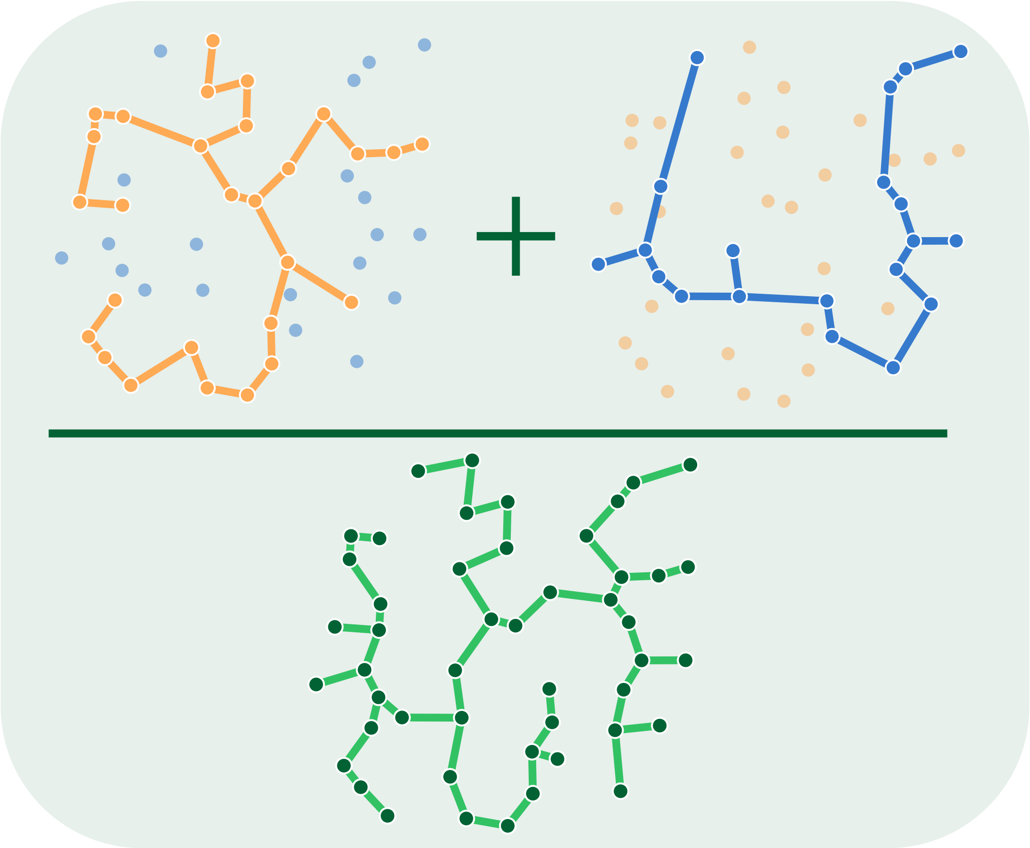
The Euclidean MST-ratio



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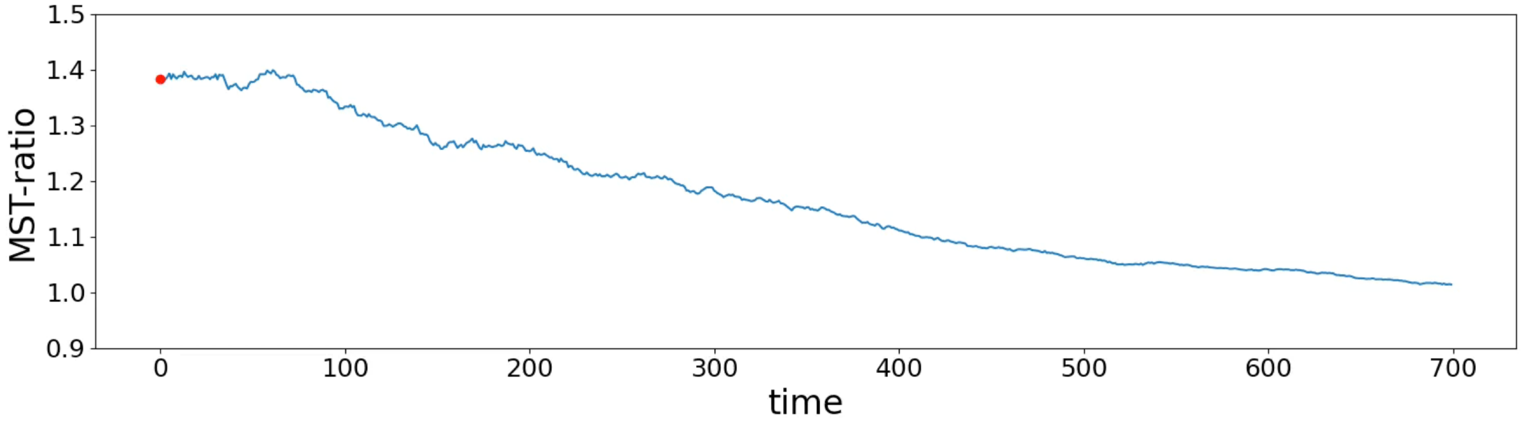
GD'24, Vienna

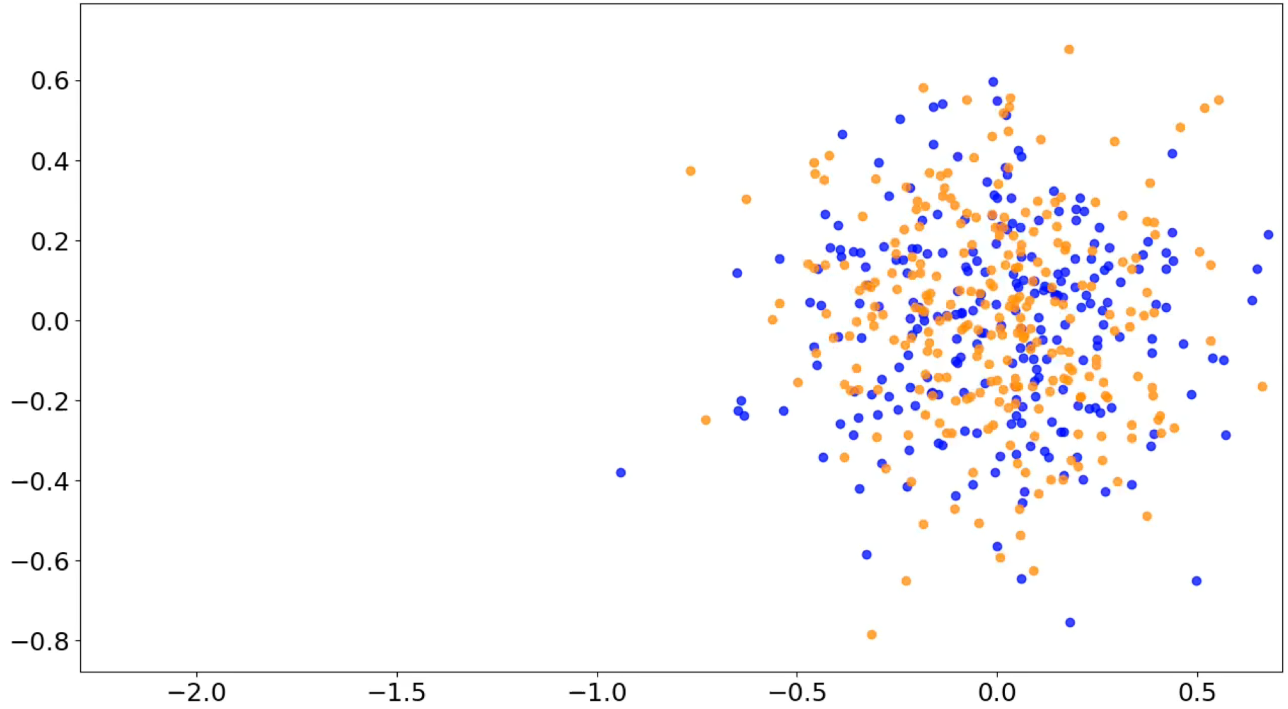
September 19, 2024

Institute of Science and Technology Austria

Joint work with S. Cultrera di Montesano, H. Edelsbrunner, and M. Saghafian

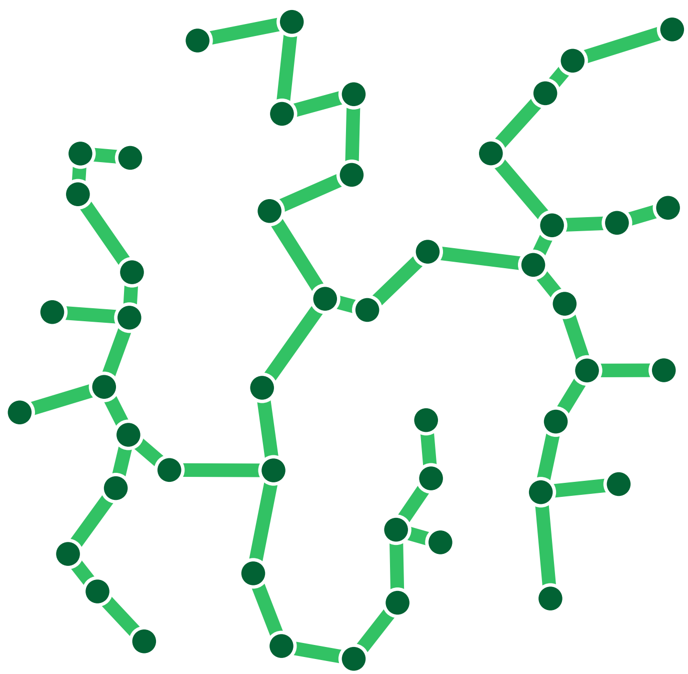
MST-ratio captures “mingling” of points





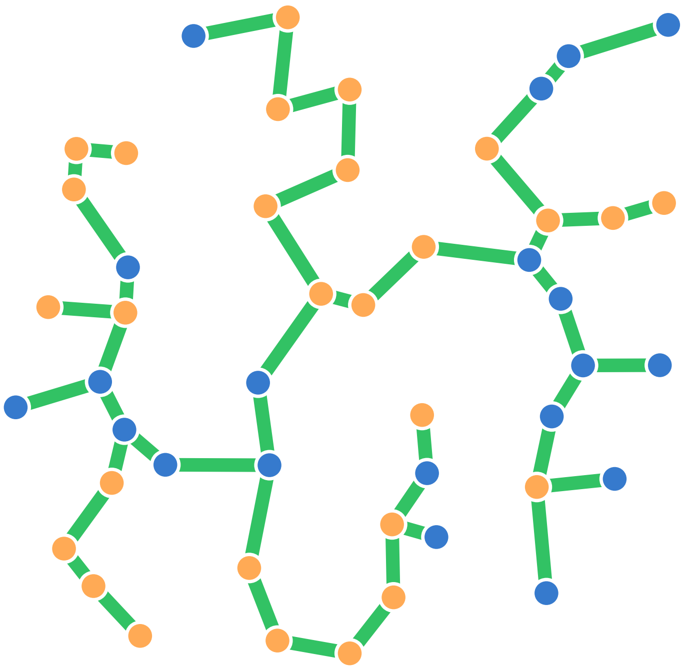
MST-ratio

- $A \subset \mathbb{R}^2$ is a finite point set
- $\mathrm{MST}(A)$ is the Euclidean minimum spanning tree



MST-ratio

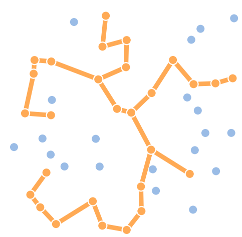
- $A \subset \mathbb{R}^2$ is a finite point set
- $\mathrm{MST}(A)$ is the Euclidean minimum spanning tree
- For $B \subseteq A$, we partition $A = B \sqcup A \setminus B$



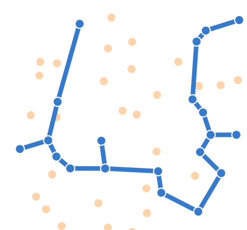
MST-ratio

- $A \subset \mathbb{R}^2$ is a finite point set
- $\mathrm{MST}(A)$ is the Euclidean minimum spanning tree
- For $B \subseteq A$, the MST-ratio of A and B is

$$\mu(A, B) = \frac{|\mathrm{MST}(B)| + |\mathrm{MST}(A \setminus B)|}{|\mathrm{MST}(A)|}$$



+



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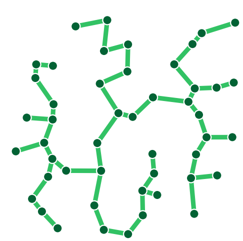
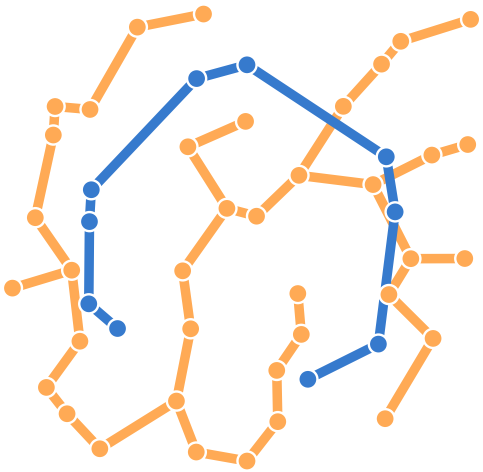
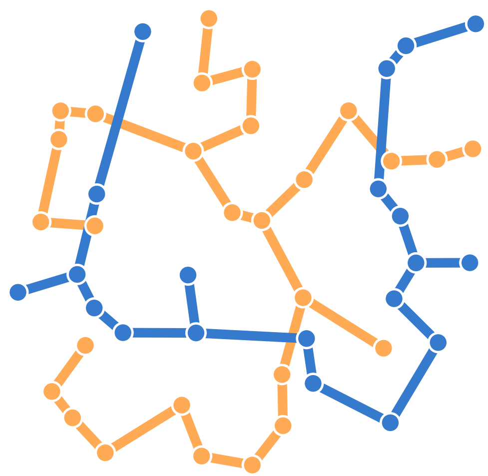


How small / large can MST-ratio get?

