Note on Min-k-Planar Drawings of Graphs

Petr Hliněný and Lili Ködmön

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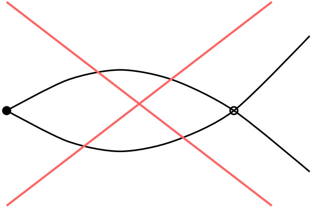
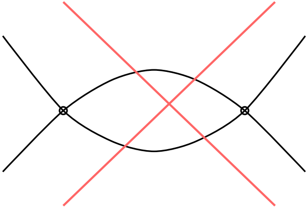
Definition (Simple drawing)

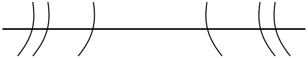
A *simple drawing* is a drawing in which any two edges intersect in at most one point, a crossing or a common endpoint.



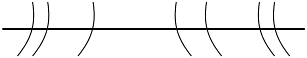


Definition (k-planar graph)

A *k-planar graph* is a graph admitting a drawing in which no edge has more than k crossings.



6-planar

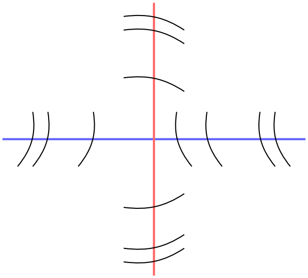


not 6-planar

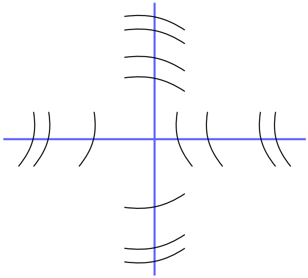
Definition (Min-k-planar graph)

General min-k-planar graphs are graphs admitting a drawing such that in any pair of crossing edges at least one has at most k crossings.

- If edge has **at most** k crossings, we call it *light*.
- If edge has **more than** k crossings, we call it *heavy*.



min-6-planar



not min-6-planar

Background

C. Binucci, A. Büngener, G. Di Battista, W. Didimo, V. Dujmović, S. Hong, M. Kaufmann, G. Liotta, P. Morin, A. Tappini. Min-k-planar drawings of graphs. (GD '23)

- introduced min-k-planar graphs
- considered only **simple min-k-planar** drawings

Theorem (M. Hoffmann, C. Liu, M. M. Reddy, C. D. Tóth (GD '20))

There exists a function $f(k)$ such that every k-planar graph admits a simple $f(k)$-planar drawing.

- $f(k) = \frac{2}{3}\sqrt{58}k^{3/2}3^k$ for $k \geq 4$

Question

Are all min-k-planar graphs simple min-k-planar or, at least, simple min-$f(k)$-planar?

YES for $k = 1$

NO for $k \geq 2$

Main Results

Proposition (Hliněný, K. 2024+)

a) *Every general min-1-planar graph admits a simple min-1-planar drawing.*

b) *Every graph with a min-2-planar drawing in which no two adjacent edges cross also admits a simple min-2-planar drawing.*

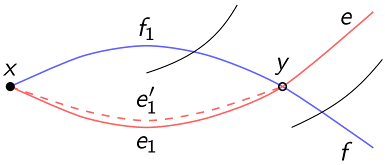
Theorem (Hliněný, K. 2024+)

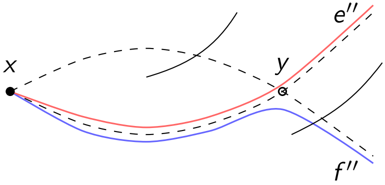
a) *For every $k \geq 2$, there exists a simple graph H_k which is general min-2-planar, but H_k has* no simple *min-k-planar drawing.*

b) *Moreover, for all $k \geq 3$, there exists a graph H'_k which has a general min-3-planar drawing in which no two adjacent edges cross, but, again, H'_k has* no simple *min-k-planar drawing.*

Proof of the Proposition

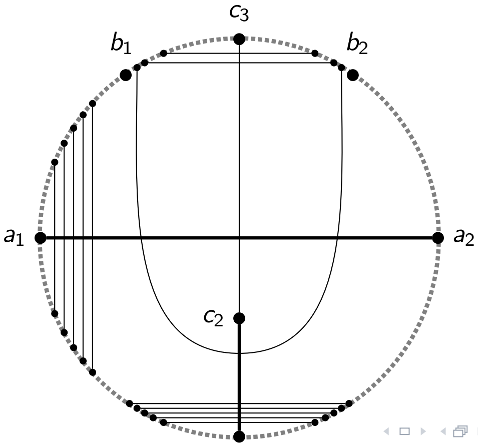
Standard redrawing along the uncrossed subarc of the light edge.





