

# The Euclidean MST-ratio

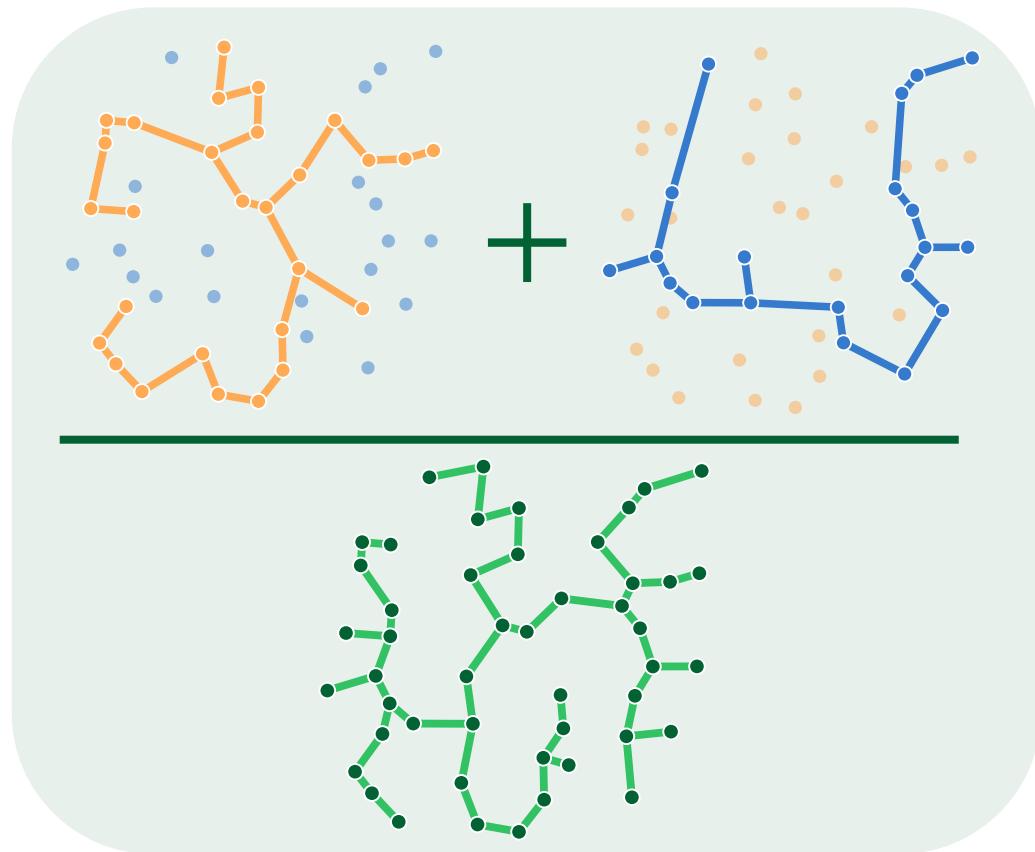
Ondřej Draganov

GD'24, Vienna

September 19, 2024

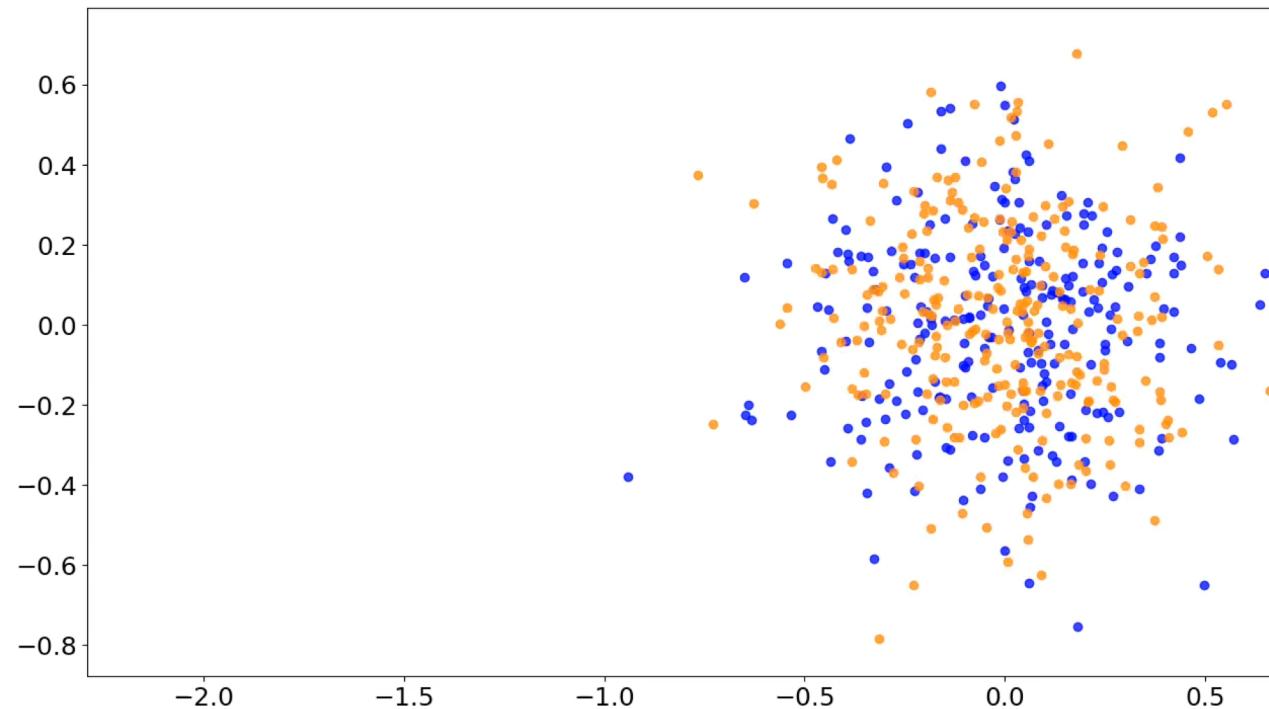
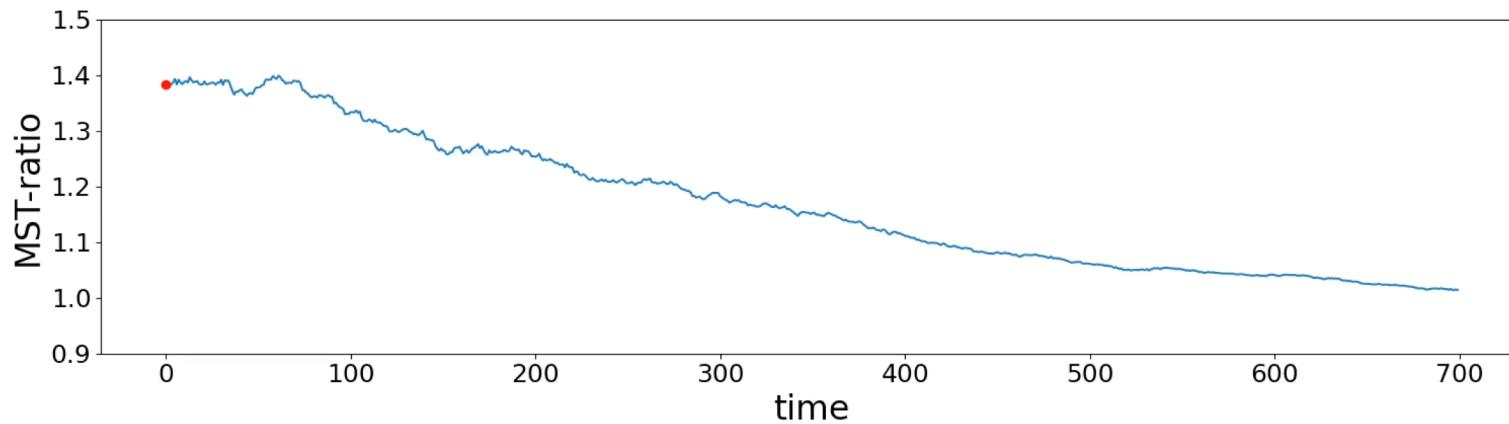


Institute of  
Science and  
Technology  
Austria



Joint work with S. Cultrera di Montesano,  
H. Edelsbrunner, and M. Saghafian

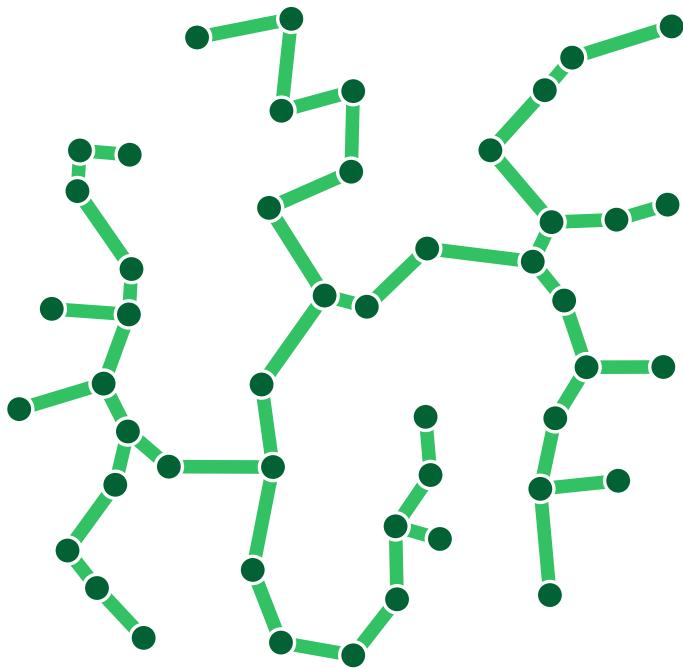
# MST-ratio captures “mingling” of points



# MST-ratio

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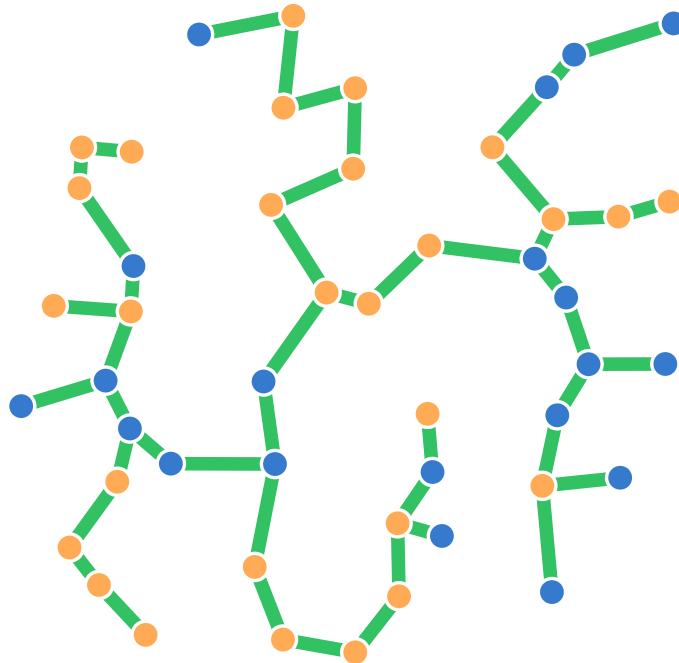
- $A \subset \mathbb{R}^2$  is a finite point set
- $\text{MST}(A)$  is the Euclidean minimum spanning tree



# MST-ratio

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- $A \subset \mathbb{R}^2$  is a finite point set
- $\text{MST}(A)$  is the Euclidean minimum spanning tree
- For  $B \subseteq A$ , we partition  $A = B \sqcup A \setminus B$

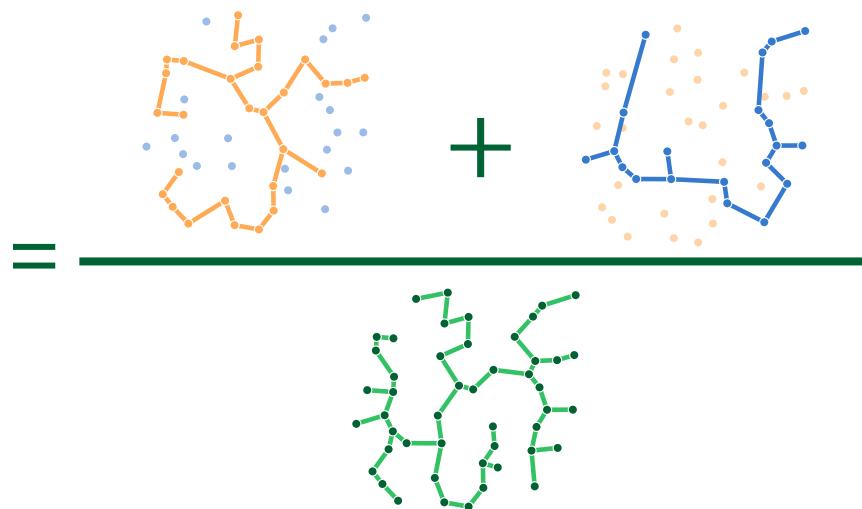


# MST-ratio

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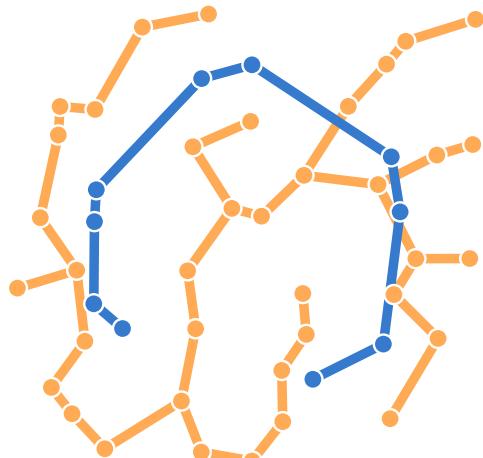
- $A \subset \mathbb{R}^2$  is a finite point set
- $\text{MST}(A)$  is the Euclidean minimum spanning tree
- For  $B \subseteq A$ , the MST-ratio of  $A$  and  $B$  is

$$\mu(A, B) = \frac{|\text{MST}(B)| + |\text{MST}(A \setminus B)|}{|\text{MST}(A)|}$$