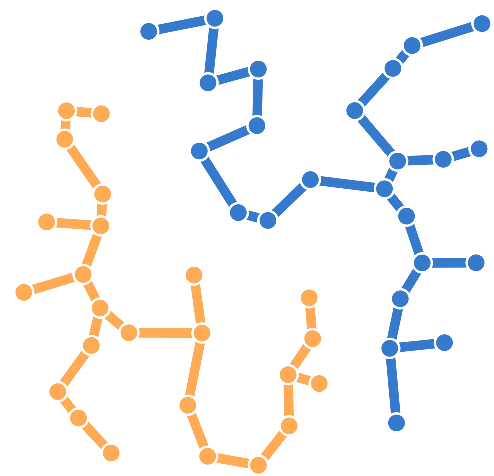
$$??\leq \mu(A,B)=\frac{|\mathrm{MST}(B)|+|\mathrm{MST}(A\setminus B)|}{|\mathrm{MST}(A)|}\leq ??$$



1.238



1.218



0.969

How small / large can MST-ratio get?

$$??\leq \mu(A,B)=\frac{|\mathrm{MST}(B)|+|\mathrm{MST}(A\setminus B)|}{|\mathrm{MST}(A)|}\leq ??$$

$$\min_{B\subseteq A}\mu(A,B)$$ ”least-mixed” coloring

$$\max_{B\subseteq A}\mu(A,B)$$ ”most-mixed” coloring

How small / large can MST-ratio get?

$$??\leq \mu(A,B)=\frac{|\mathrm{MST}(B)|+|\mathrm{MST}(A\setminus B)|}{|\mathrm{MST}(A)|}\leq ??$$

$$\inf_{A}\ \min_{B\subseteq A}\mu(A,B)\quad\leq \min_{B\subseteq A}\mu(A,B)\leq\quad \sup_{A}\ \min_{B\subseteq A}\mu(A,B)$$

$$\inf_{A}\ \max_{B\subseteq A}\mu(A,B)\quad\leq \max_{B\subseteq A}\mu(A,B)\leq\quad \sup_{A}\ \max_{B\subseteq A}\mu(A,B)$$

How small / large can MST-ratio get?

$$??\leq \mu(A,B) = \frac{|\mathrm{MST}(B)| + |\mathrm{MST}(A\setminus B)|}{|\mathrm{MST}(A)|} \leq ??$$

$$\inf_{A} \min_{B\subseteq A} \mu(A,B) \leq \min_{B\subseteq A} \mu(A,B) \leq \sup_{A} \min_{B\subseteq A} \mu(A,B)$$

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$$\inf_{A} \min_{B\subseteq A} \mu(A,B) = ?$$

$$\sup_{A} \min_{B\subseteq A} \mu(A,B) = ?$$

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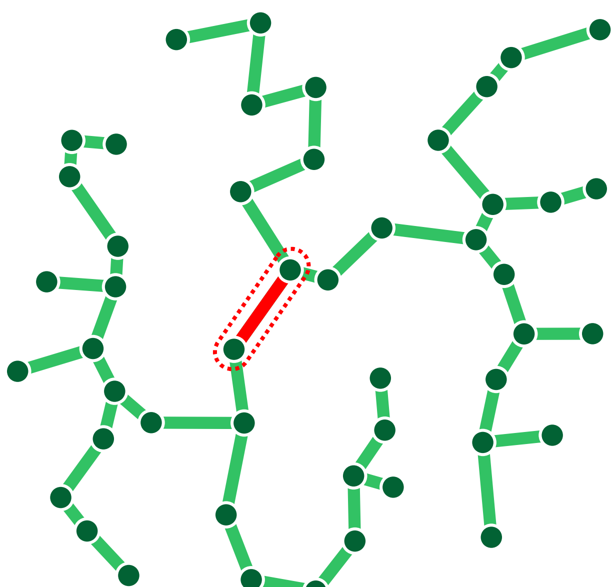
The minimum MST-ratio for a given A

- $\min_{B \subseteq A} \mu(A, B) = \dfrac{|\mathrm{MST}(A)| - |\omega|}{|\mathrm{MST}(A)|}$

  where ω is the longest edge in $|\mathrm{MST}(A)|$



$\inf_{A} \min_{B \subseteq A} \mu(A, B)$

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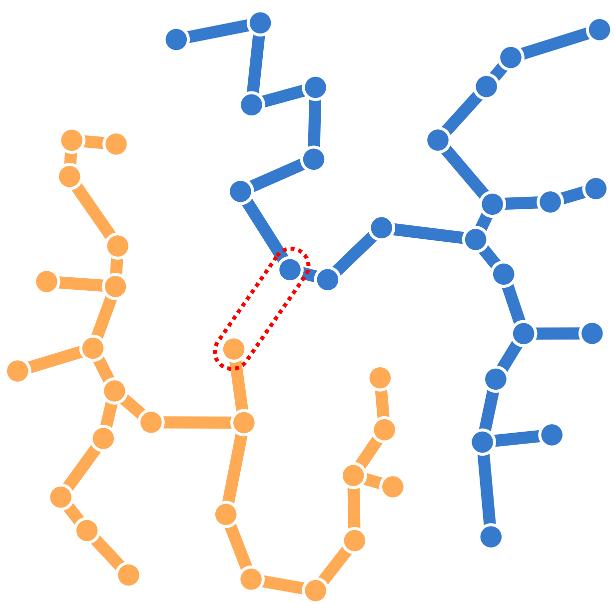
  where ω is the longest edge in $|\mathrm{MST}(A)|$
- we can always achieve this



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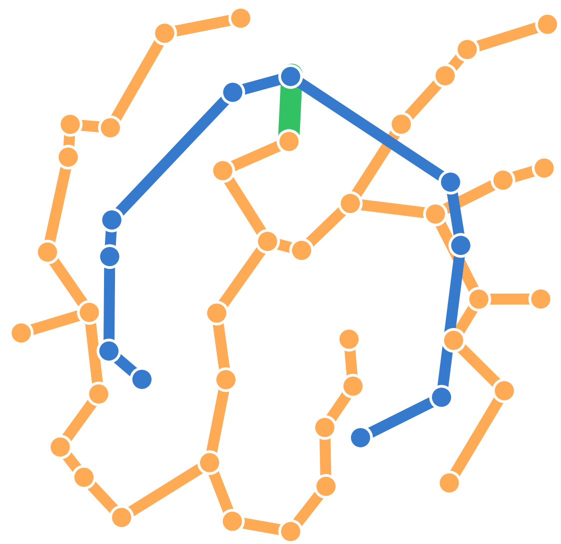
- it is the minimum:

$$|\mathrm{MST}(A)| \qquad \le |\mathrm{MST}(B)| + |\mathrm{MST}(A \setminus B)| + |e|$$



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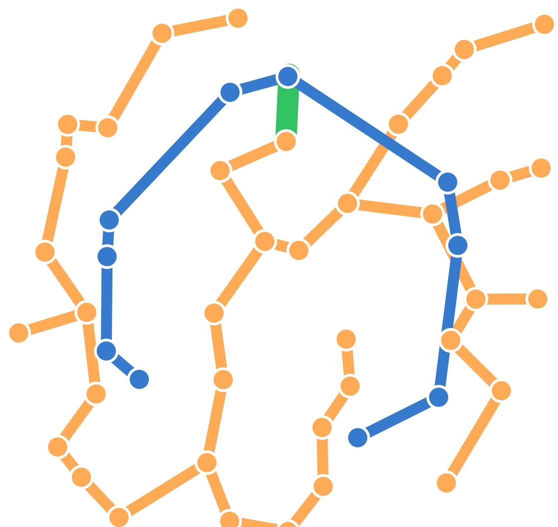

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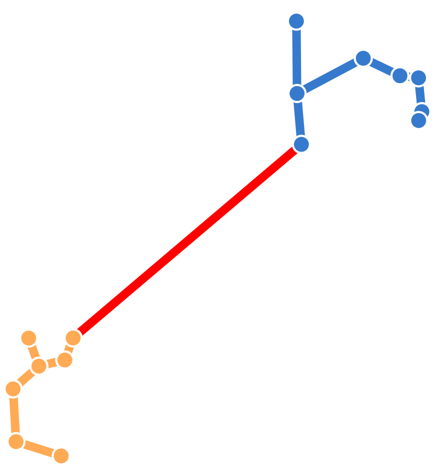
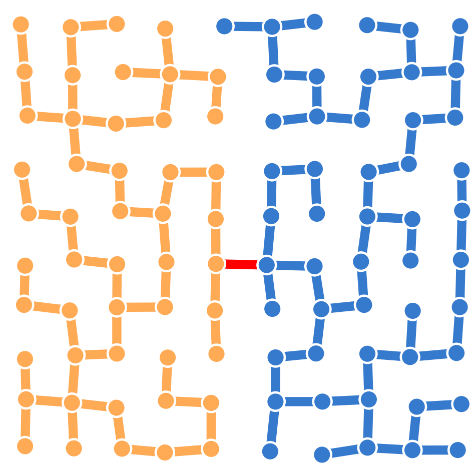


“only $|\omega|$ matters”



“$|\omega|$ negligible”

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The maximum MST-ratio for a given A

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NP-hard ??

A. J. Ameli, F. Motiei, M. Saghafian:
The Complexity of Maximizing the MST-ratio
[arXiv:2409.11079]

more general
formulation is
NP-hard

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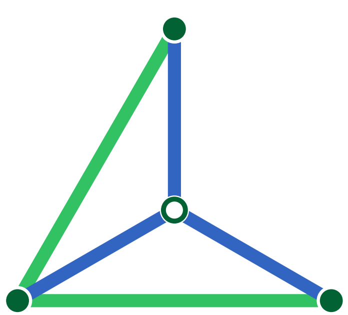
The maximum MST-ratio for a given A

- computing $\max_{B \subseteq A} \mu(A, B)$ is difficult
- the supremum of $\mu(A, B)$ closely relates to Steiner ratio
  - Steiner tree of A
    - $\mathrm{MST}(A \cup S)$ for some extra points S
  - Minimum Steiner tree $\mathrm{StT}(A)$
    - Steiner tree minimizing the length
  - Steiner ratio
    - $\alpha \coloneqq \sup_{A} \frac{\mathrm{MST}(A)}{\mathrm{StT}(A)}$



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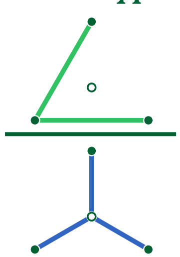
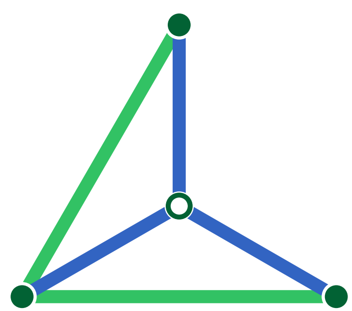


$= \frac{2}{\sqrt{3}} \approx 1.1547$



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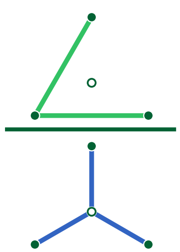
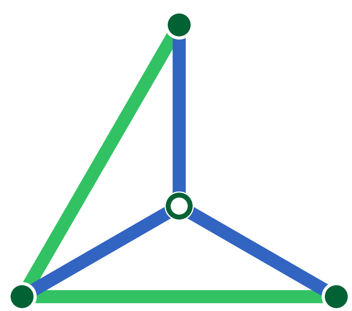
    - $\alpha := \sup_A \frac{\mathrm{MST}(A)}{\mathrm{StT}(A)} \overset{[1]}{\leq} 1.213\cdots$



$$= \frac{2}{\sqrt{3}} \approx 1.1547$$



[1] F. Chung, R. Grham, *"A new bound for Euclidean Steiner minimal trees,"* (1985)