Proof of the Theorem (sketch)

We restrict the problem to graphs with a special structure, called *anchored* graphs. We then enforce the anchored property of the graphs with a special frame construction.



Anchored Graphs

Definition

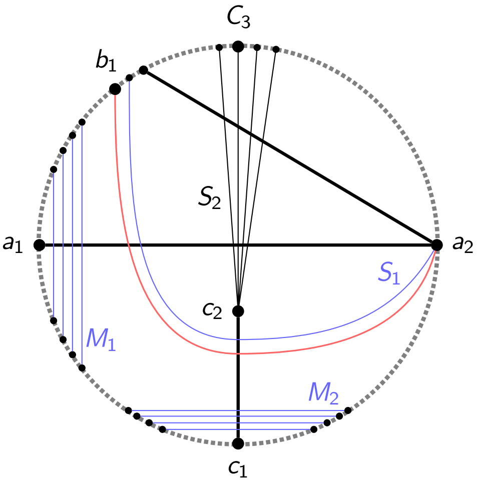
An *anchored graph* is a pair (G, A) where $A \subseteq V(G)$ is an ordered tuple of vertices. An *anchored drawing* of (G, A) in the unit disk $D \subseteq \mathbb{R}^2$ is a drawing $\mathcal{G} \subseteq D$ of G such that $\mathcal{G}$ intersects the boundary of D precisely in the points of A (the *anchors*) in this clockwise order. We naturally extend the adjective *anchored* to min-k-planar drawings.

Lemma (Hliněný, K. 2024+)

For every $k \geq 2$, there exists a simple anchored graph (G_k, A_k) which has an anchored general min-2-planar drawing, but (G_k, A_k) has no *anchored* simple *min-k-planar drawing. Furthermore, for any $k \geq 3$, there exists a simple anchored graph (G'_k, A'_k) which has an anchored general min-3-planar drawing in which no pair of adjacent edges cross, but (G'_k, A'_k) has* no *anchored* simple *min-k-planar drawing.*

Proof of the Lemma I

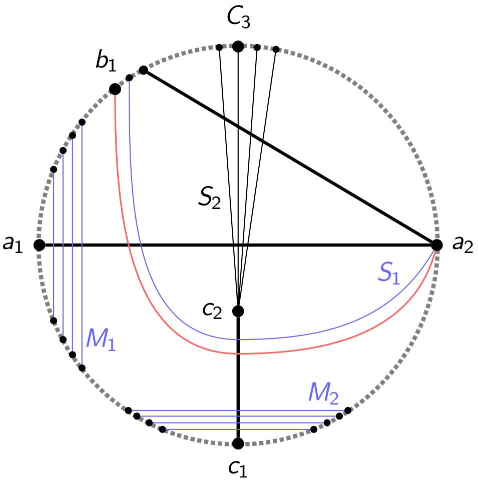
Min-2-Planar Case:



- M_1, M_2: disjoint matchings of $k+1$ edges
- S_1: induced star with center a_2 and $k+1$ leaves including a_1, b_1
- S_2: induced star with center c_2 and $k+2$ leaves in the set $C_3 \cup \{c_1\}$ where $|C_3| = k+1$
- $A_k = V(G_k) \setminus \{c_2\}$: anchor set

Proof of the Lemma II

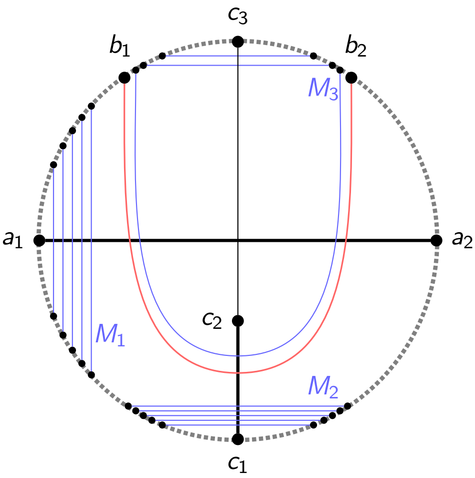
Min-2-Planar Case:



- By the Jordan Curve Theorem, a_1a_2 must cross all edges of M_1, so it is heavy.
- If c_1c_2 crosses a_1a_2, it must be light. Then, some edge of M_2 crosses a_1a_2 twice, contradiction.
- The edges going from c_2 to C_3 cross a_1a_2 and are light, so none of them can cross all $k+1$ edges of S_1. Some edge of S_1 crosses c_1c_2, but then it must cross a_1a_2, contradiction.