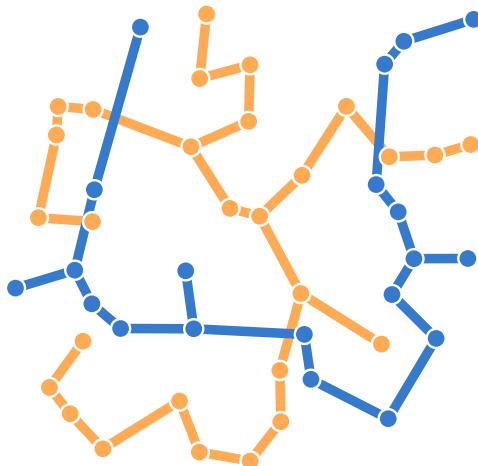


# How small / large can MST-ratio get?

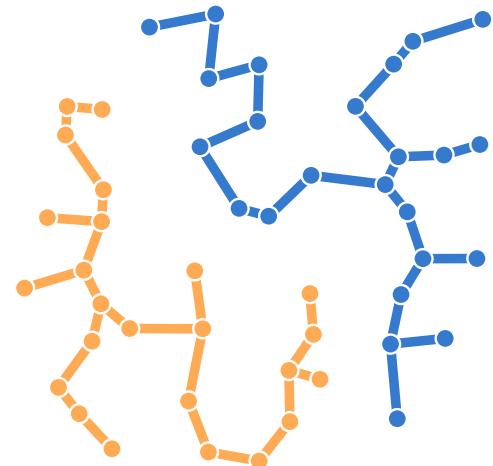
$$?? \leq \mu(A, B) = \frac{|\text{MST}(B)| + |\text{MST}(A \setminus B)|}{|\text{MST}(A)|} \leq ??$$



1.238



1.218



0.969

# How small / large can MST-ratio get?

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$$\text{? ?} \leq \mu(A, B) = \frac{|\text{MST}(B)| + |\text{MST}(A \setminus B)|}{|\text{MST}(A)|} \leq \text{? ?}$$

$$\min_{B \subseteq A} \mu(A, B) \quad \text{"least-mixed" coloring}$$

$$\max_{B \subseteq A} \mu(A, B) \quad \text{"most-mixed" coloring}$$

# How small / large can MST-ratio get?

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$$\text{? ?} \leq \mu(A, B) = \frac{|\text{MST}(B)| + |\text{MST}(A \setminus B)|}{|\text{MST}(A)|} \leq \text{? ?}$$

$$\inf_A \min_{B \subseteq A} \mu(A, B) \leq \min_{B \subseteq A} \mu(A, B) \leq \sup_A \min_{B \subseteq A} \mu(A, B)$$

$$\inf_A \max_{B \subseteq A} \mu(A, B) \leq \max_{B \subseteq A} \mu(A, B) \leq \sup_A \max_{B \subseteq A} \mu(A, B)$$

# How small / large can MST-ratio get?

$$\text{?} \leq \mu(A, B) = \frac{|\text{MST}(B)| + |\text{MST}(A \setminus B)|}{|\text{MST}(A)|} \leq \text{?}$$

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$$\inf_A \min_{B \subseteq A} \mu(A, B) = ?$$

$$\sup_A \min_{B \subseteq A} \mu(A, B) = ?$$

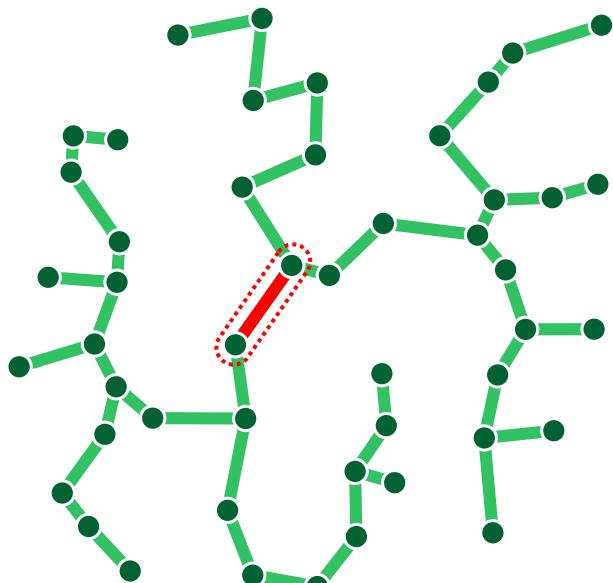
$$\inf_A \max_{B \subseteq A} \mu(A, B) = ?$$

$$\sup_A \max_{B \subseteq A} \mu(A, B) = ?$$

# The minimum MST-ratio for a given $A$

$$\min_{B \subseteq A} \mu(A, B) = \frac{|\text{MST}(A)| - |\omega|}{|\text{MST}(A)|}$$

where  $\omega$  is the longest edge in  $|\text{MST}(A)|$



$$\inf_A \min_{B \subseteq A} \mu(A, B)$$

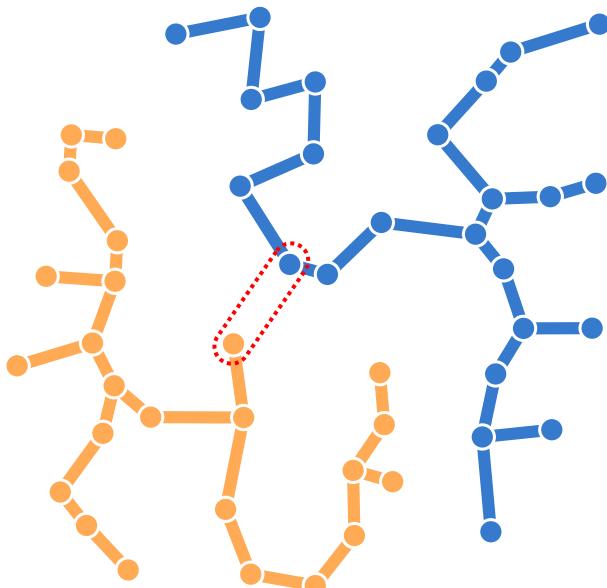
$$\sup_A \min_{B \subseteq A} \mu(A, B)$$

$$\inf_A \max_{B \subseteq A} \mu(A, B)$$

$$\sup_A \max_{B \subseteq A} \mu(A, B)$$

# The minimum MST-ratio for a given $A$

- $\min_{B \subseteq A} \mu(A, B) = \frac{|\text{MST}(A)| - |\omega|}{|\text{MST}(A)|} = \frac{|\text{MST}(B_\omega)| + |\text{MST}(A \setminus B_\omega)|}{|\text{MST}(A)|}$   
where  $\omega$  is the longest edge in  $|\text{MST}(A)|$
- we can always achieve this



$$\inf_A \min_{B \subseteq A} \mu(A, B)$$

$$\sup_A \min_{B \subseteq A} \mu(A, B)$$

$$\inf_A \max_{B \subseteq A} \mu(A, B)$$

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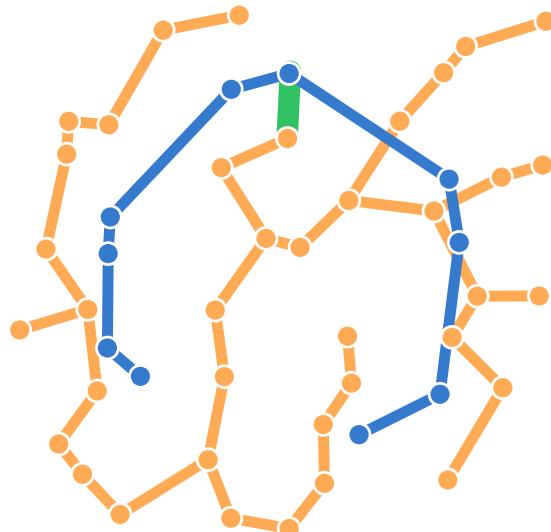
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where  $\omega$  is the longest edge in  $|\text{MST}(A)|$

- we can always achieve this
- it is the minimum:

$$|\text{MST}(A)| \leq |\text{MST}(B)| + |\text{MST}(A \setminus B)| + |e|$$



$$\inf_A \min_{B \subseteq A} \mu(A, B)$$

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$$\inf_A \max_{B \subseteq A} \mu(A, B)$$

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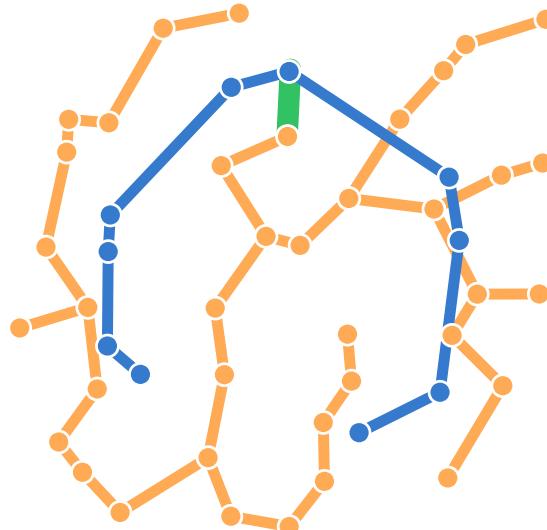
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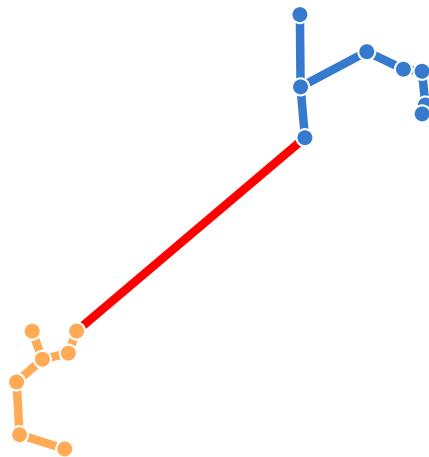
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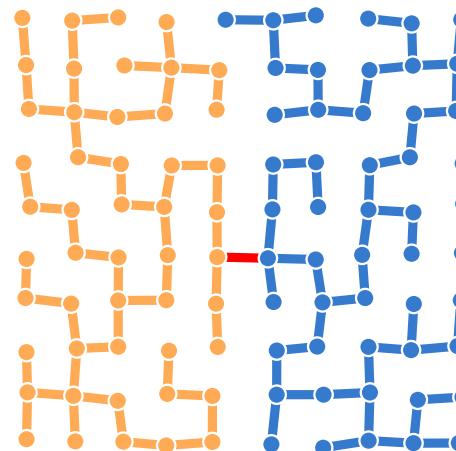
where  $\omega$  is the longest edge in  $|\text{MST}(A)|$

- we can always achieve this
- it is the minimum:

$$|\text{MST}(A)| - |\omega| \leq |\text{MST}(A)| - |e| \leq |\text{MST}(B)| + |\text{MST}(A \setminus B)|$$



“only  $|\omega|$  matters”



“ $|\omega|$  negligible”

$$\inf_A \min_{B \subseteq A} \mu(A, B) = 0$$

$$\sup_A \min_{B \subseteq A} \mu(A, B) = 1$$

$$\inf_A \max_{B \subseteq A} \mu(A, B)$$

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# The maximum MST-ratio for a given $A$

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- computing  $\max_{B \subseteq A} \mu(A, B)$  is difficult

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A. J. Ameli, F. Motiei, M. Saghafian:

*The Complexity of Maximizing the MST-ratio*

[arXiv:2409.11079]

A light green speech bubble with a triangular point on the left side. Inside the bubble, the text "more general formulation is NP-hard" is written in a dark green font.