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$\alpha \leq 1.213 \cdots$

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$$\frac{|\mathrm{MST}(B)| + |\mathrm{MST}(A \setminus B)|}{|\mathrm{MST}(A)|} \leq \alpha + \alpha \leq 2.426 \cdots$$

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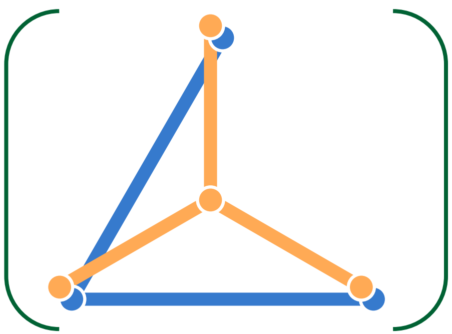
μ  $= \frac{\sqrt{3} + 2}{\sqrt{3} + 3\epsilon} \to 2.154\cdots$

$$\inf_A \min_{B \subseteq A} \mu(A, B) = 0$$

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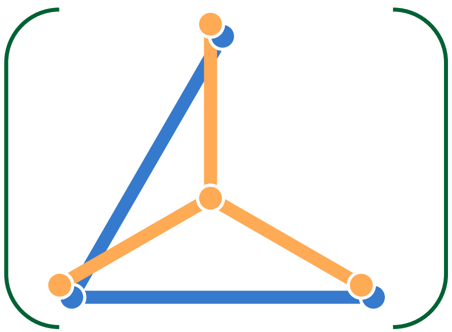
μ  $= \frac{\sqrt{3} + 2}{\sqrt{3} + 3\epsilon} \rightarrow 2.154 \cdots$

$$2.154 \cdots \leq \sup_A \max_{B \subseteq A} \mu(A, B) \leq 2.426 \cdots$$

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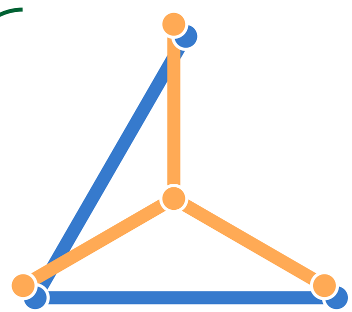
$\mu\Big($  $\Big) = \dfrac{\overbrace{\sqrt{3}}^{1} + 2}{\sqrt{3} + 3\epsilon} \to 2.154\cdots$ ($\leq \alpha$)

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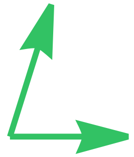
$$1 + \alpha \leq \sup_A \max_{B\subseteq A} \mu(A,B) \leq 2\alpha$$

Lattices

The supremum MST-ratio for lattices

- a lattice is $\Lambda(\mathbf{u}, \mathbf{v}) = \{i\mathbf{u} + j\mathbf{v} \mid i, j \in \mathbb{Z}\}$



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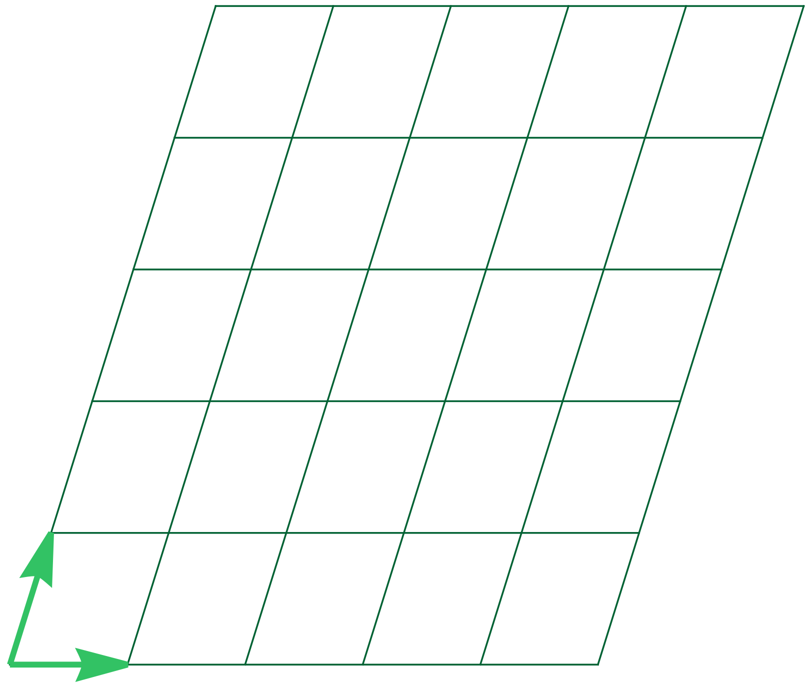

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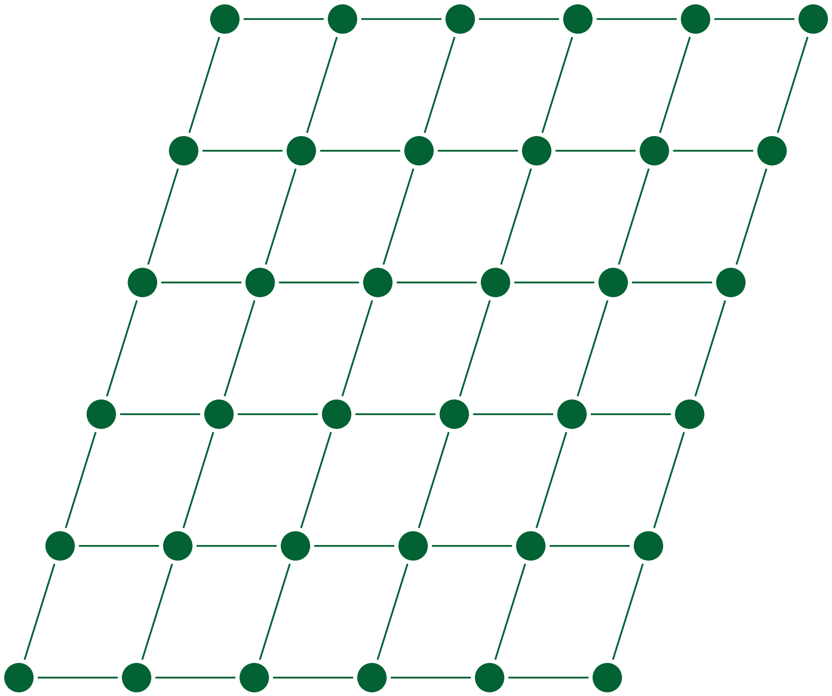

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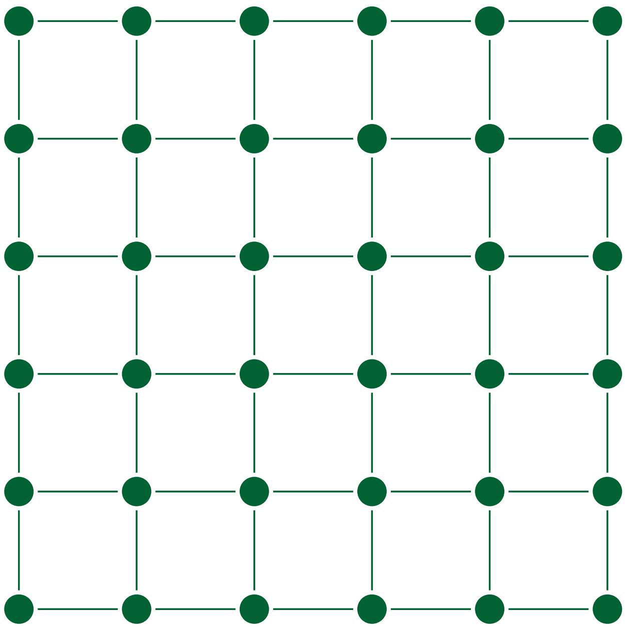
- for example: square



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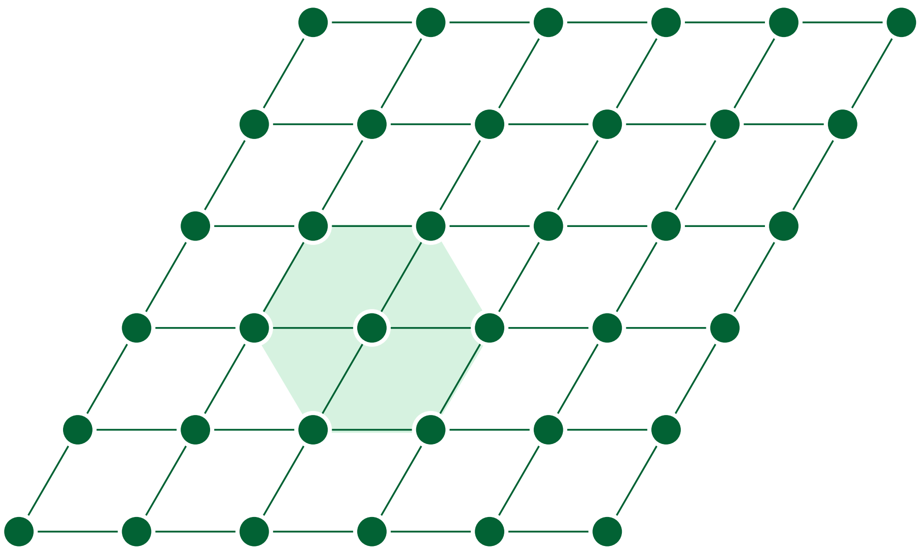
- for example: square, hexagonal



The supremum MST-ratio for lattices

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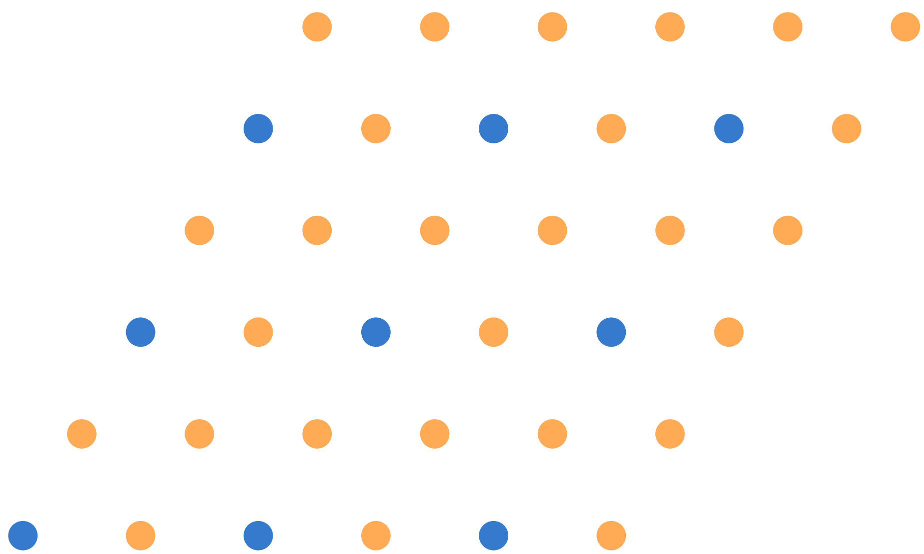
- for example: square, hexagonal
- MST-ratio is the limit of finite rhombi: $\mu(\Lambda, \mathrm{B}) = \lim_{n \to \infty} \mu(\Lambda_n, B_n)$



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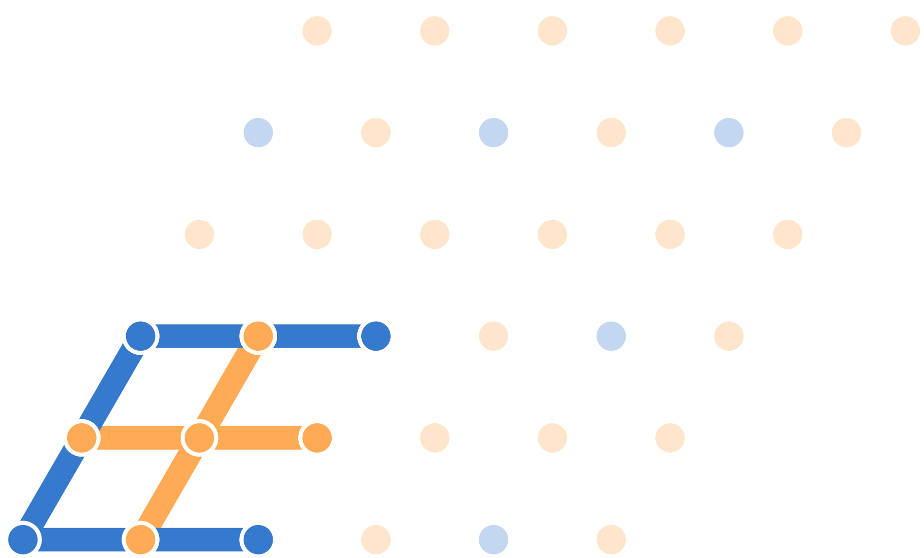

