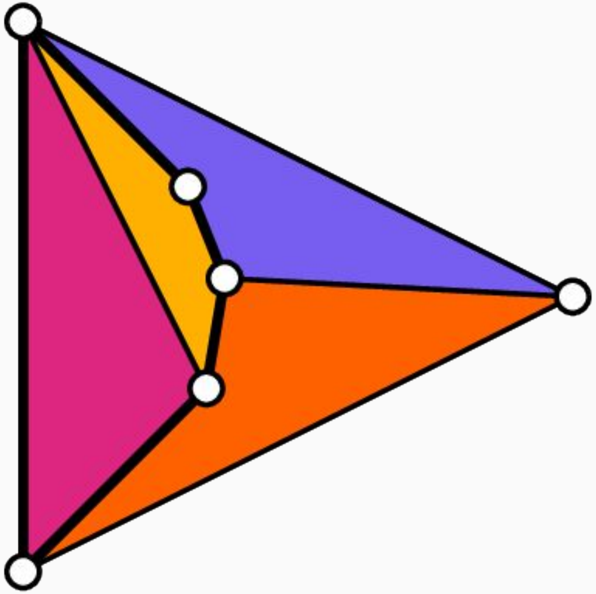
Theorem (Mchedlidze, Nöllenburg, Rutter GD 2013): In a maximal plane graph, a drawing of a cycle as a convex “one-sided” polygon can be extended to the whole graph if all its chords are internal.

Each region is star-shaped, so it can be extended.
(Hong, Nagamochi 2008)



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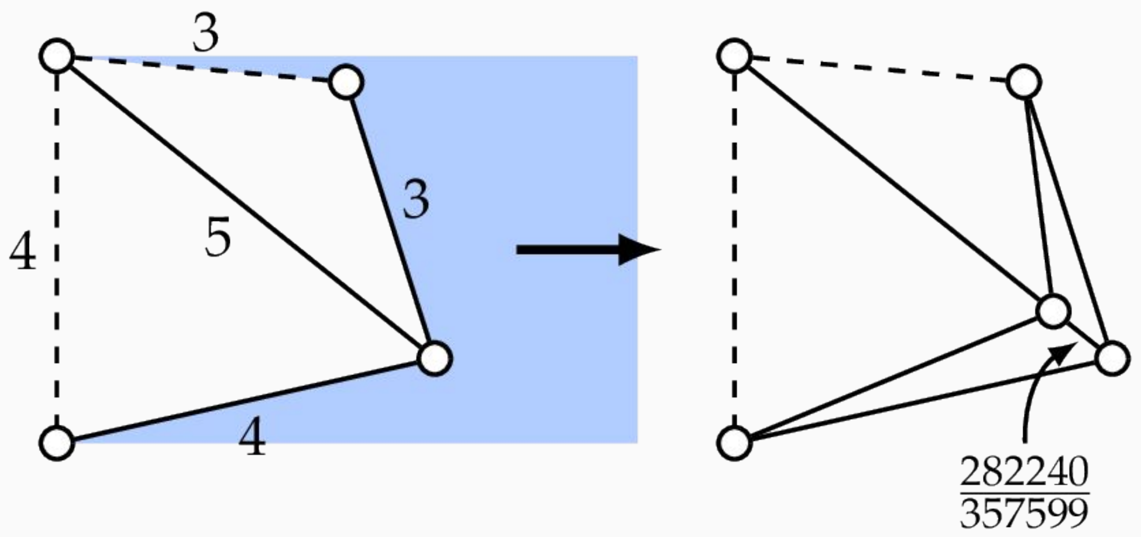
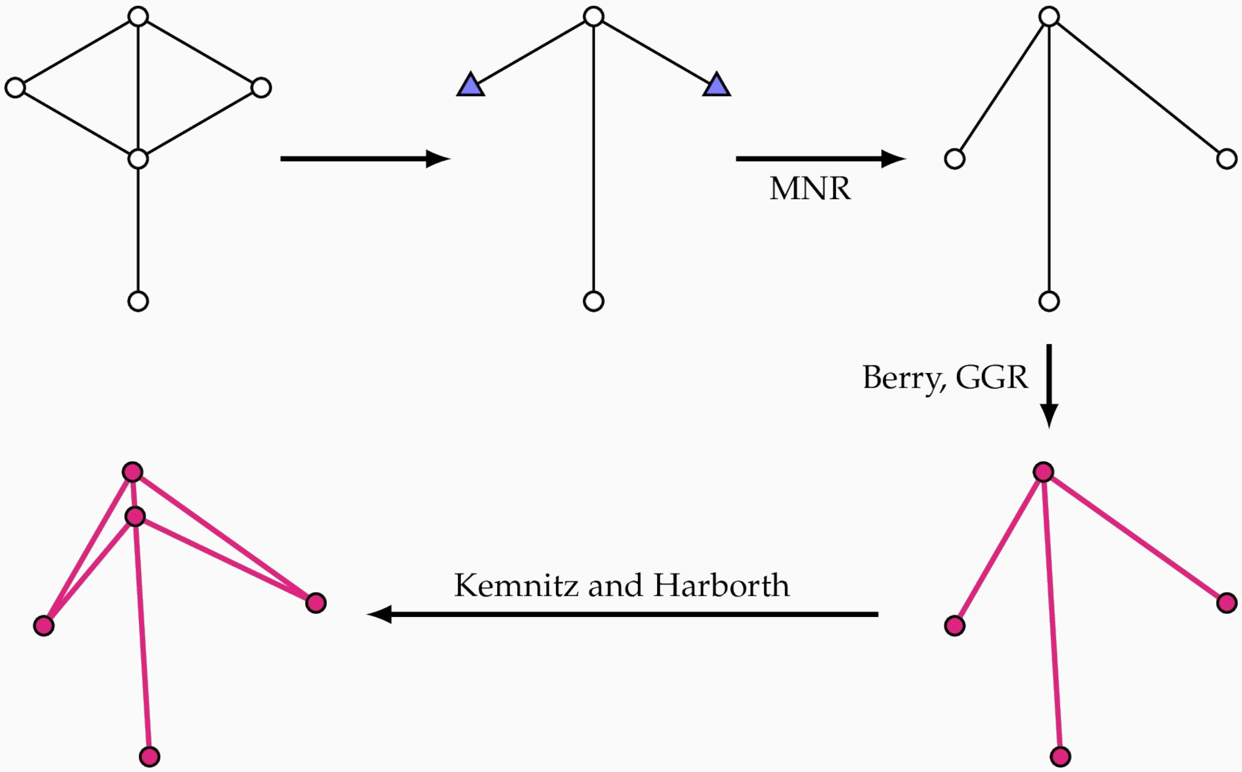