$$\inf_A \min_{B \subseteq A} \mu(A, B) = 0$$

$$\sup_A \min_{B \subseteq A} \mu(A, B) = 1$$

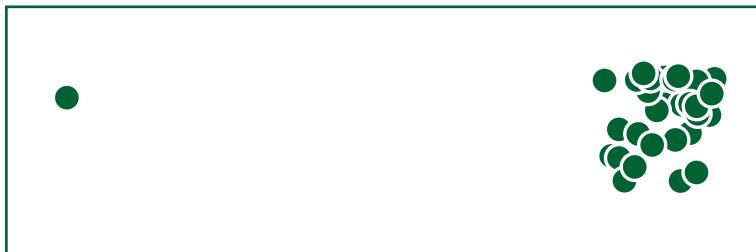
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- the supremum of  $\mu(A, B)$  closely relates to Steiner ratio

$$\inf_A \min_{B \subseteq A} \mu(A, B) = 0$$

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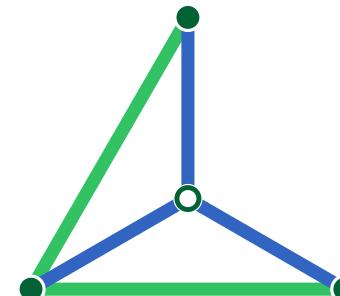
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- computing  $\max_{B \subseteq A} \mu(A, B)$  is difficult
- the supremum of  $\mu(A, B)$  closely relates to Steiner ratio

- Steiner tree of  $A$ 
  - $\text{MST}(A \cup S)$  for some extra points  $S$
- Minimum Steiner tree  $\text{StT}(A)$ 
  - Steiner tree minimizing the length
- Steiner ratio
  - $\alpha := \sup_A \frac{\text{MST}(A)}{\text{StT}(A)}$



$$\inf_A \min_{B \subseteq A} \mu(A, B) = 0$$

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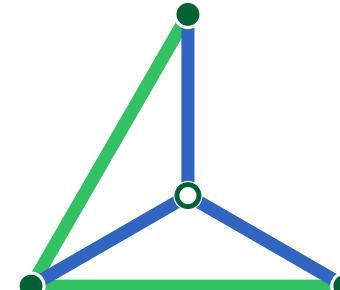
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$$\bullet \alpha := \sup_A \frac{\text{MST}(A)}{\text{StT}(A)}$$

$$= \frac{2}{\sqrt{3}} \approx 1.1547$$



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- $\alpha := \sup_A \frac{\text{MST}(A)}{\text{StT}(A)} \stackrel{[1]}{\leq} 1.213 \dots$

The diagram shows two sets of points. On the left, three points are arranged horizontally, with a Steiner tree consisting of a green vertical segment connecting the top point to a blue horizontal segment that splits into two lines to the bottom points. An angle symbol indicates the internal angle at the bottom center point. On the right, four points are arranged in a diamond shape, with a Steiner tree consisting of a green vertical segment from the top point to a blue triangle connecting all four points.

$$= \frac{2}{\sqrt{3}} \approx 1.1547$$

[1] F. Chung, R. Graham, "A new bound for Euclidean Steiner minimal trees," (1985)

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$$\alpha \leq 1.213 \dots$$

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$$\frac{|\text{MST}(B)| + |\text{MST}(A \setminus B)|}{|\text{MST}(A)|} \leq \alpha + \alpha \leq 2.426 \dots$$

$$\alpha := \sup_A \frac{\text{MST}(A)}{\text{StT}(A)}$$
$$\alpha \leq 1.213 \dots$$

$$\inf_A \min_{B \subseteq A} \mu(A, B) = 0$$

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$$\mu \left( \begin{array}{c} \text{Diagram of a triangle with one internal edge (blue) and two Steiner points (orange).} \\ \text{The blue edge connects two vertices, while the orange edges connect the blue edge to the third vertex and to each other.} \end{array} \right) = \frac{\sqrt{3} + 2}{\sqrt{3} + 3\epsilon} \rightarrow 2.154 \dots$$

$$\inf_A \min_{B \subseteq A} \mu(A, B) = 0$$

$$\sup_A \min_{B \subseteq A} \mu(A, B) = 1$$

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$$\frac{|\text{MST}(B)| + |\text{MST}(A \setminus B)|}{|\text{MST}(A)|} \leq \alpha + \alpha \leq 2.426 \dots$$

$$\mu \left( \begin{array}{c} \text{Diagram of a triangle with vertices and edges colored orange and blue.} \\ \text{An orange edge connects the top vertex to the bottom-left vertex.} \\ \text{An orange edge connects the top vertex to the bottom-right vertex.} \\ \text{A blue edge connects the bottom-left vertex to the bottom-right vertex.} \end{array} \right) = \frac{\sqrt{3} + 2}{\sqrt{3} + 3\epsilon} \rightarrow 2.154 \dots$$

$$2.154 \dots \leq \sup_A \max_{B \subseteq A} \mu(A, B) \leq 2.426 \dots$$

$$\alpha := \sup_A \frac{\text{MST}(A)}{\text{StT}(A)}$$
$$\alpha \leq 1.213 \dots$$

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$$\frac{|\text{MST}(B)| + |\text{MST}(A \setminus B)|}{|\text{MST}(A)|} \leq \alpha + \alpha \leq 2.426 \dots$$

$$\mu \left( \begin{array}{c} \text{Diagram of a triangle with vertices and edges highlighted in orange and blue.} \\ \end{array} \right) = \frac{1}{\sqrt{3} + 2} \leq \alpha \rightarrow 2.154 \dots$$

$$1 + \alpha \leq \sup_A \max_{B \subseteq A} \mu(A, B) \leq 2\alpha$$

$$\alpha := \sup_A \frac{\text{MST}(A)}{\text{StT}(A)}$$
$$\alpha \leq 1.213 \dots$$

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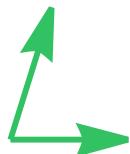
# Lattices

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# The supremum MST-ratio for lattices

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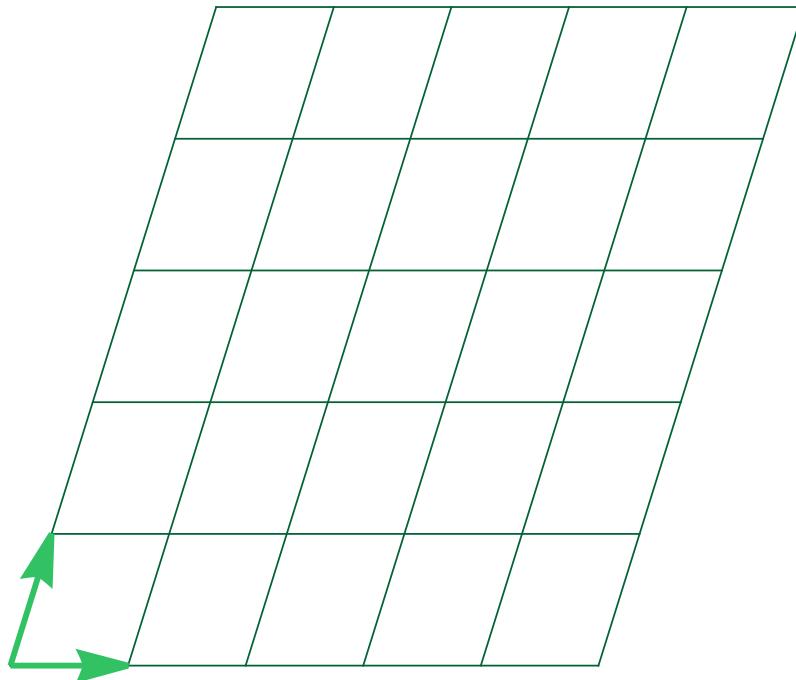
- a **lattice** is  $\Lambda(\mathbf{u}, \mathbf{v}) = \{i\mathbf{u} + j\mathbf{v} \mid i, j \in \mathbb{Z}\}$



# The supremum MST-ratio for lattices

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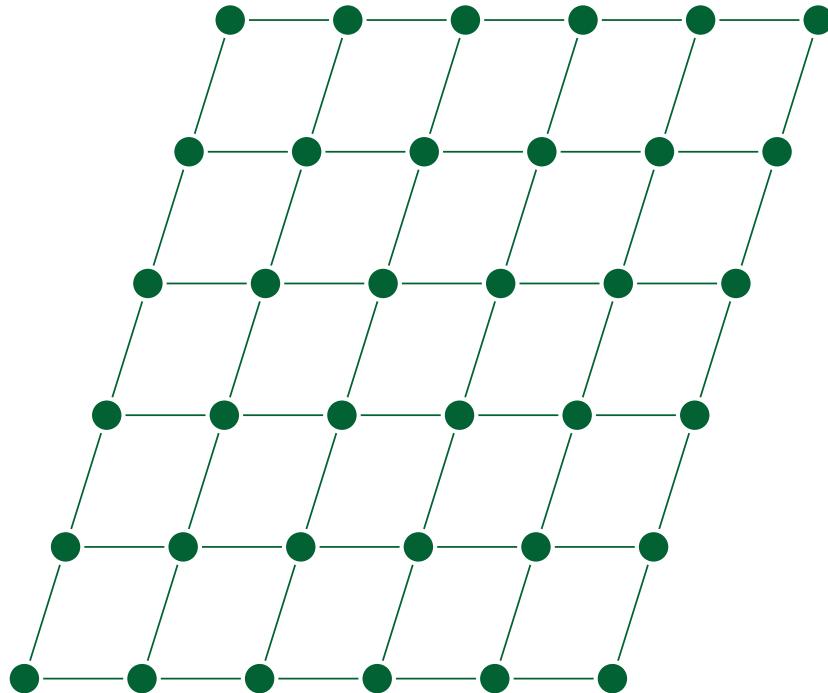
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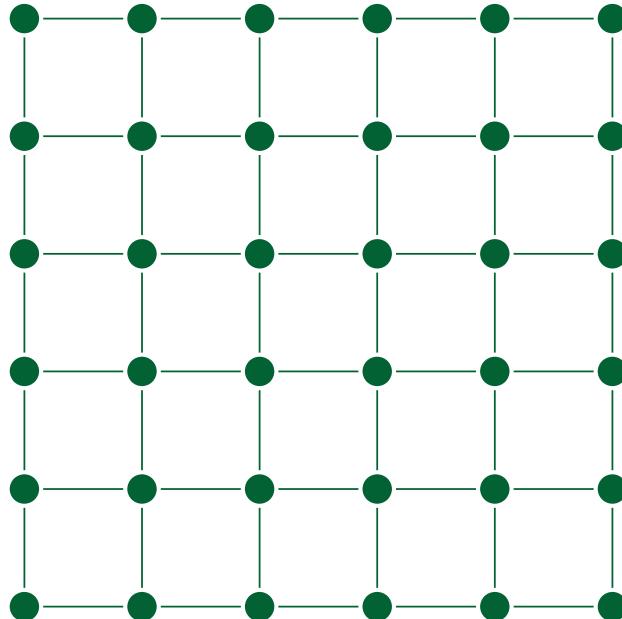
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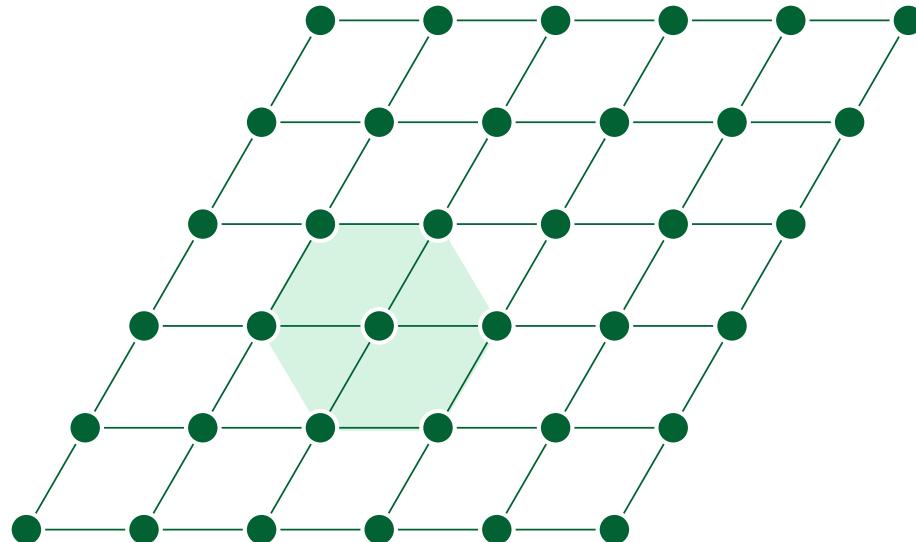
- a **lattice** is  $\Lambda(\mathbf{u}, \mathbf{v}) = \{i\mathbf{u} + j\mathbf{v} \mid i, j \in \mathbb{Z}\}$
- for example: square