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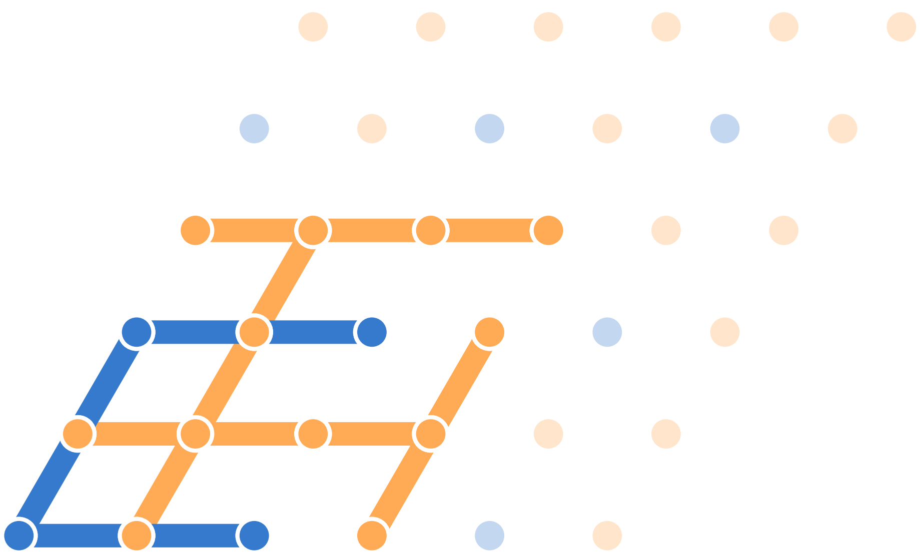

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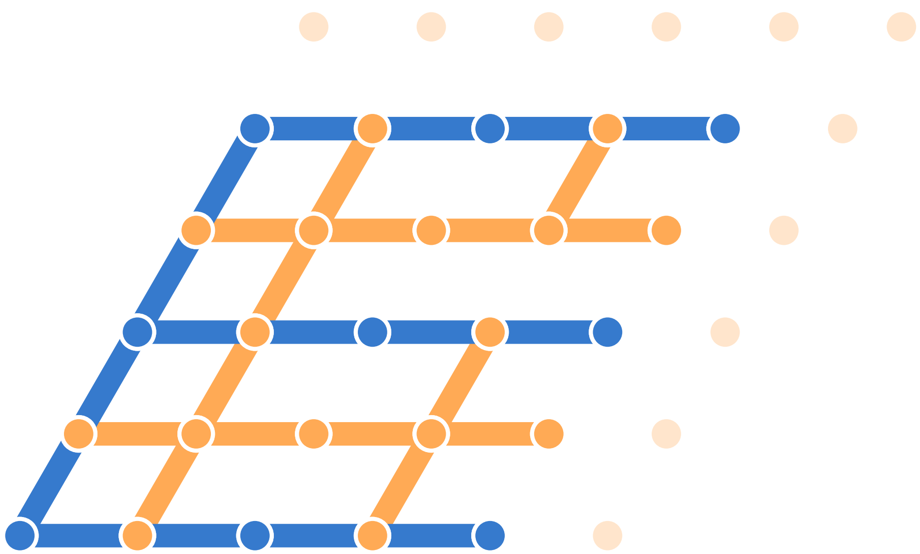

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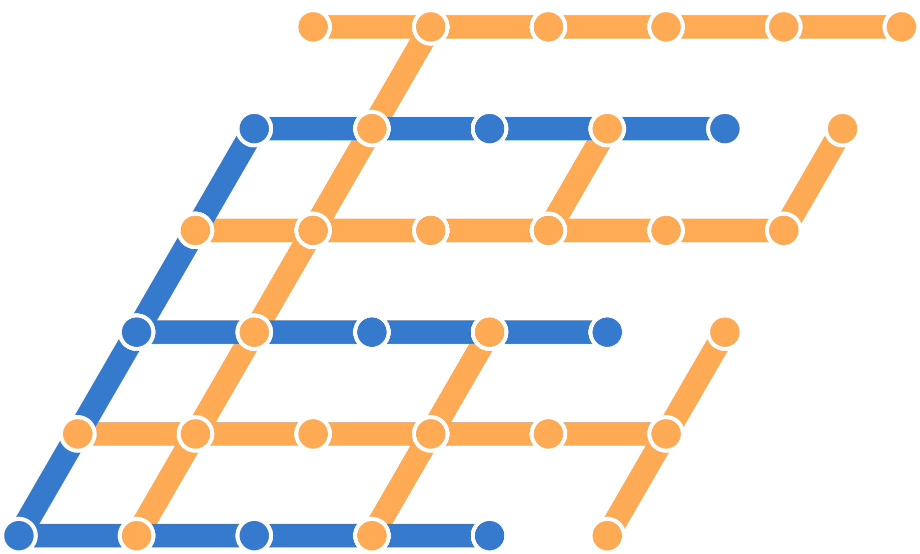

The supremum MST-ratio for lattices

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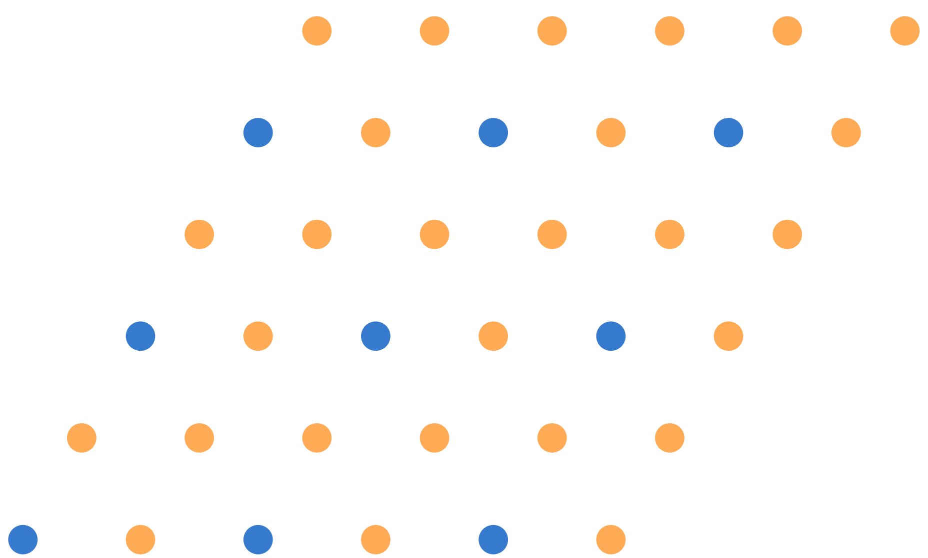
- $\mu(\Lambda_n, B_n) = 1.25, 1.13, 1.29, 1.2, \ldots \xrightarrow[n\to\infty]{} 1.25$



The supremum MST-ratio for lattices

Theorem

$$1.25 \leq \inf_{\Lambda} \sup_{B \subseteq \Lambda} \mu(\Lambda, B) \leq 1.25$$

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The supremum MST-ratio for lattices

Theorem

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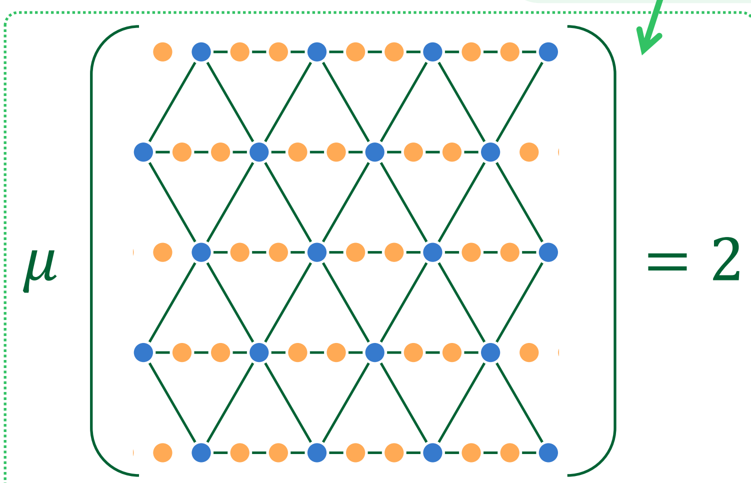

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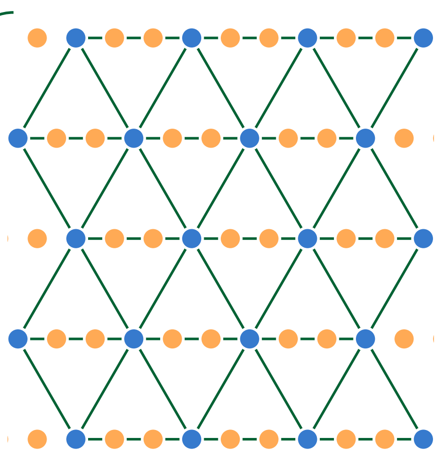
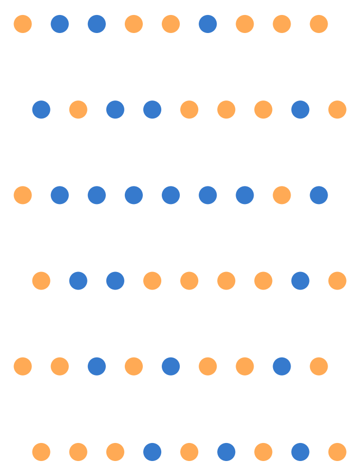
μ () $= 2$



$$\frac{|\mathrm{MST}(\Lambda_n)| + |\mathrm{MST}(\Lambda_n \setminus B_n)|}{|\mathrm{MST}(A)|}$$

$\le$

The supremum MST-ratio for lattices

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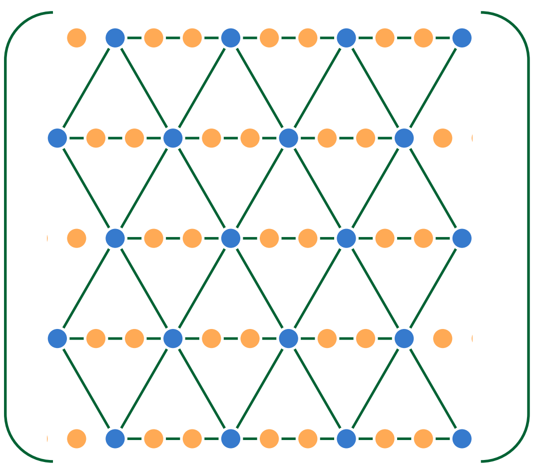
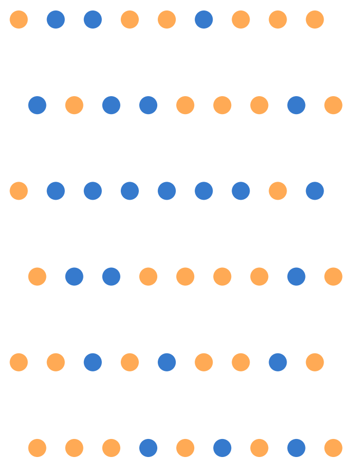
μ



$= 2$



$$\le \frac{\dfrac{|\mathrm{MST}(\Lambda_n)| + |\mathrm{MST}(\Lambda_n \setminus B_n)|}{|\mathrm{MST}(A)|}}{\sim n^2}$$

The supremum MST-ratio for lattices

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$$1.25 \le \inf_{\Lambda} \sup_{B \subseteq \Lambda} \mu(\Lambda, B) \le 1.25$$

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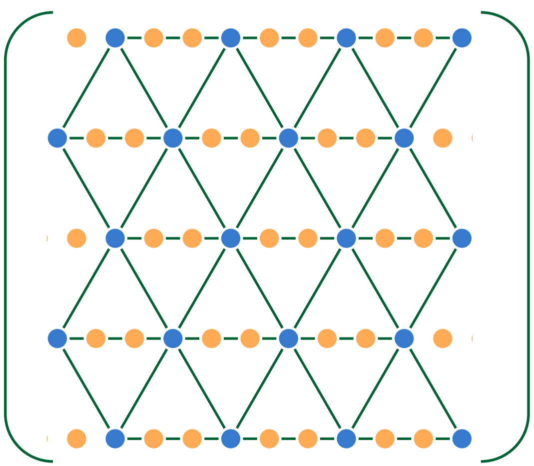