Proof of the Lemma III

Min-3-Planar Case:



- Same as before, except:
  - M_3: induced matching of $k+1$ edges
  - vertex c_3 replaces C_3

The proof of this case is almost identical to the previous one. We have to consider whether a_1a_2 crosses c_1c_2 or c_2c_3.

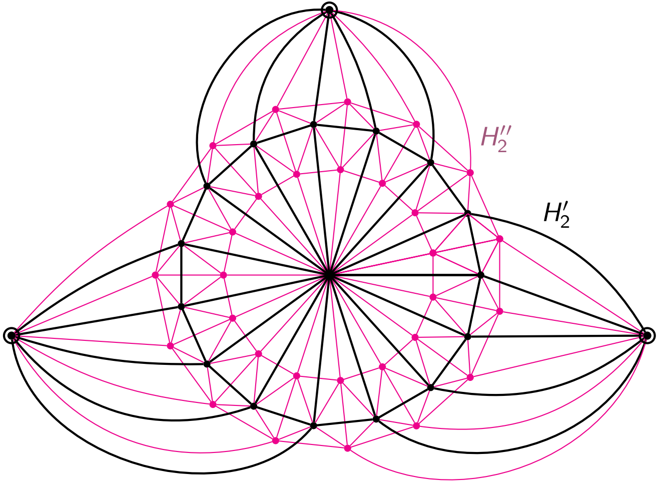
Constructing the Frame

Lemma (Hliněný, K. 2024+)

For any integers a, k and simple graph G with an ordered subset $A \subseteq V(G)$, $|A| = a$, there exists a simple anchored graph (H, A) disjoint from G except in the anchors A, such that the following hold:

a) (H, A) *has an anchored simple min-1-planar drawing.*

b) *In every general min-k-planar drawing $\mathcal{H}$ of $H \cup G$, the subdrawing $\mathcal{G} \subseteq \mathcal{H}$ of G is (spherically) homeomorphic to an anchored drawing of (G, A) or its mirror image.*



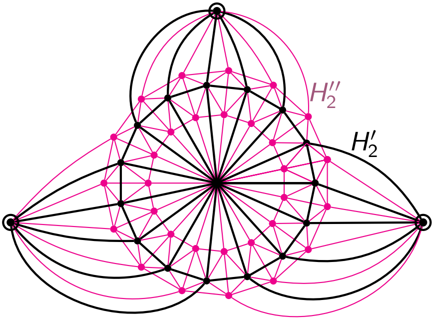
H_2''
H_2'

Open Problems

Which results in **Min-k-planar drawings of graphs** hold also for general min-k-planar drawings?

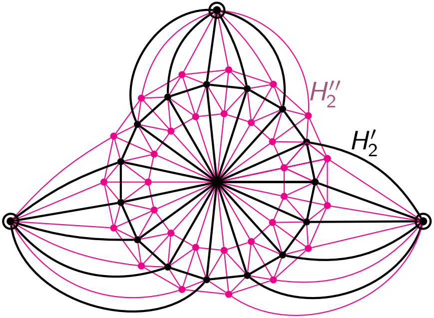
Thank you!

Proof of the Lemma (sketch) I



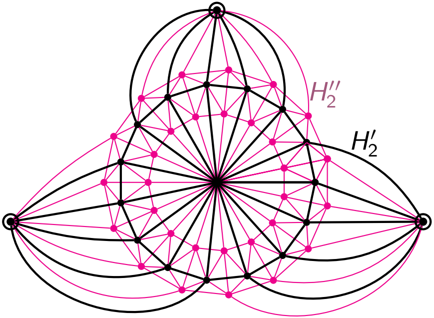
- H_0: double wheel with central vertices w_1, w_2
- H_0^*: take planar dual of H_0
- H_1: $H_0 \cup H_0^*$ with incident vertex-face pairs connected
- H_2: split w_2 into a vertices incident to equally split consecutive spoke edges of H_0 (anchor set A)
- H_2'': subgraph formed by magenta edges
- H: replace each edge of H_2'' with t paths of length 2

Proof of the Lemma (sketch) II



- (H, A) has an anchored simple min-1-planar drawing
- in any min-k-planar drawing $\mathcal{H}$ of $H \cup G$ there is a planar subdrawing of a $(k+1)$-amplification of H_2'' if t is sufficiently large (Ramsey-type argument)
- this crosses every black edge in $\mathcal{H}$ at least $(k+1)$-times (they are heavy)
- w_1 is connected to anchor by an uncrossed path in $\mathcal{H}$

Proof of the Lemma (sketch) III



- "invert" a local neighbourhood of w_1 into the outer face
- get an anchored subdrawing of (G, A) or its mirror image

To complete the proof of the main theorem, plug (G_k, A_k) and (G'_k, A'_k) into this lemma!