# The supremum MST-ratio for lattices

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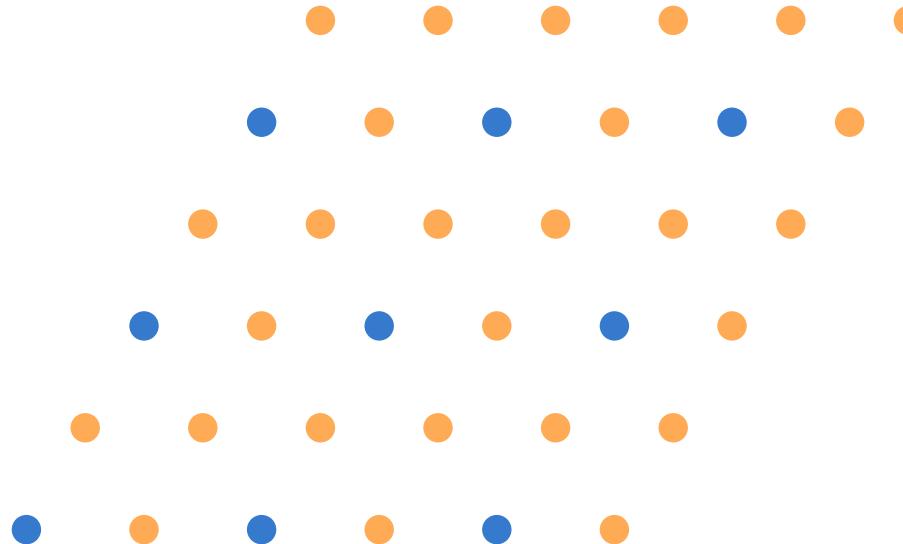
- a **lattice** is  $\Lambda(\mathbf{u}, \mathbf{v}) = \{i\mathbf{u} + j\mathbf{v} \mid i, j \in \mathbb{Z}\}$
- for example: square, hexagonal



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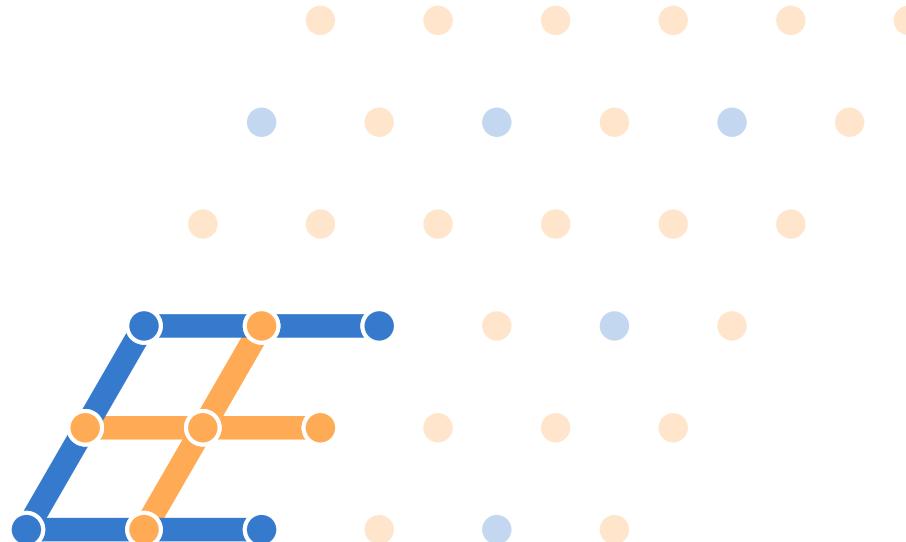
- a **lattice** is  $\Lambda(\mathbf{u}, \mathbf{v}) = \{i\mathbf{u} + j\mathbf{v} \mid i, j \in \mathbb{Z}\}$
- for example: square, hexagonal
- MST-ratio is the limit of finite rhombi:  $\mu(\Lambda, B) = \lim_{n \rightarrow \infty} \mu(\Lambda_n, B_n)$



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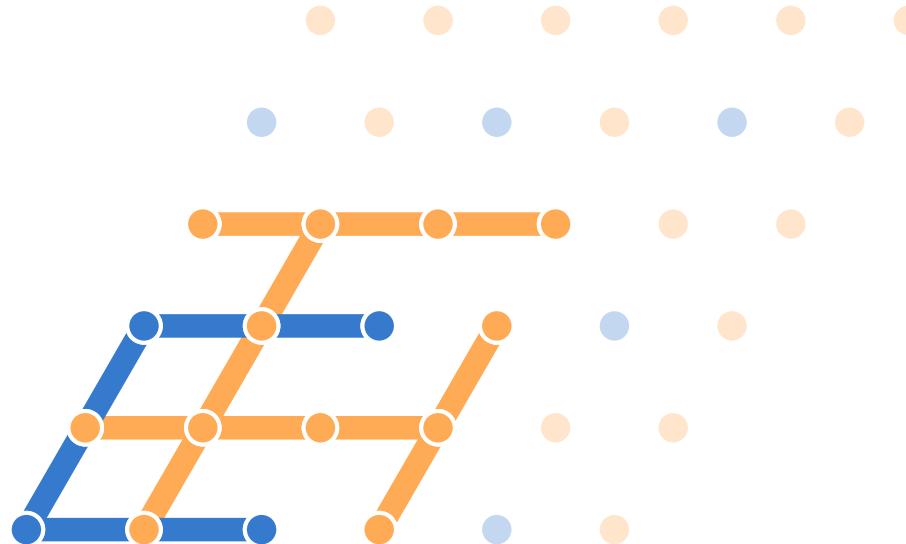
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- MST-ratio is the limit of finite rhombi:  $\mu(\Lambda, B) = \lim_{n \rightarrow \infty} \mu(\Lambda_n, B_n)$
- $\mu(\Lambda_n, B_n) = 1.25$



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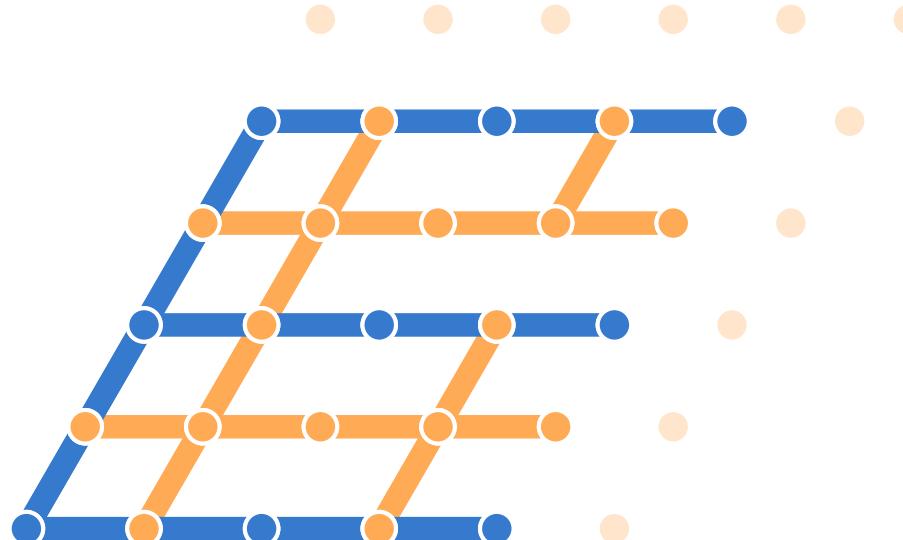
- a **lattice** is  $\Lambda(\mathbf{u}, \mathbf{v}) = \{i\mathbf{u} + j\mathbf{v} \mid i, j \in \mathbb{Z}\}$
- for example: square, hexagonal
- MST-ratio is the limit of finite rhombi:  $\mu(\Lambda, B) = \lim_{n \rightarrow \infty} \mu(\Lambda_n, B_n)$
- $\mu(\Lambda_n, B_n) = 1.25, 1.13$



# The supremum MST-ratio for lattices

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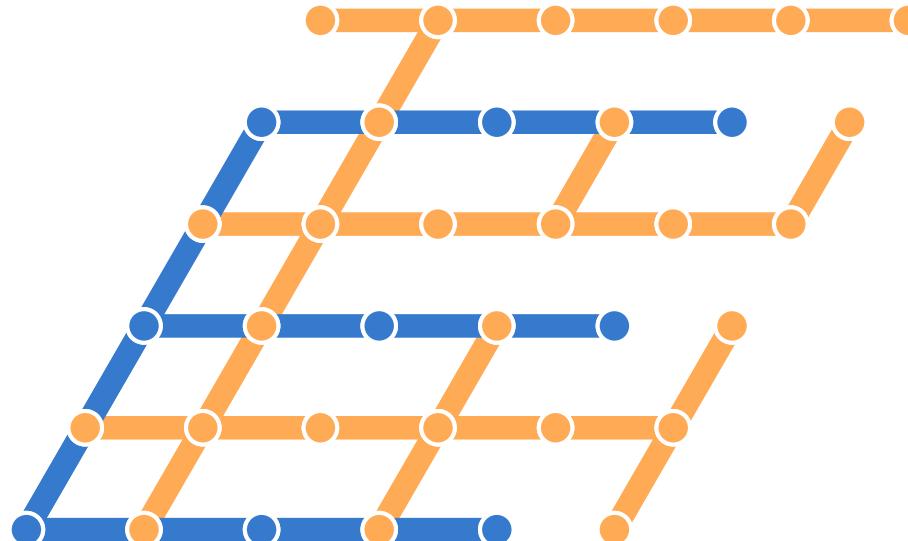
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# The supremum MST-ratio for lattices

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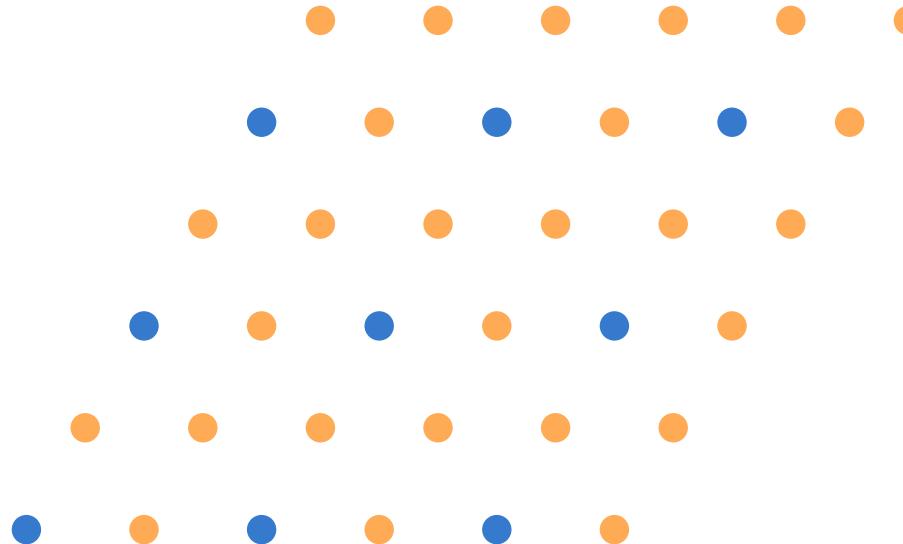
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# The supremum MST-ratio for lattices

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- MST-ratio is the limit of finite rhombi:  $\mu(\Lambda, B) = \lim_{n \rightarrow \infty} \mu(\Lambda_n, B_n)$
- $\mu(\Lambda_n, B_n) = 1.25, 1.13, 1.29, 1.2, \dots \xrightarrow[n \rightarrow \infty]{} 1.25$



# The supremum MST-ratio for lattices

---

## Theorem

$$1.25 \leq \inf_{\Lambda} \sup_{B \subseteq \Lambda} \mu(\Lambda, B) \leq 1.25$$

$$2 \leq \sup_{\Lambda} \sup_{B \subseteq \Lambda} \mu(\Lambda, B) \leq 2$$

# The supremum MST-ratio for lattices

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## Theorem

$$1.25 \leq \inf_{\Lambda} \sup_{B \subseteq \Lambda} \mu(\Lambda, B) \stackrel{!}{\leq} 1.25$$

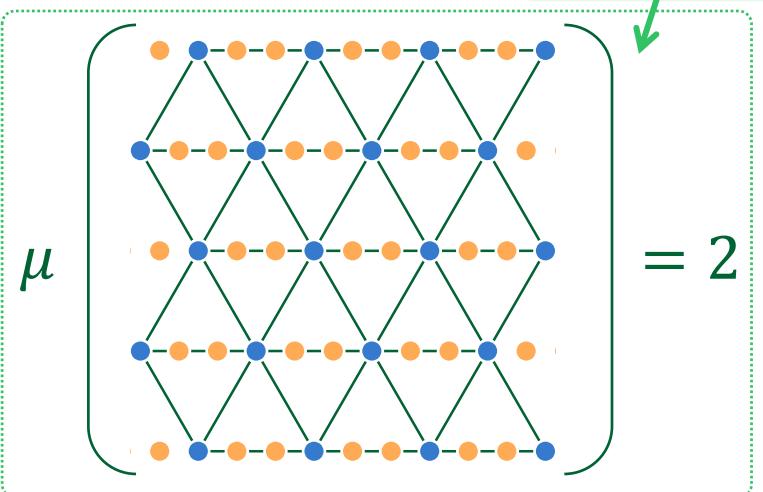
$$2 \leq \sup_{\Lambda} \sup_{B \subseteq \Lambda} \mu(\Lambda, B) \leq 2$$