Adding vertices of degree 4

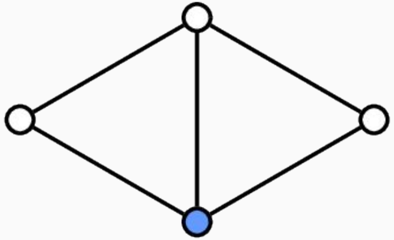
If 4-regular graph has a ***diamond***, then we can directly apply these results:



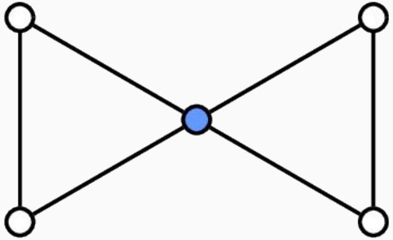
Lebesgue’s criterion

What if the graph doesn’t have a diamond?

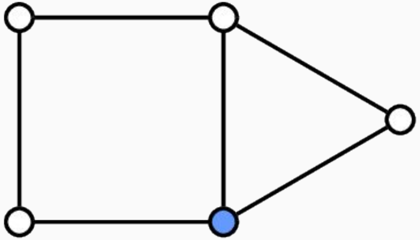
Then it must have a ***bowtie*** or a ***house***. (Proof: discharging)