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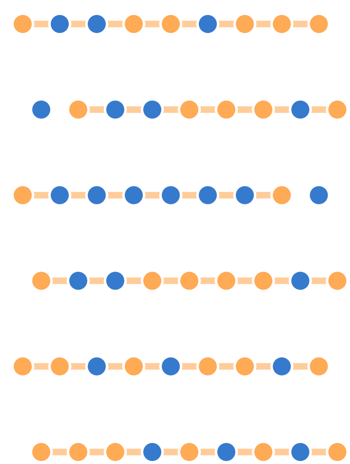

$$\frac{|\mathrm{MST}(\Lambda_n)| + |\mathrm{MST}(\Lambda_n \setminus B_n)|}{|\mathrm{MST}(A)|}$$

$$\le \frac{\sim n^2}{\sim n^2}$$

The supremum MST-ratio for lattices

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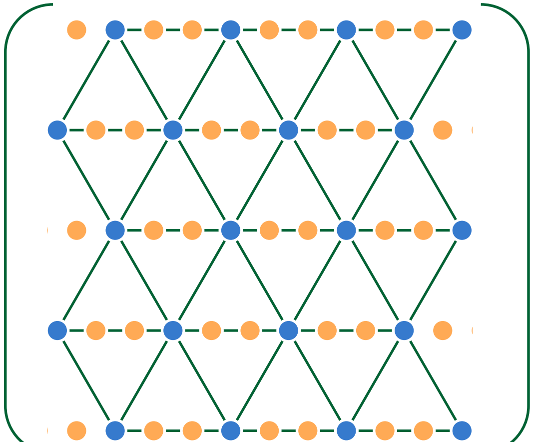
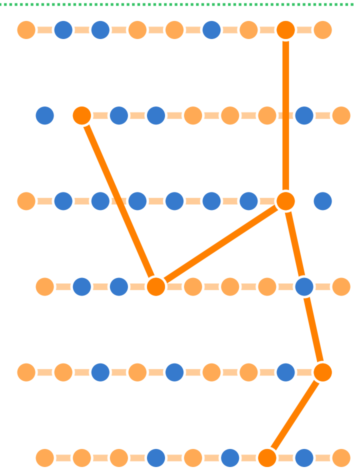
μ  $= 2$



$$\frac{|\mathrm{MST}(\Lambda_n)| + |\mathrm{MST}(\Lambda_n \setminus B_n)|}{|\mathrm{MST}(A)|} \le \frac{\sim n^2 + O(n\sqrt{n})}{\sim n^2}$$

The supremum MST-ratio for lattices

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$$1.25 \le \inf_{\Lambda} \sup_{B \subseteq \Lambda} \mu(\Lambda, B) \le 1.25$$

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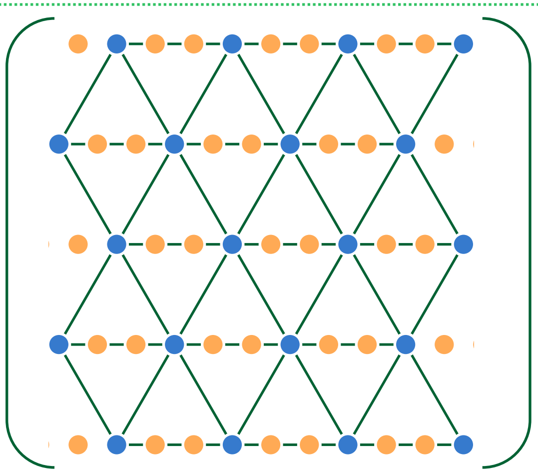
μ



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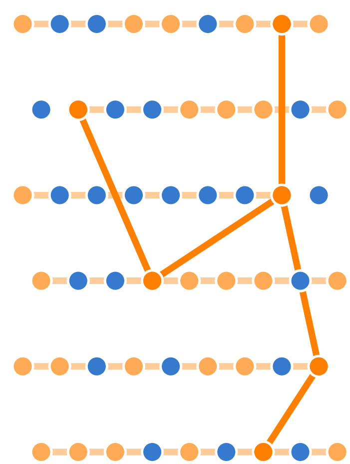

$$\frac{|\mathrm{MST}(\Lambda_n)| + |\mathrm{MST}(\Lambda_n \setminus B_n)|}{|\mathrm{MST}(A)|}$$

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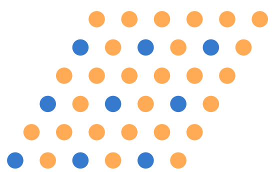
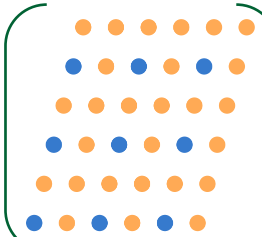
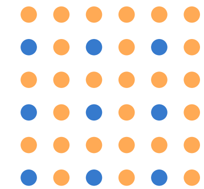
$$\xrightarrow{n \to \infty} 2$$

The supremum MST-ratio for lattices

$1.25 \leq \mu\Big($  $\Big), \ \mu\Big($  $\Big), \ \mu\Big($  $\Big)$

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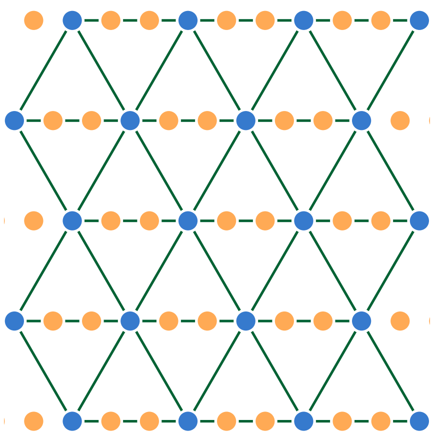
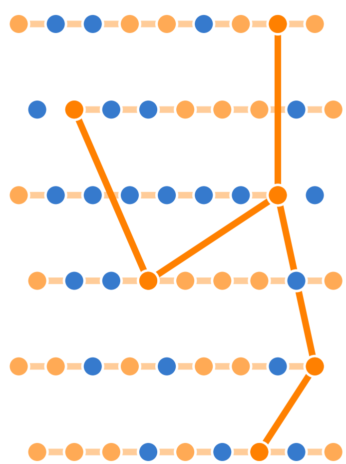
$\mu\Big($  $\Big) = 2$



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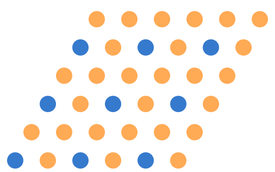
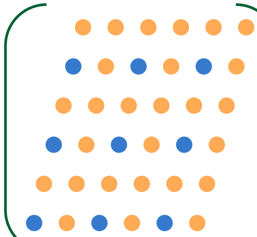
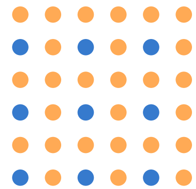
The supremum MST-ratio for lattices

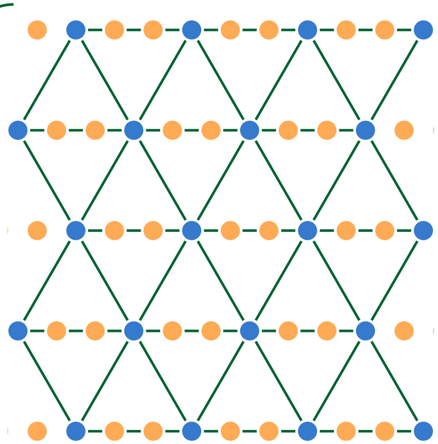
$1.25 \le \mu$  $, \mu$  $, \mu$ 

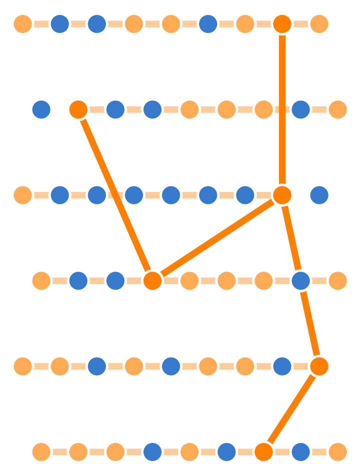
Analyze the hexagonal lattice

Theorem

$$1.25 \le \inf_{\Lambda} \sup_{B \subseteq \Lambda} \mu(\Lambda, B) \le 1.25$$

$$2 \le \sup_{\Lambda} \sup_{B \subseteq \Lambda} \mu(\Lambda, B) \le 2$$

μ  $= 2$



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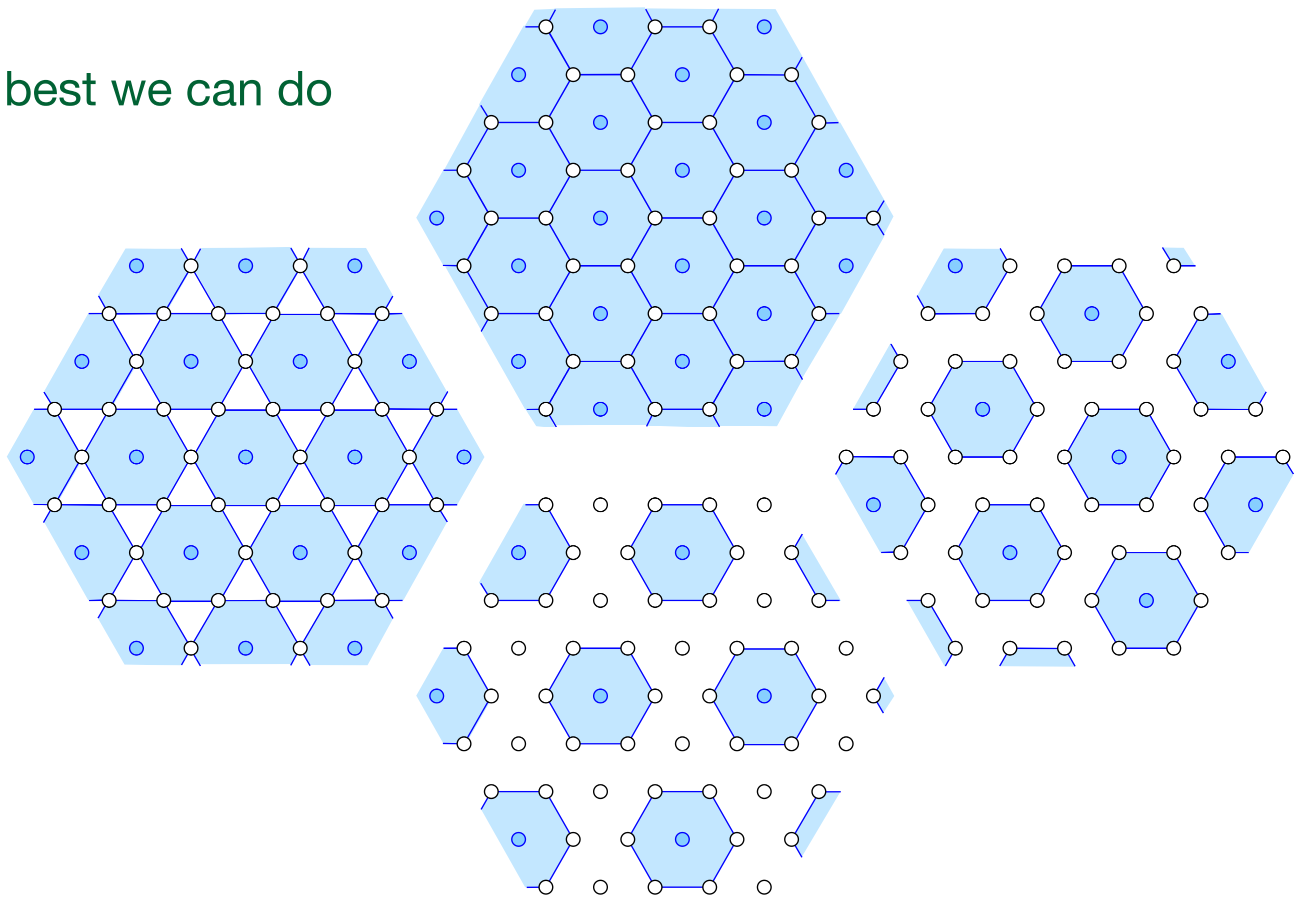
$$\xrightarrow{n \to \infty} 2$$