# The supremum MST-ratio for lattices

## Theorem

$$1.25 \leq \inf_{\Lambda} \sup_{B \subseteq \Lambda} \mu(\Lambda, B) \leq 1.25$$

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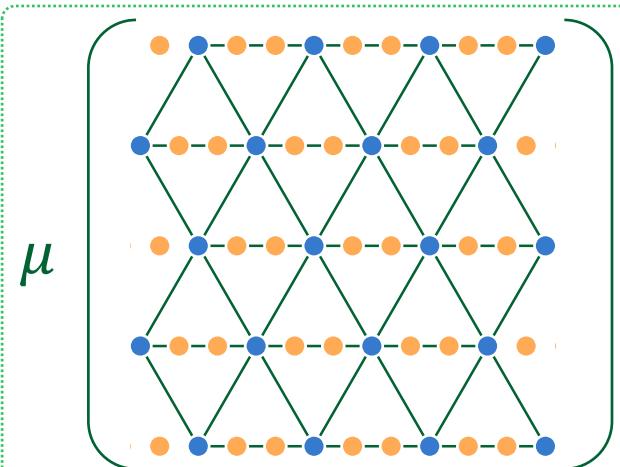


# The supremum MST-ratio for lattices

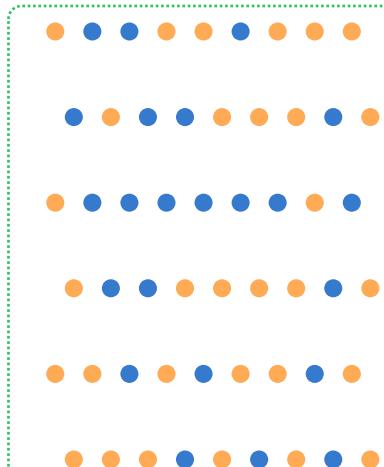
## Theorem

$$1.25 \leq \inf_{\Lambda} \sup_{B \subseteq \Lambda} \mu(\Lambda, B) \leq 1.25$$

$$2 \leq \sup_{\Lambda} \sup_{B \subseteq \Lambda} \mu(\Lambda, B) \leq 2$$



$$= 2$$



$$\frac{|\text{MST}(\Lambda_n)| + |\text{MST}(\Lambda_n \setminus B_n)|}{|\text{MST}(A)|}$$

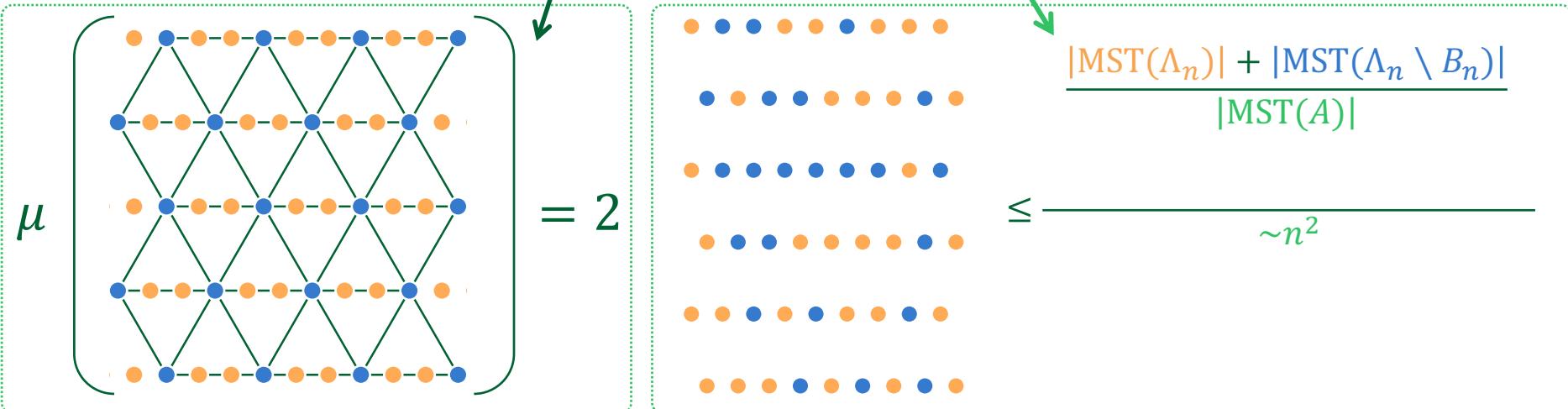
$$\leq \text{_____}$$

# The supremum MST-ratio for lattices

## Theorem

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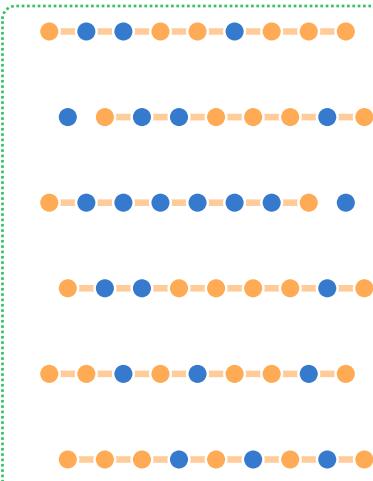
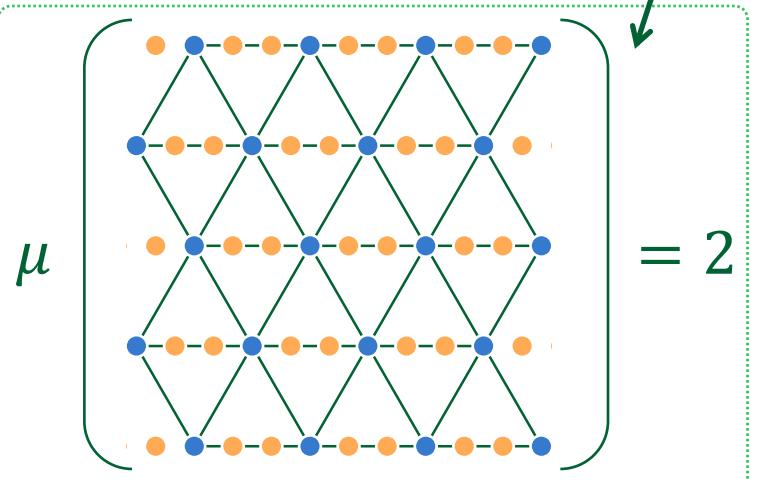


# The supremum MST-ratio for lattices

## Theorem

$$1.25 \leq \inf_{\Lambda} \sup_{B \subseteq \Lambda} \mu(\Lambda, B) \leq 1.25$$

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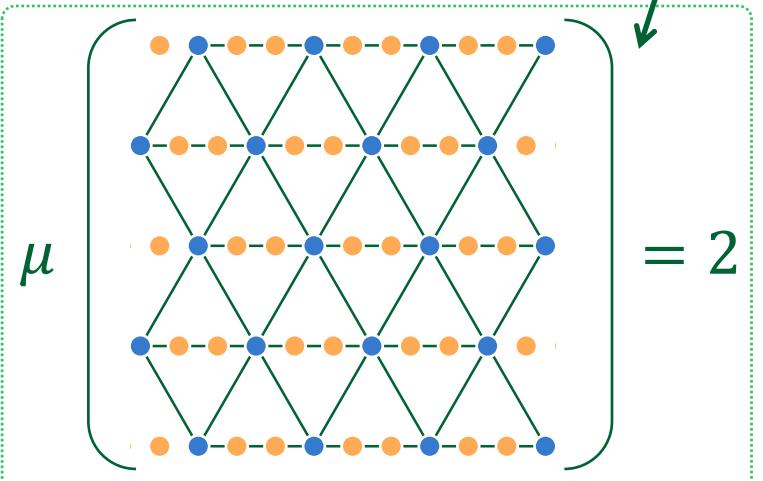


# The supremum MST-ratio for lattices

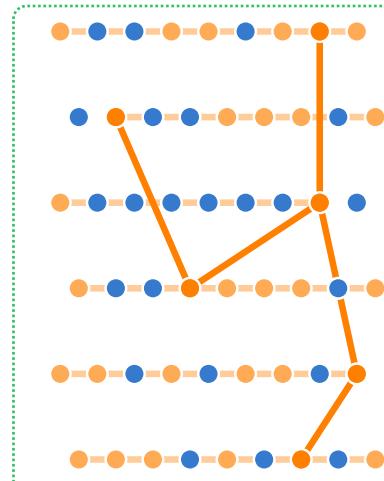
## Theorem

$$1.25 \leq \inf_{\Lambda} \sup_{B \subseteq \Lambda} \mu(\Lambda, B) \leq 1.25$$

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$$= 2$$



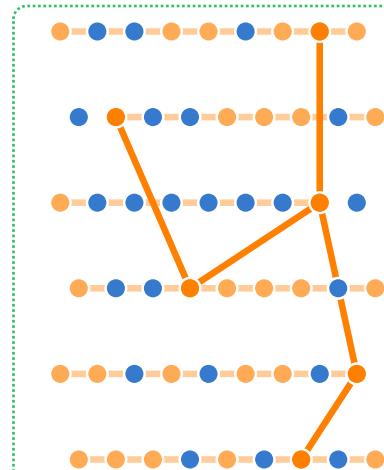
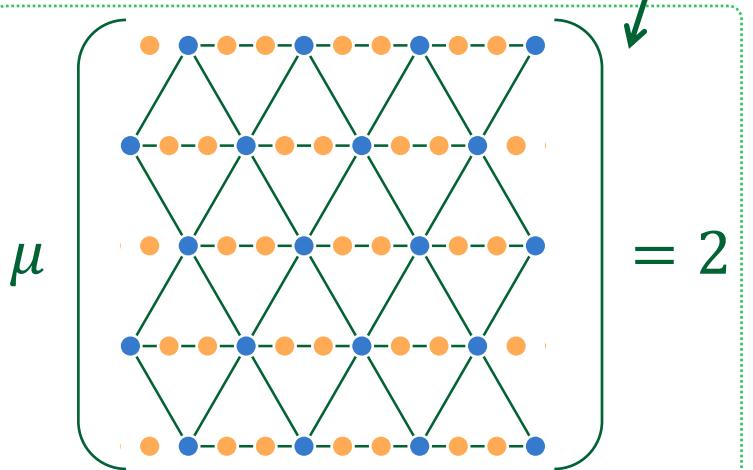
$$\frac{|\text{MST}(\Lambda_n)| + |\text{MST}(\Lambda_n \setminus B_n)|}{|\text{MST}(A)|} \leq \frac{\sim n^2 + O(n\sqrt{n})}{\sim n^2}$$

# The supremum MST-ratio for lattices

## Theorem

$$1.25 \leq \inf_{\Lambda} \sup_{B \subseteq \Lambda} \mu(\Lambda, B) \leq 1.25$$

$$2 \leq \sup_{\Lambda} \sup_{B \subseteq \Lambda} \mu(\Lambda, B) \leq 2$$

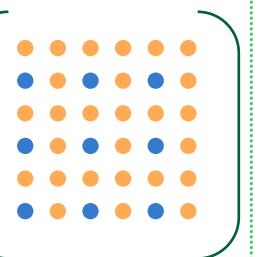
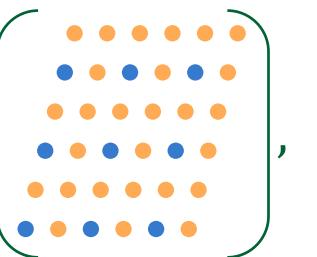
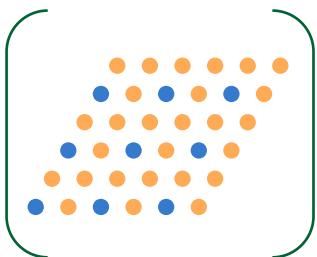


$$\frac{|\text{MST}(\Lambda_n)| + |\text{MST}(\Lambda_n \setminus B_n)|}{|\text{MST}(A)|} \leq \frac{\sim n^2 + O(n\sqrt{n}) + \sim n^2 + O(n\sqrt{n})}{\sim n^2}$$