diamond



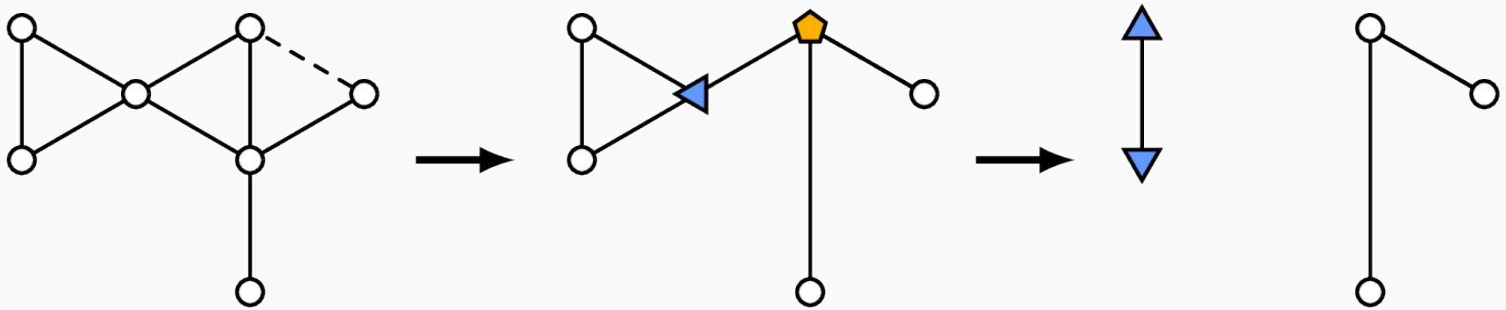
bowtie



house

3-elimination orders for bowties and houses

- Add an edge near a triangular face to get a diamond.
- Show that there is still a 3-elimination order (reduce until subquartic).



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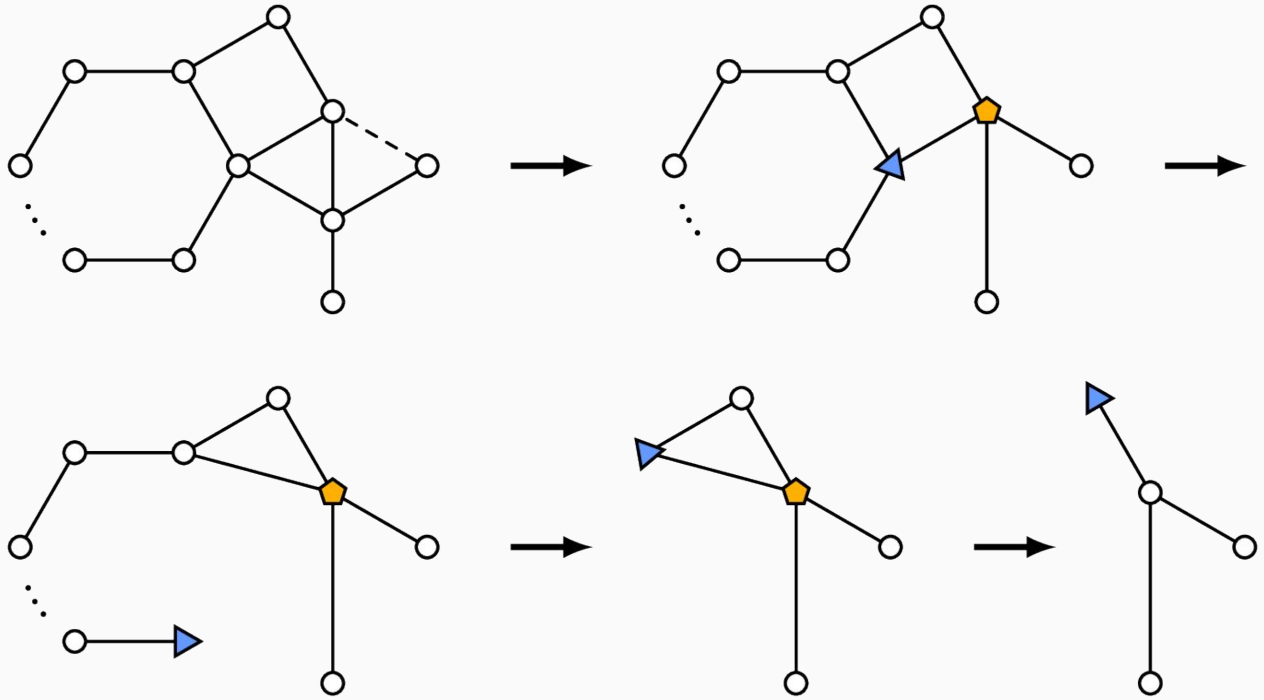
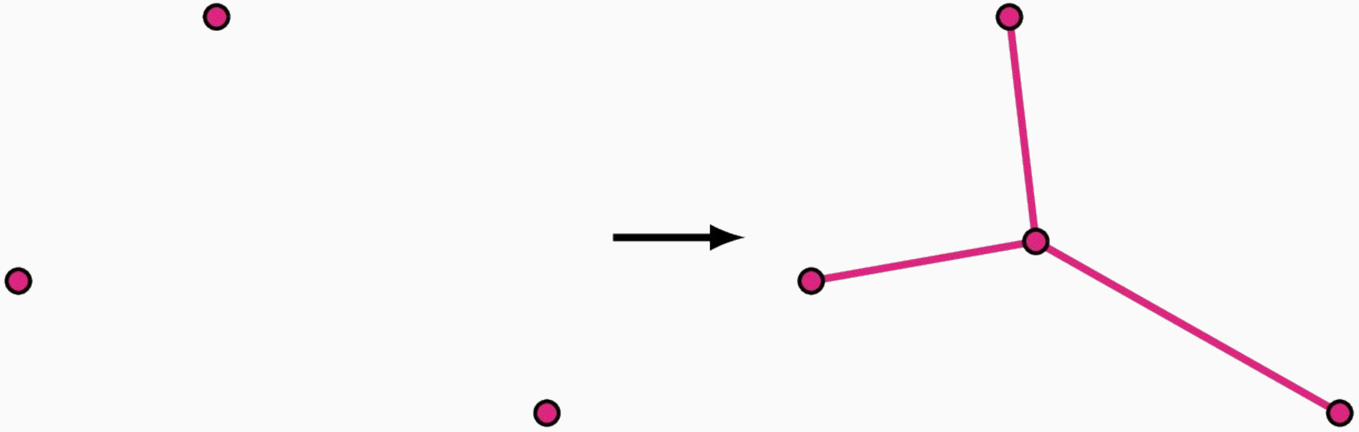