The supremum MST-ratio for hexagonal

Goal:
1.25 is the best we can do

1.245

1.25



1.236

1.222

The supremum MST-ratio for hexagonal

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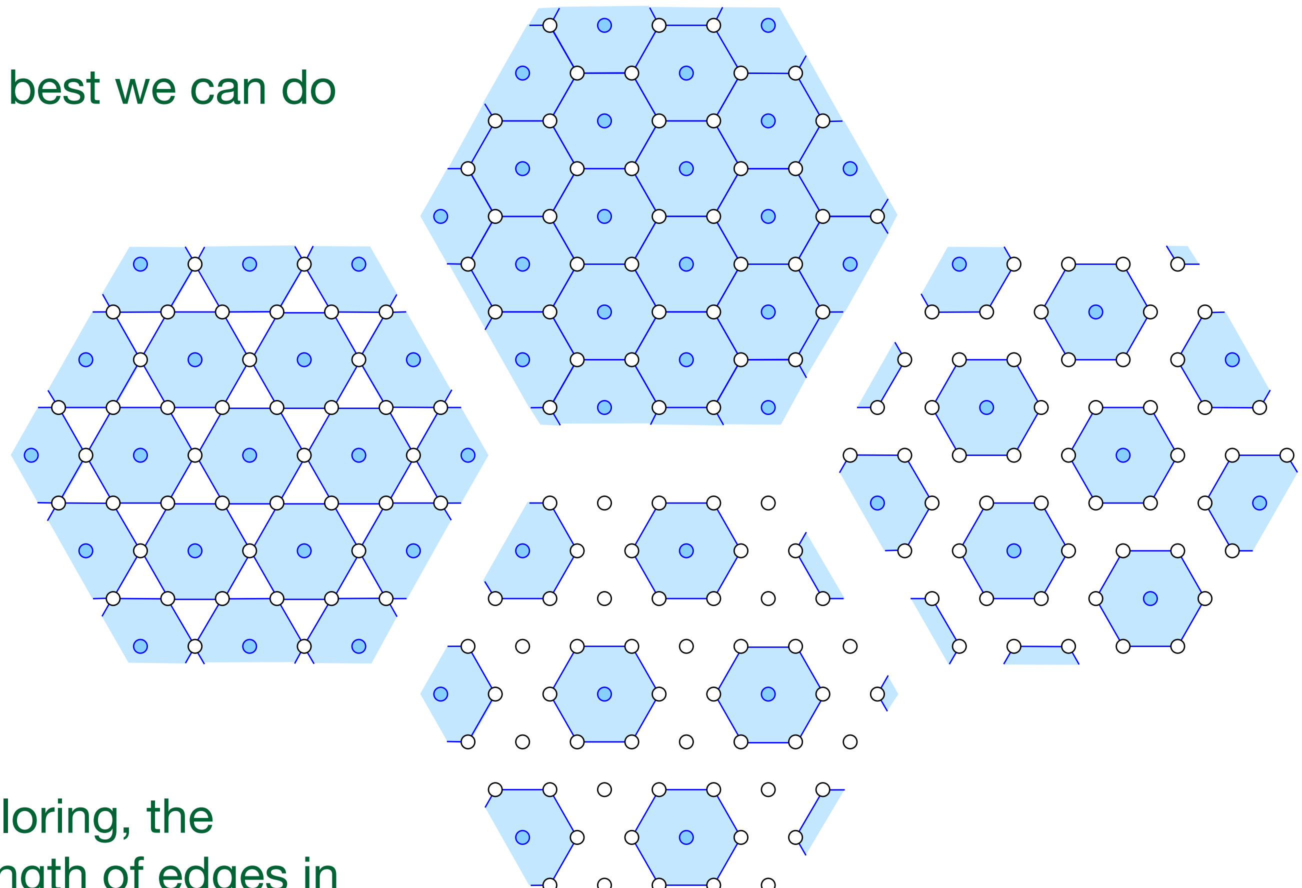
1.25 is the best we can do

1.245

1.25

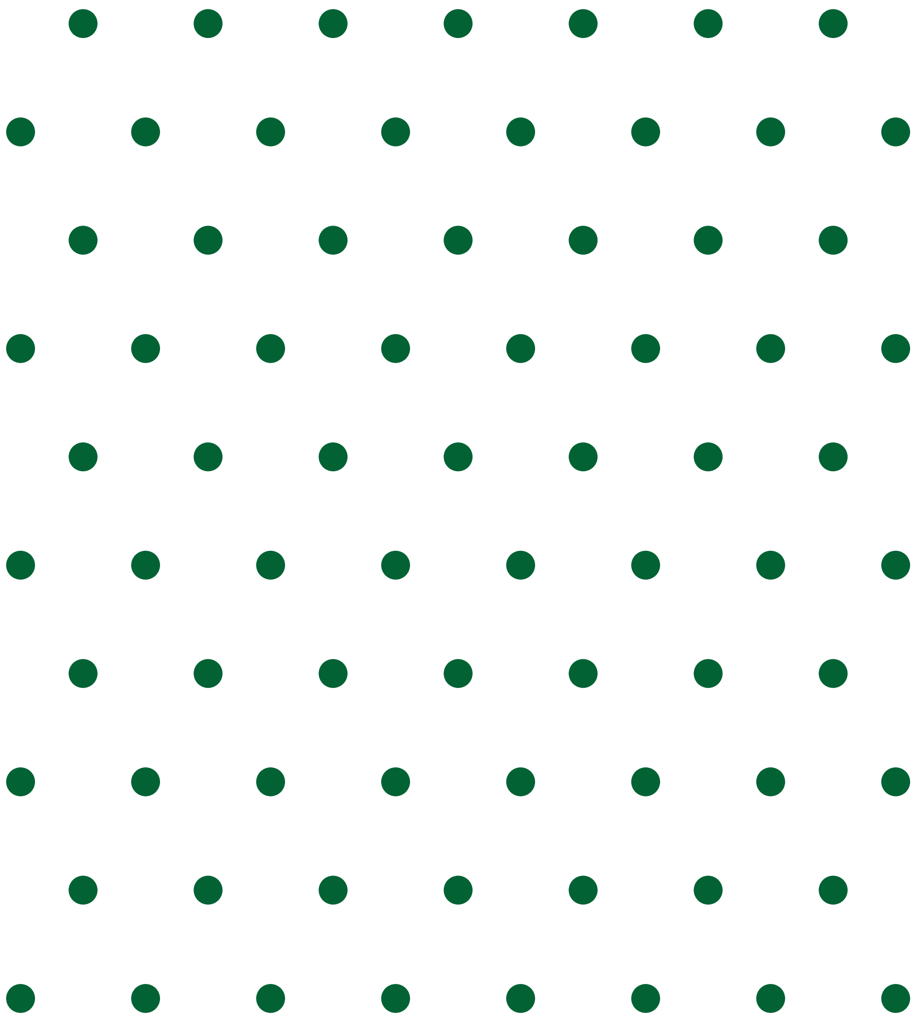


1.236

1.222

Idea:
For any coloring, the average length of edges in $|\mathrm{MST(B)}|$, $|\mathrm{MST}(\Lambda \setminus B)|$ is at most 1.25

Average colored edge length ≤ 1.25



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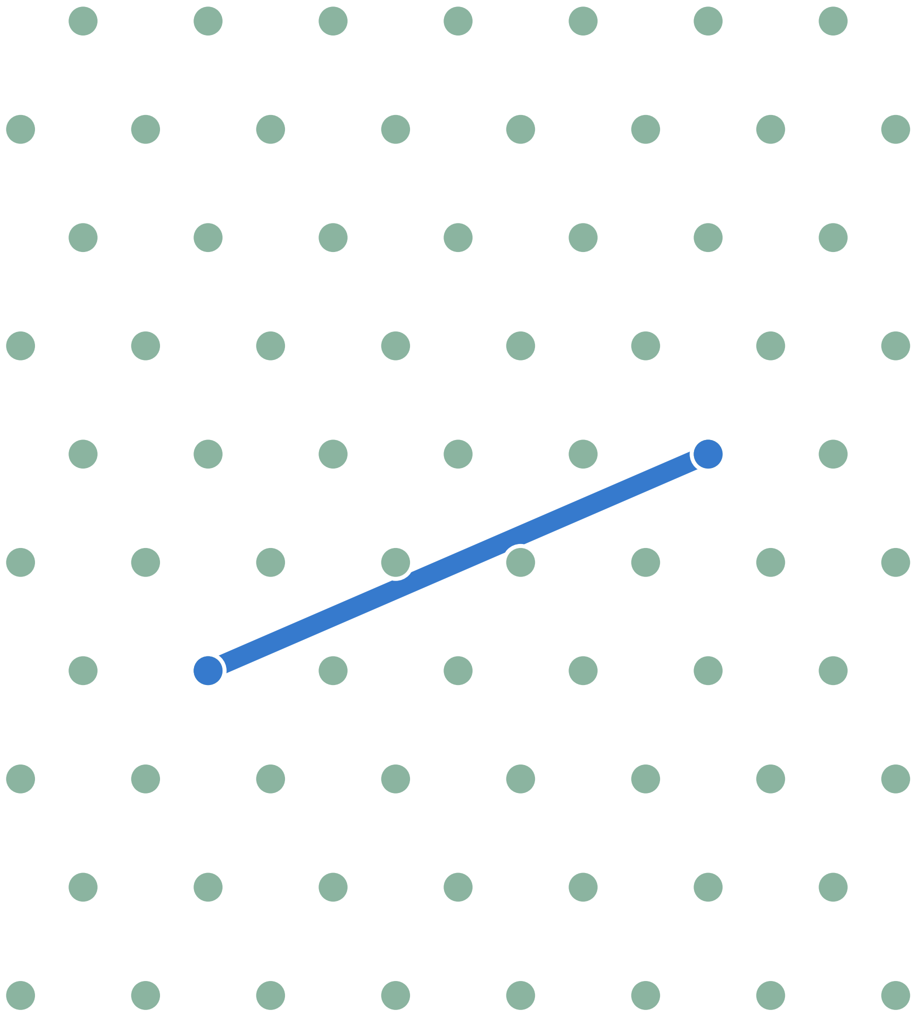

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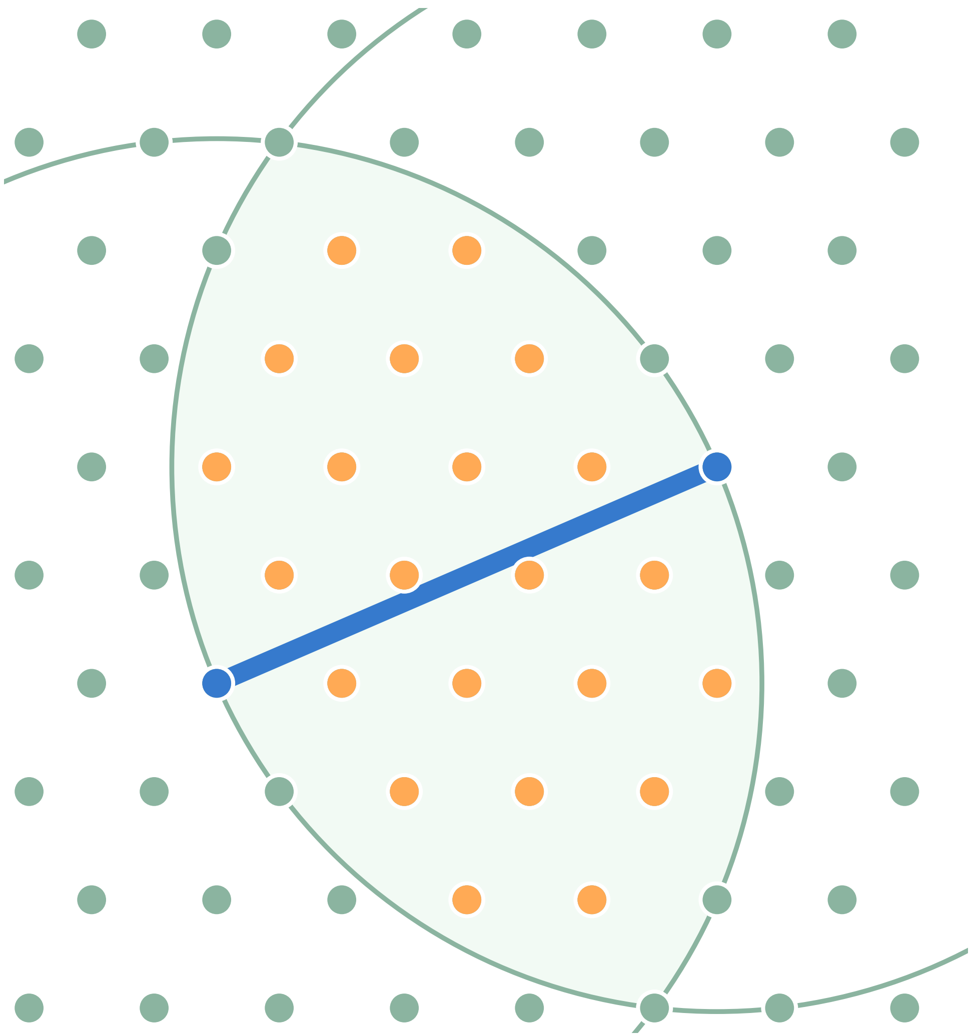