$n \rightarrow \infty \rightarrow 2$

# The supremum MST-ratio for lattices

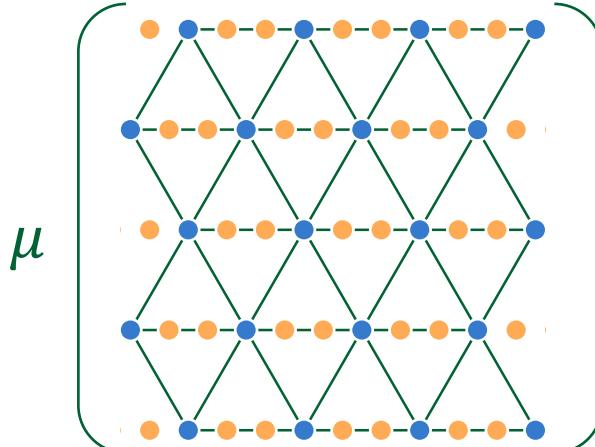
$$1.25 \leq \mu$$



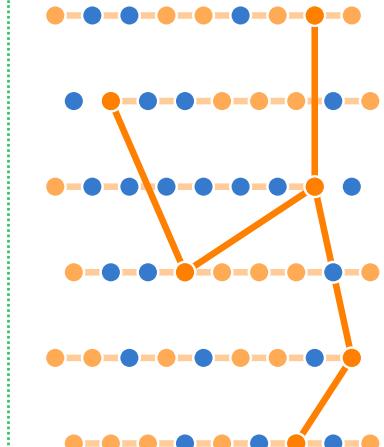
Theorem

$$1.25 \leq \inf_{\Lambda} \sup_{B \subseteq \Lambda} \mu(\Lambda, B) \leq 1.25$$

$$2 \leq \sup_{\Lambda} \sup_{B \subseteq \Lambda} \mu(\Lambda, B) \leq 2$$



$$= 2$$

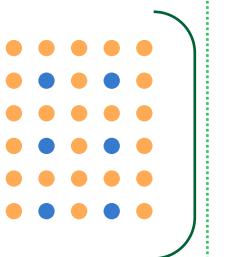
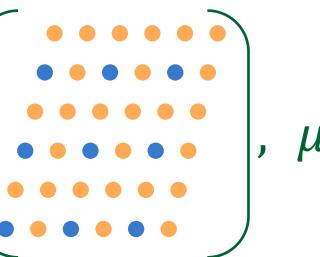
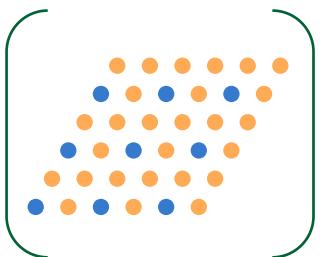


$$\frac{|\text{MST}(\Lambda_n)| + |\text{MST}(\Lambda_n \setminus B_n)|}{|\text{MST}(A)|} \leq \frac{\sim n^2 + O(n\sqrt{n}) + \sim n^2 + O(n\sqrt{n})}{\sim n^2}$$

$$\xrightarrow{n \rightarrow \infty} 2$$

# The supremum MST-ratio for lattices

$$1.25 \leq \mu$$

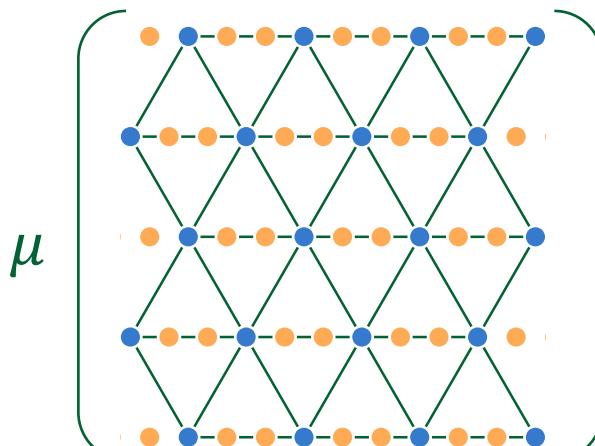


Theorem

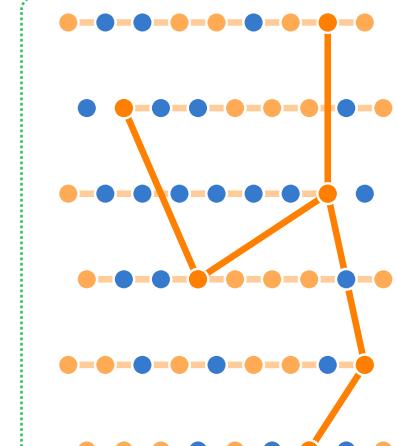
$$1.25 \leq \inf_{\Lambda} \sup_{B \subseteq \Lambda} \mu(\Lambda, B) \leq 1.25$$

Analyze the hexagonal lattice

$$2 \leq \sup_{\Lambda} \sup_{B \subseteq \Lambda} \mu(\Lambda, B) \leq 2$$



$$= 2$$



$$\frac{|\text{MST}(\Lambda_n)| + |\text{MST}(\Lambda_n \setminus B_n)|}{|\text{MST}(A)|} \leq \frac{\sim n^2 + O(n\sqrt{n}) + \sim n^2 + O(n\sqrt{n})}{\sim n^2}$$

$$\xrightarrow{n \rightarrow \infty} 2$$

# The supremum MST-ratio for hexagonal

**Goal:**

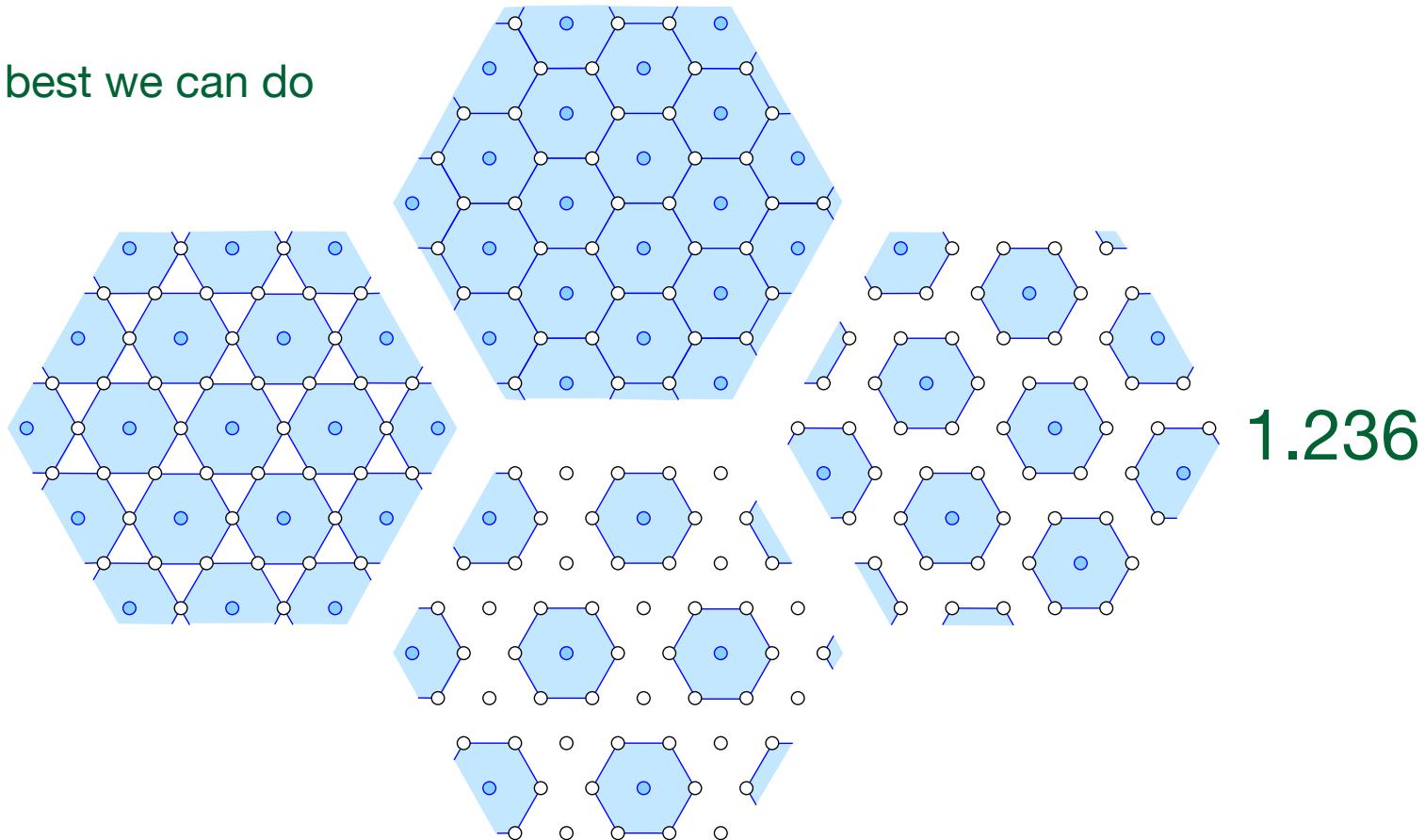
1.25 is the best we can do

1.245

1.25

1.236

1.222

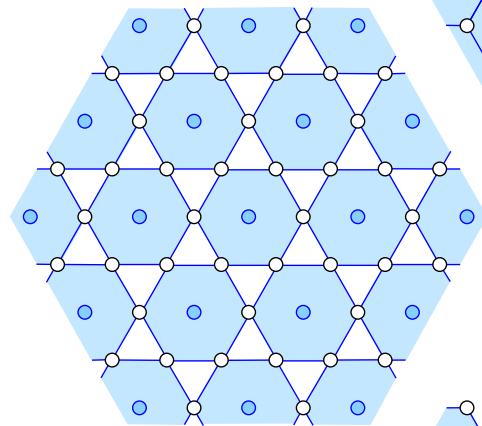


# The supremum MST-ratio for hexagonal

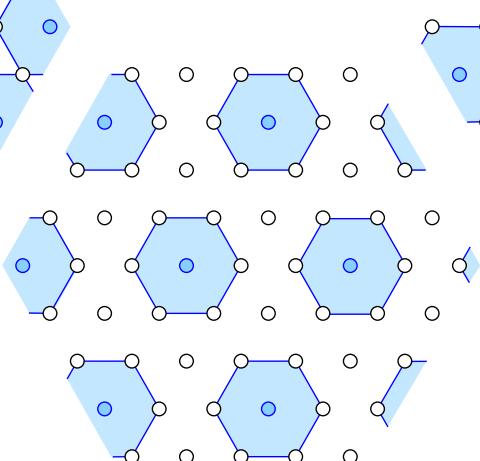
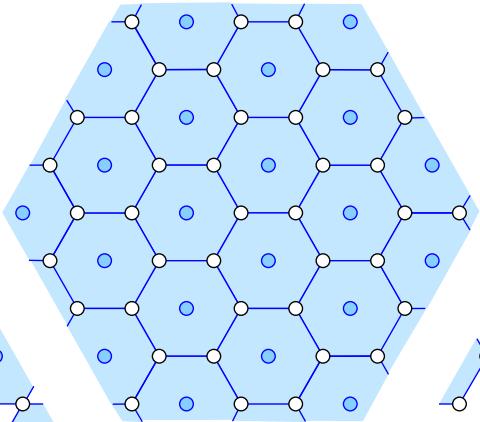
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1.25



1.236

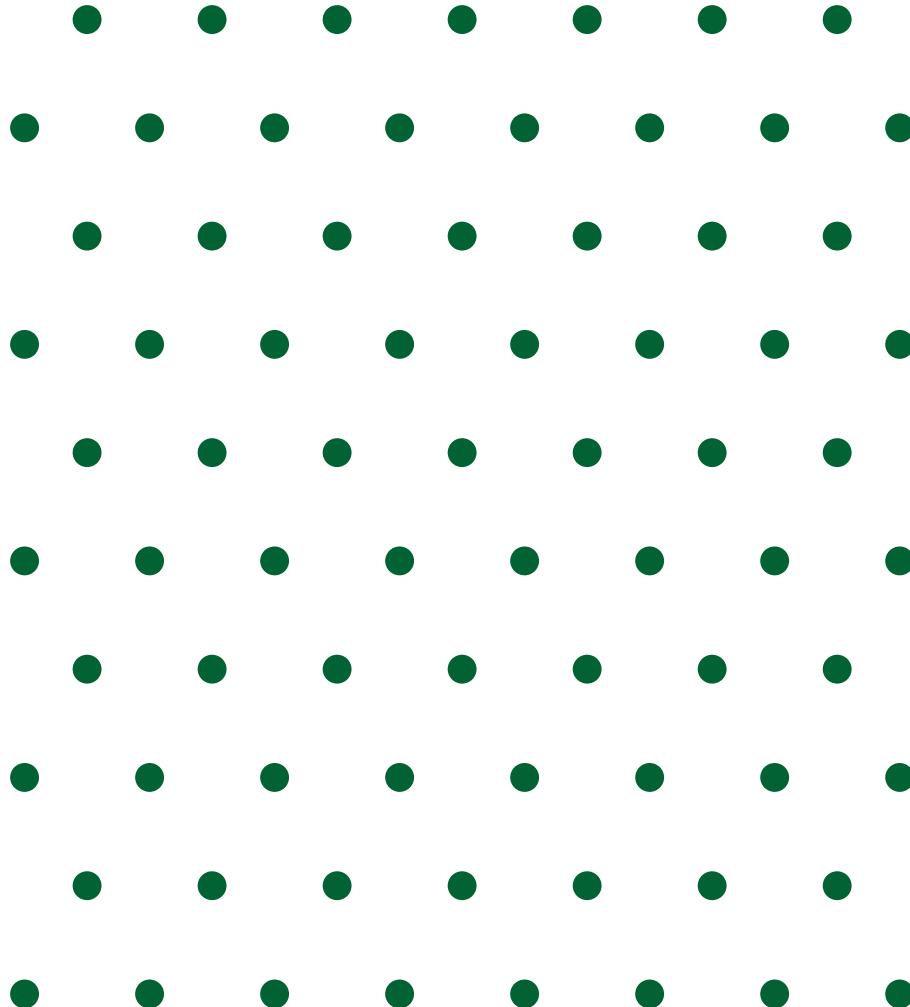
1.222

**Idea:**

For any coloring, the average length of edges in  $|\text{MST}(B)|$ ,  $|\text{MST}(\Lambda \setminus B)|$  is at most 1.25