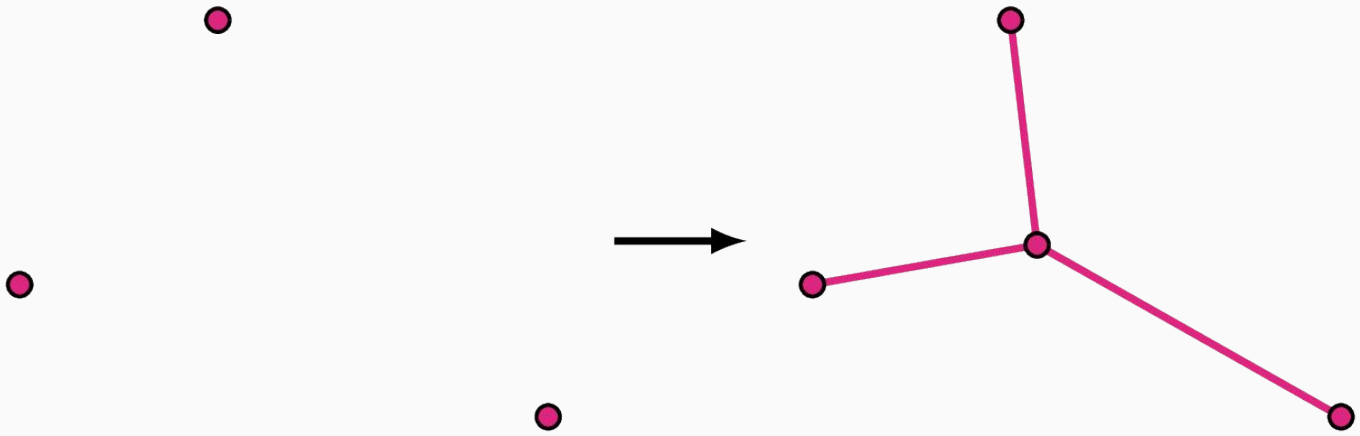
Addendum

Theorem (Corvaja, Turchet, Zannier 2024): The set of points at rational distance to three rational points are dense in the plane.



Addendum

Corollary: all 3-degenerate graphs can be “approximated” by rational drawings with rational vertex coordinates.



Conclusion

Harborth's conjecture is now known for all graphs of maximum degree 4.

- Can these proofs be made algorithmic?
- Are there any methods for finding a point at rational distance to a "special" set of 5 points?