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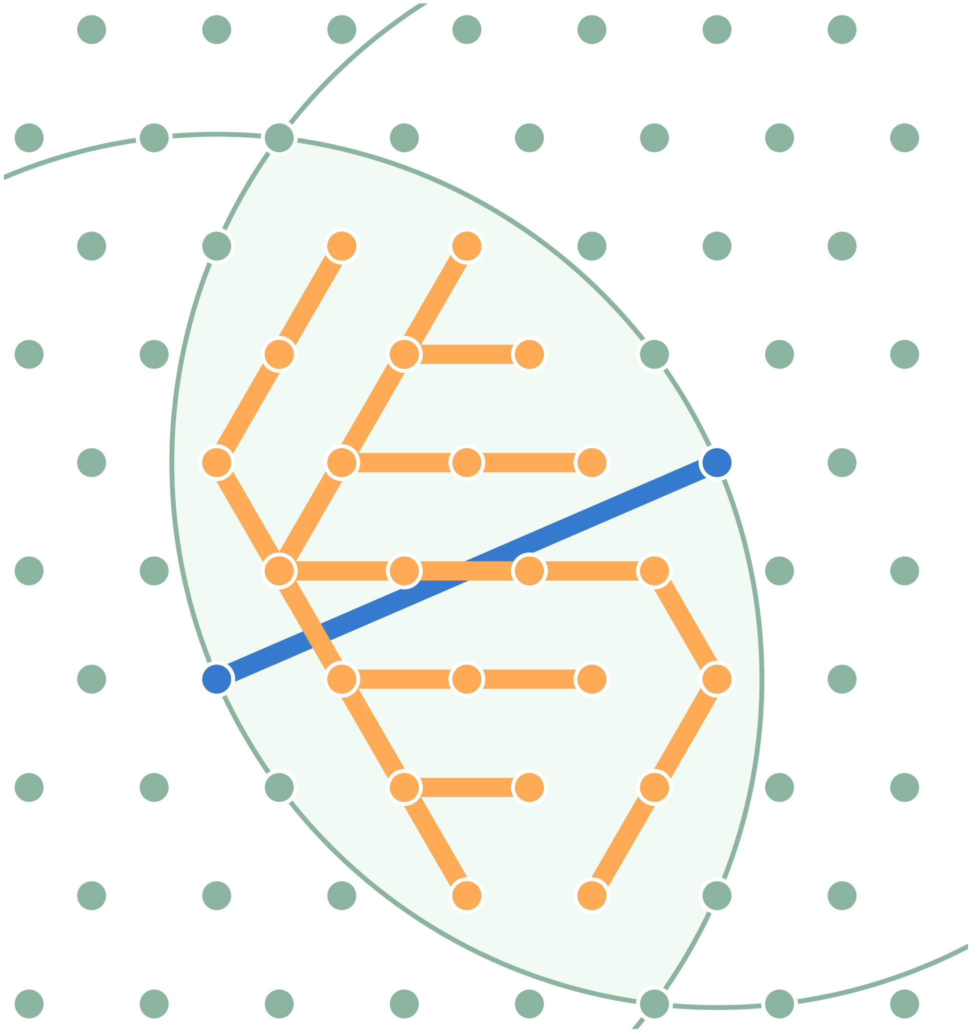

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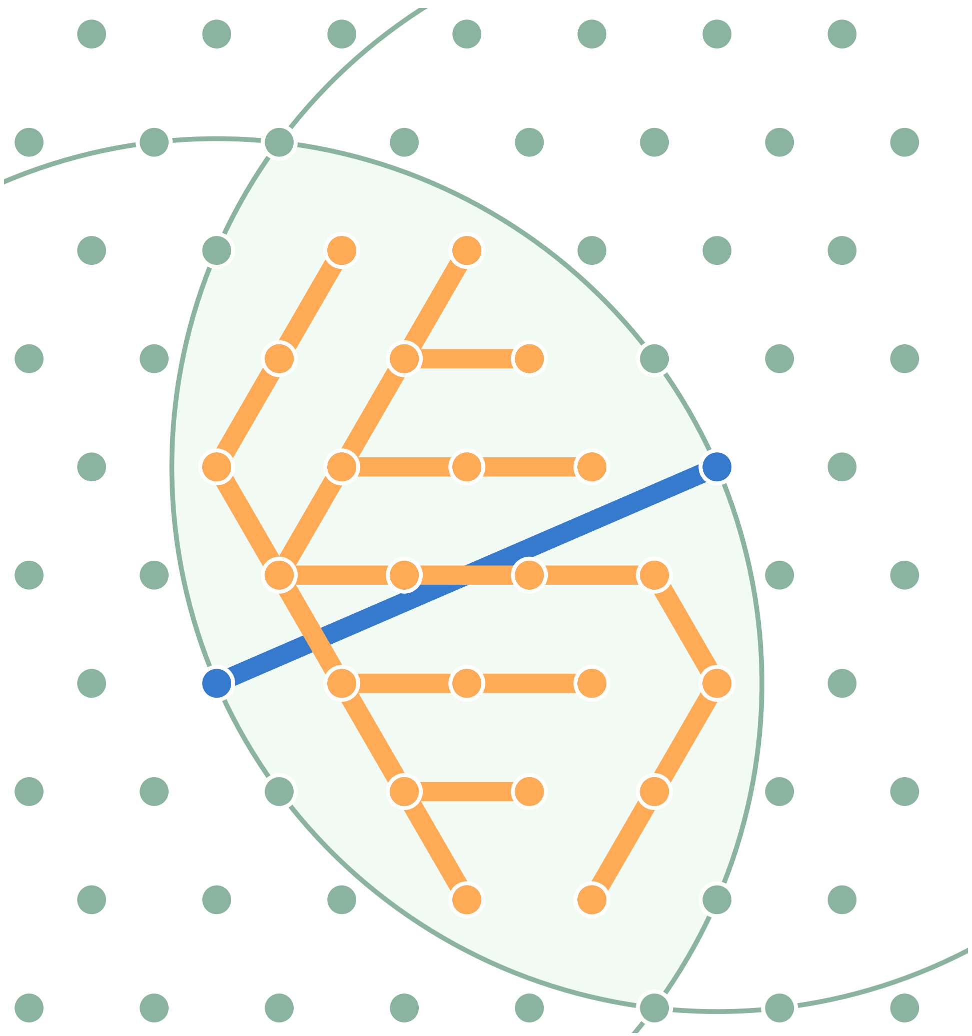


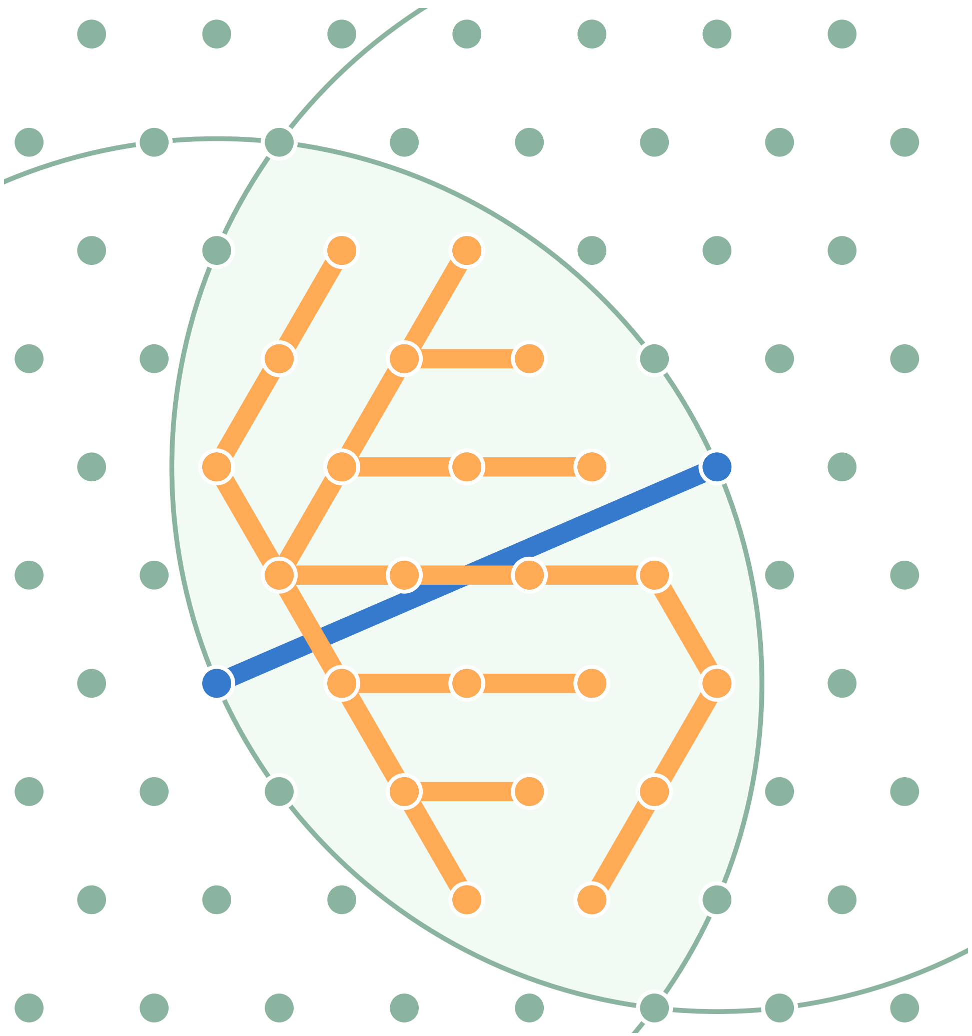
Budget:

- 0.25 for each orange edge of length 1

Cost:

- $|\text{edge}| - 1.25$

Average colored edge length ≤ 1.25



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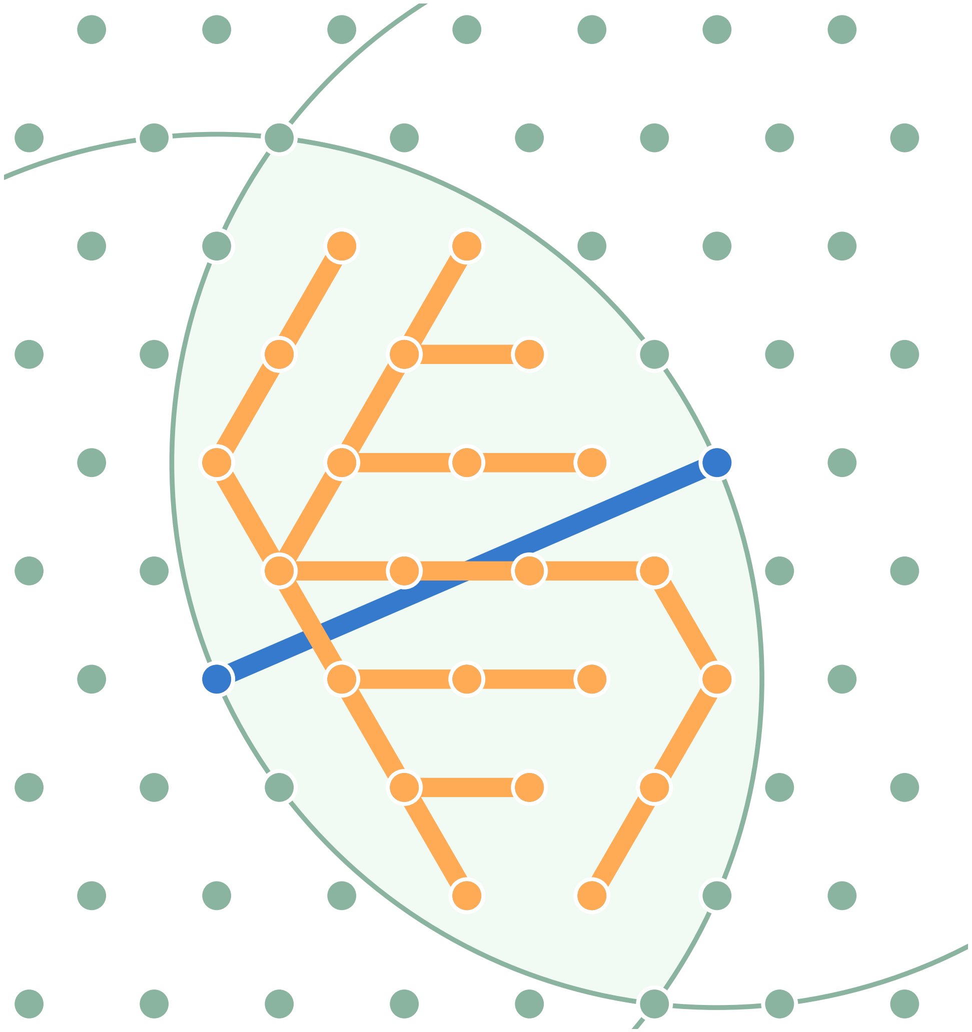
Problem:

- Can we afford long edges with available short edges?