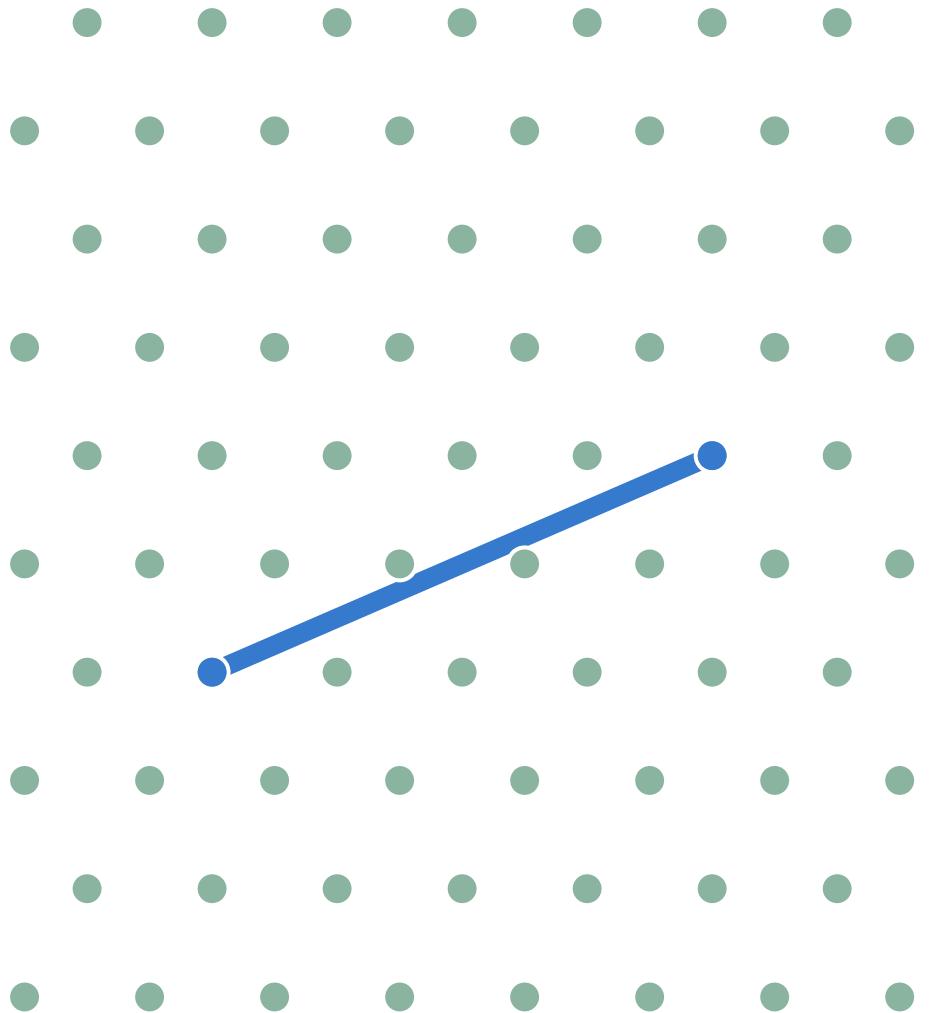
# Average colored edge length $\leq 1.25$

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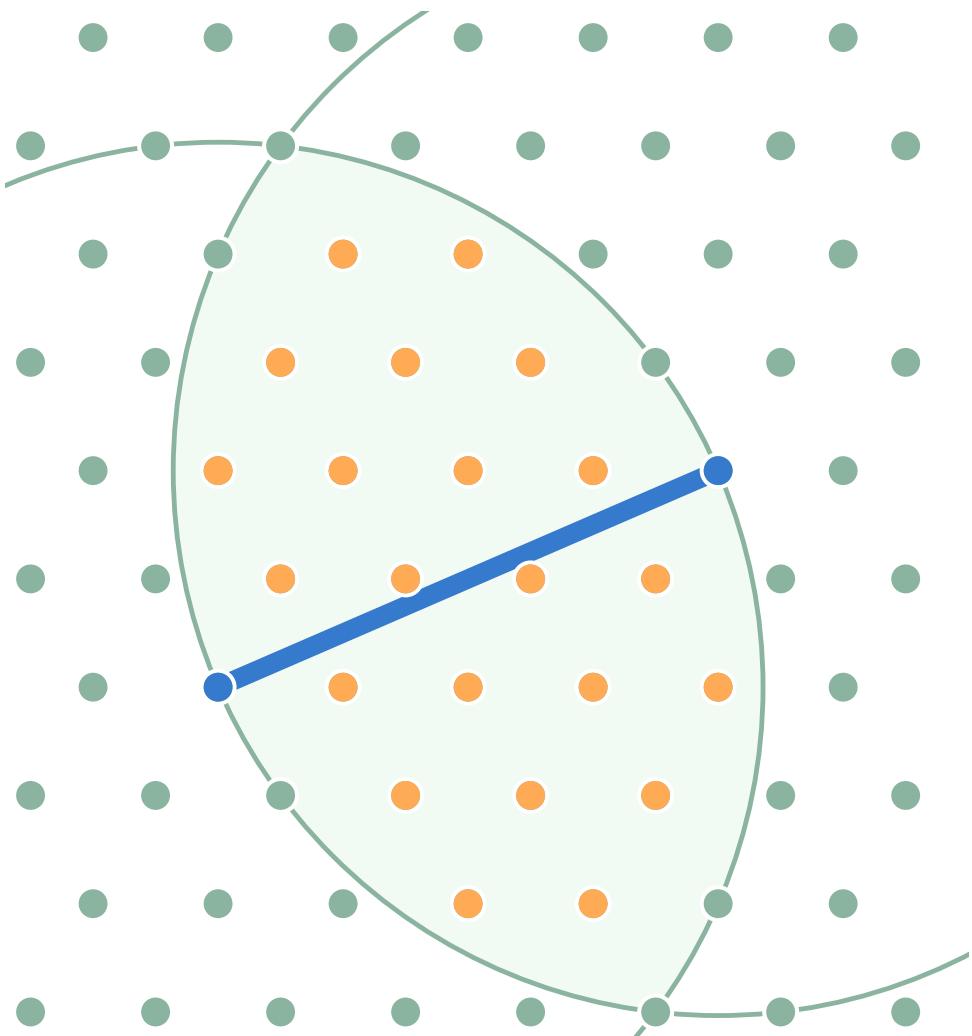
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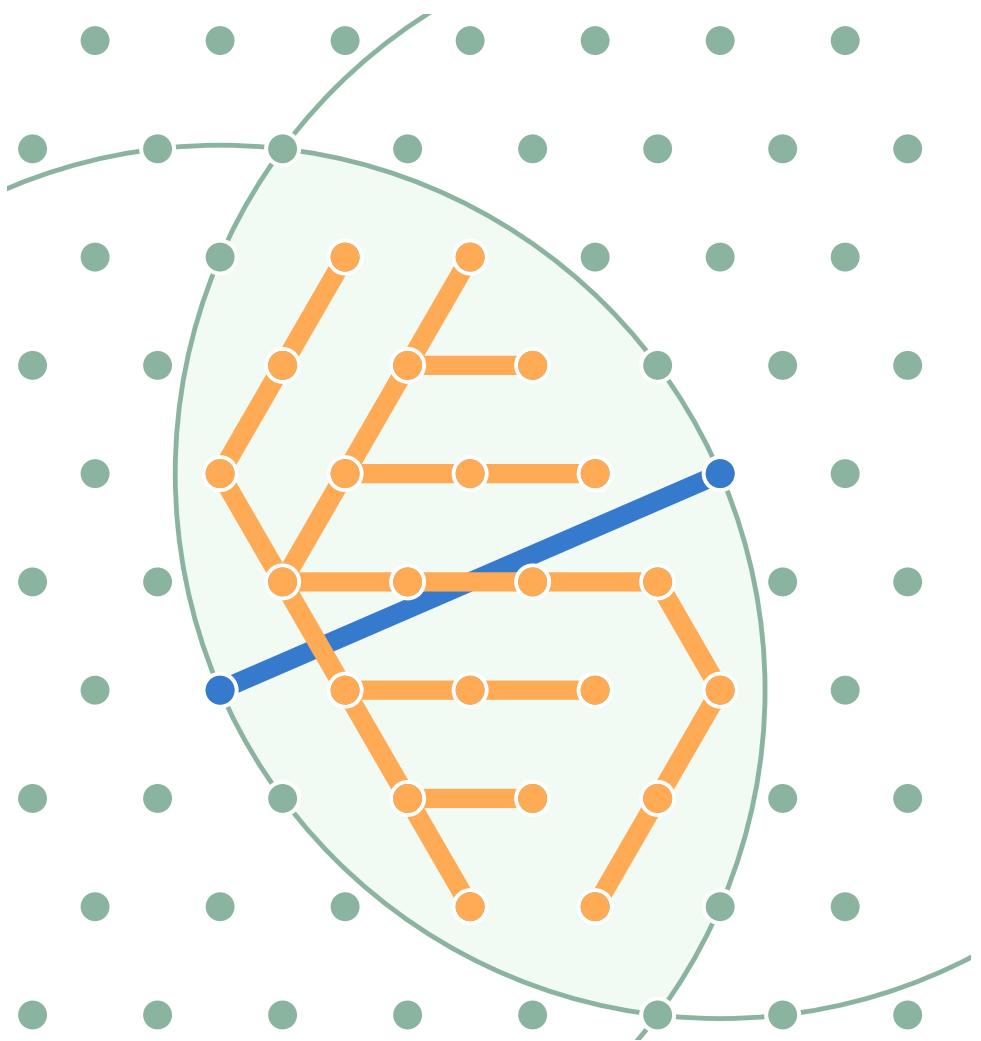
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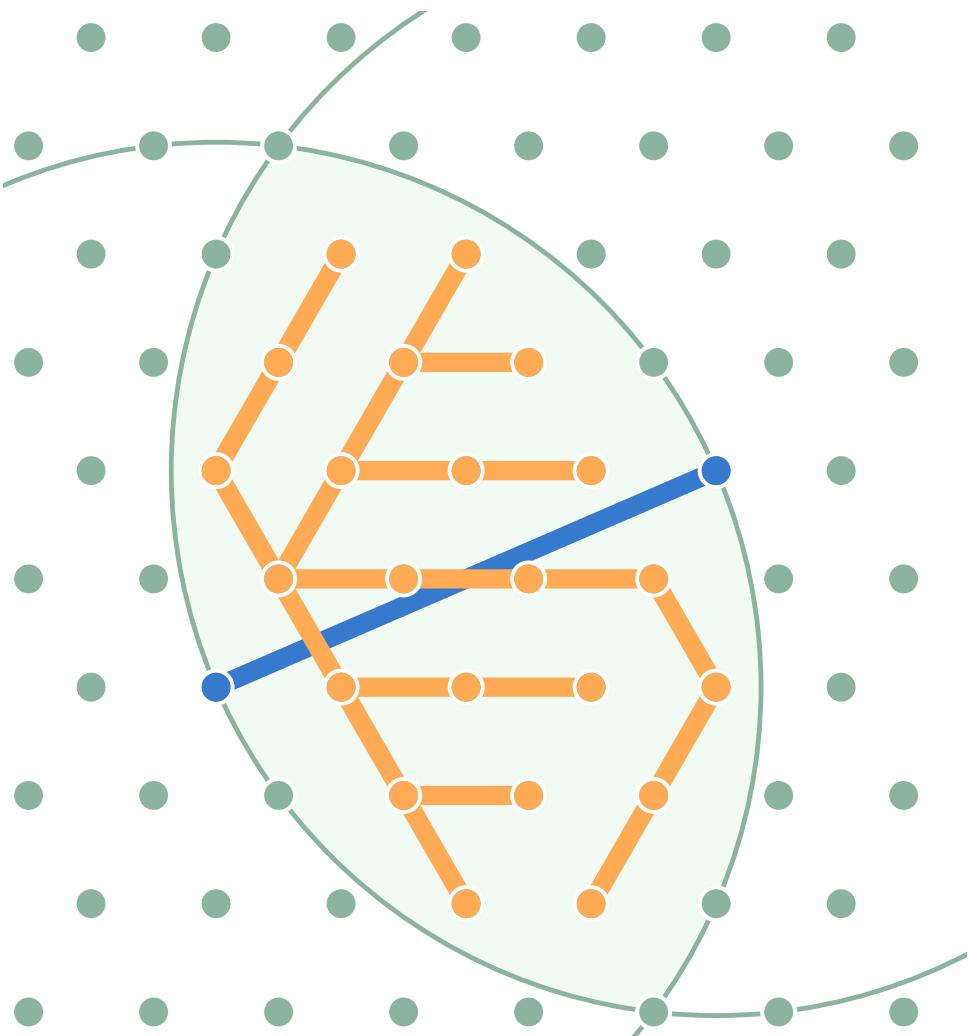