Average colored edge length ≤ 1.25



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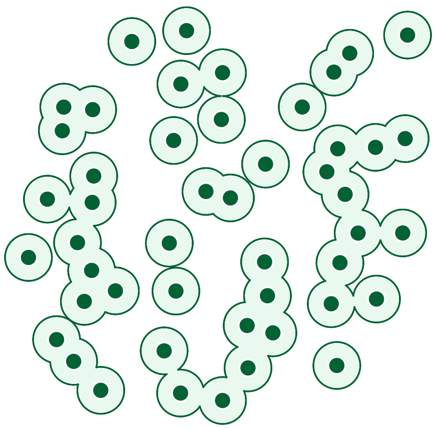
Solution:

- 13 pages of accounting

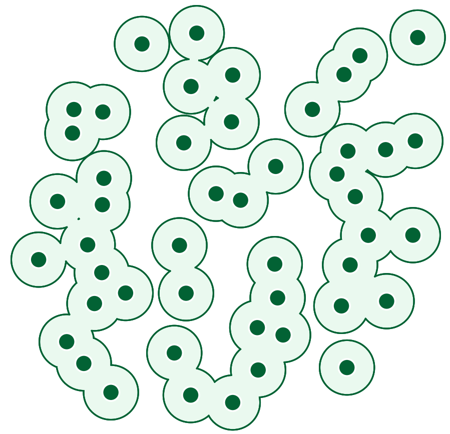
Topological Data Analysis

Sequence of topological spaces

growing
disks



$\subseteq$



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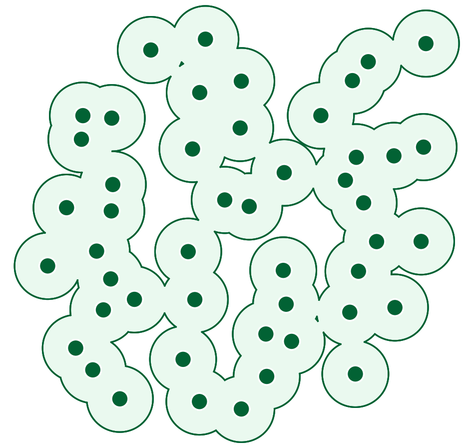
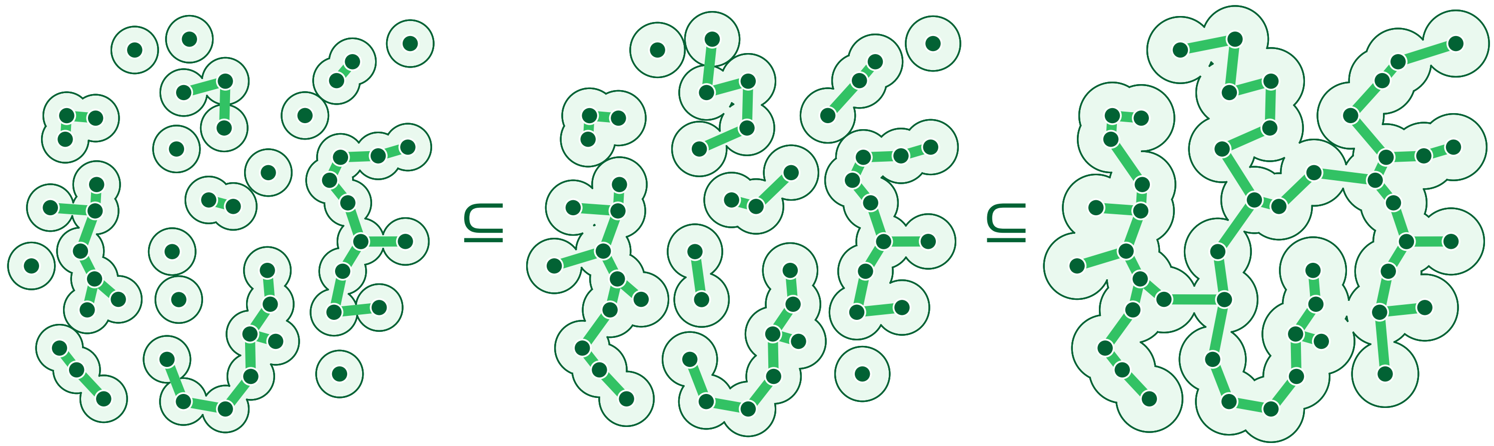


Sequence of topological spaces

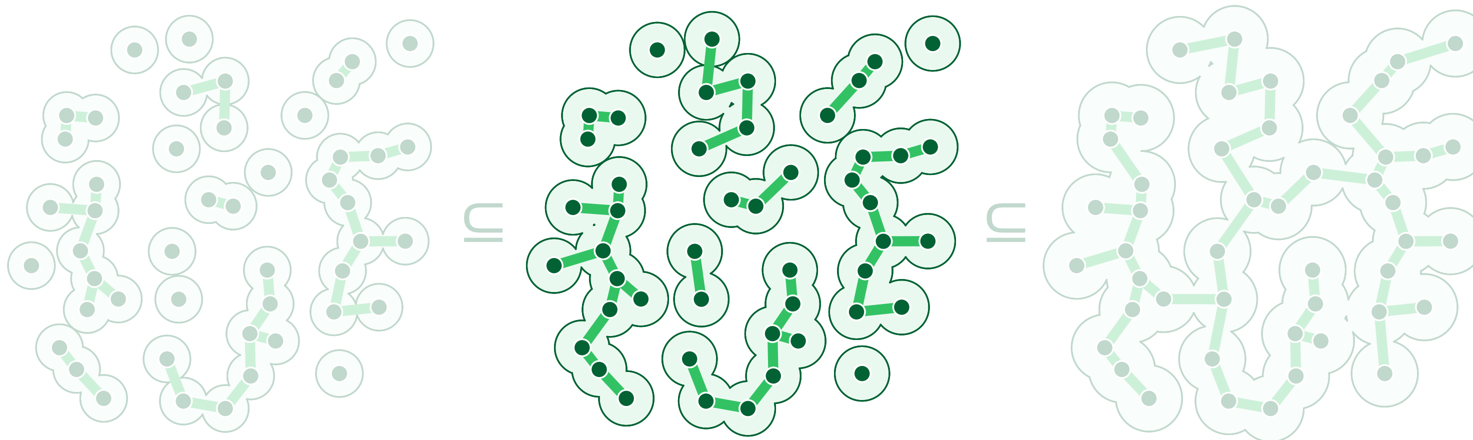
components
of
connectivity

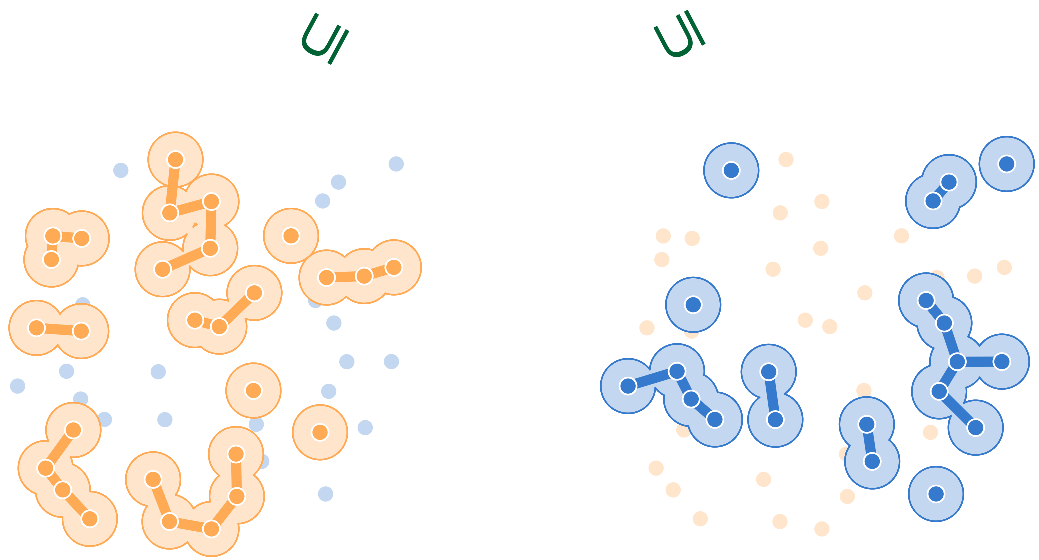


components of connectivity ←→ minimal spanning trees

Sequence of topological spaces

components
of
connectivity

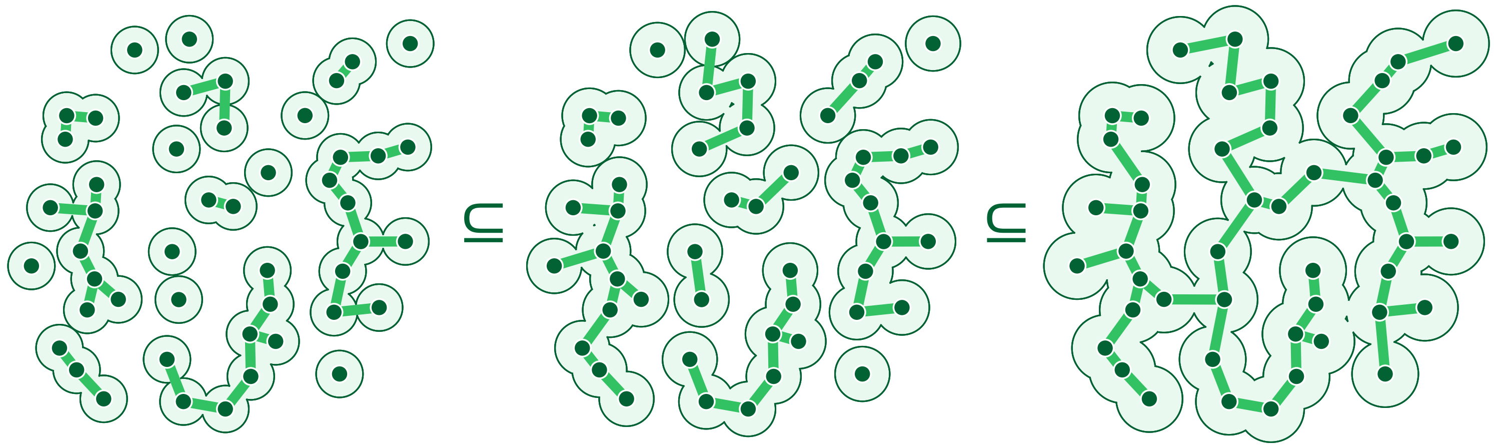




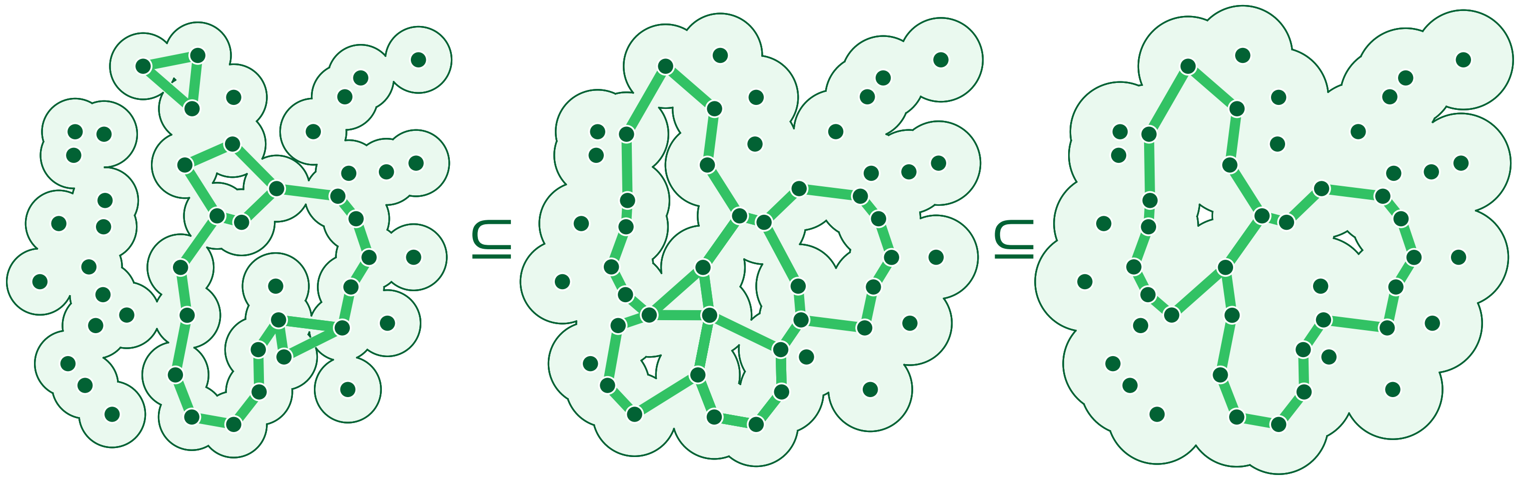
How do colors
spatially interact?

Sequence of topological spaces

components of connectivity



loops



Generalizations, open questions

- more colors, more dimensions
  - finite sets → straightforward generalization of presented results
  - ? lattices
- average MST-ratio over colorings
  - for n random points: $\sqrt{2}$ with high probability as $n \to \infty$
  - ? bounds for arbitrary sets, variance (also for maximum MST-ratio), …
- other summaries from topological data analysis
  - ? loops, …
  - ? nice geometric reformulations
  - → discrete geometry, stochastic geometry

Follow up research:

- Ameli, Motiei, Saghafian: *The Complexity of Maximizing the MST-ratio* [arXiv:2409.11079]
- Dumitrescu, Pach, Tóth: *Two trees are better than one* [arXiv:2312.09916]
- D., Edelsbrunner, Rosenmeier, Saghafian:
  Expected 1-norms of various TDA summaries for two colors [preprint soon to appear]

Thank you for your attention!

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  - for n random points: $\sqrt{2}$ with high probability as $n \to \infty$
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