# Average colored edge length $\leq 1.25$

---



# Average colored edge length $\leq 1.25$



## Budget:

- 0.25 for each orange edge of length 1

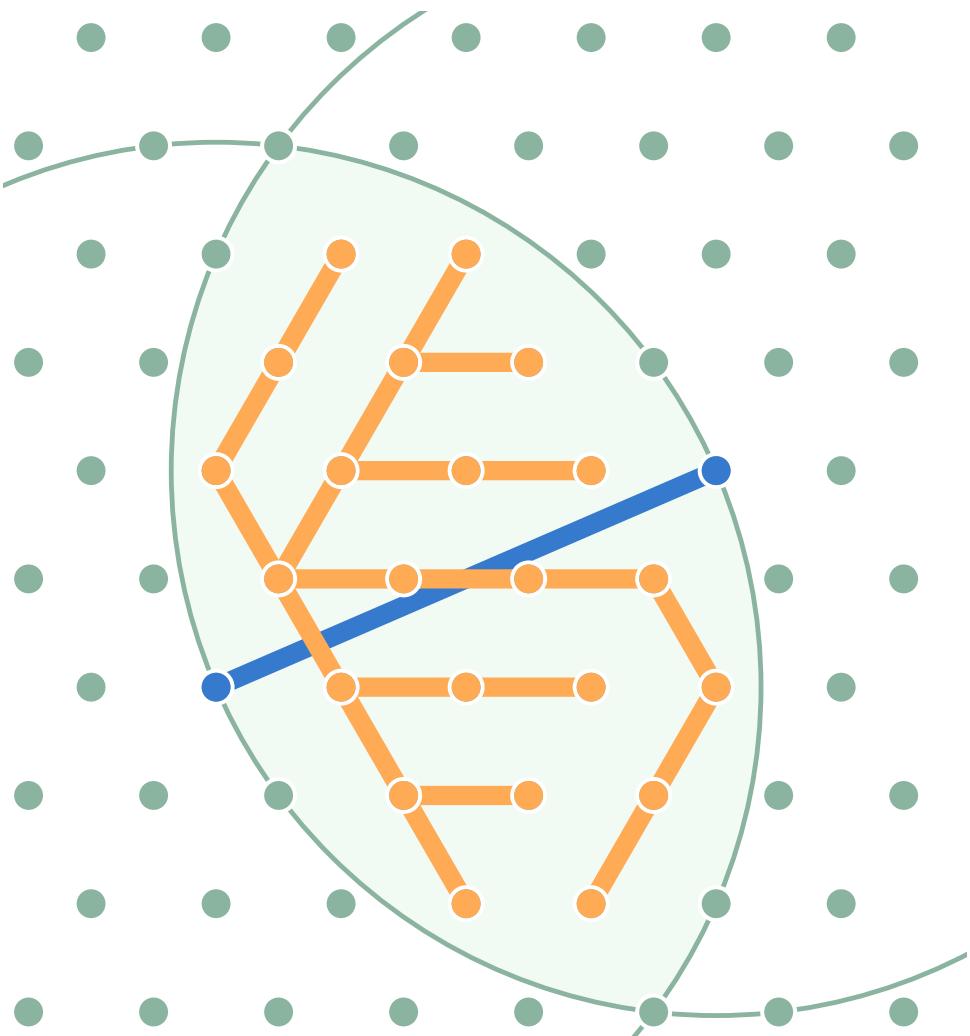


## Cost:

- $|edge| - 1.25$



# Average colored edge length $\leq 1.25$



## Budget:

- 0.25 for each orange edge of length 1



## Cost:

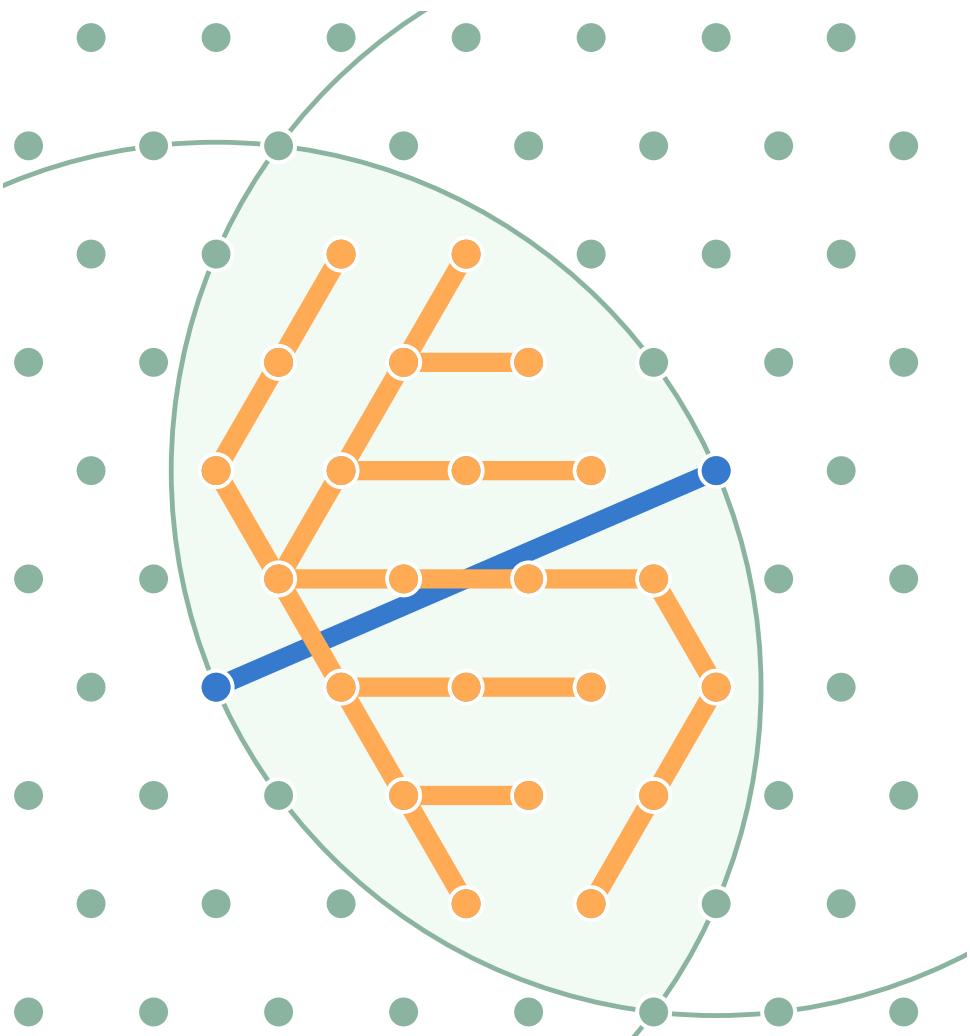
- $|edge| - 1.25$



## Problem:

- Can we afford long edges with available short edges?

# Average colored edge length $\leq 1.25$



## Budget:

- 0.25 for each orange edge of length 1



## Cost:

- $|edge| - 1.25$



## Problem:

- Can we afford long edges with available short edges?

## Solution:

- 13 pages of accounting

---

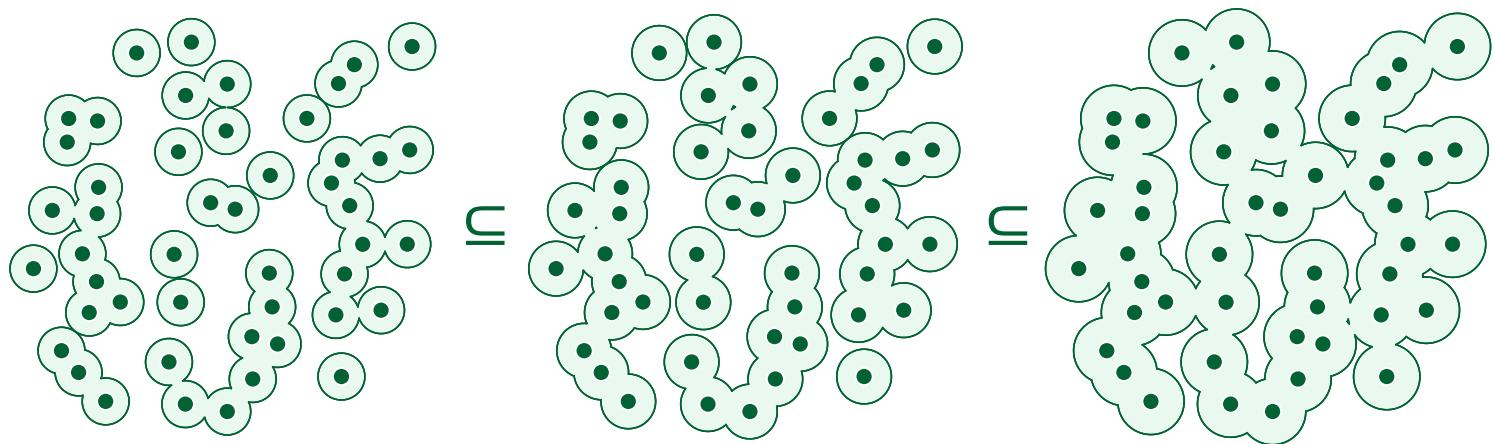
# Topological Data Analysis

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# Sequence of topological spaces

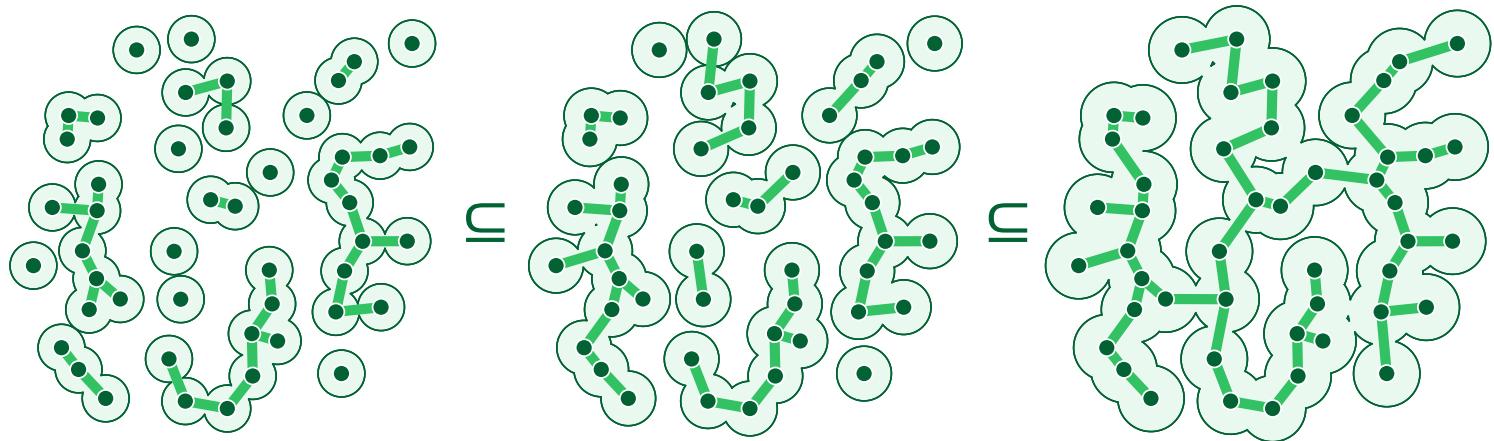
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growing  
disks



# Sequence of topological spaces

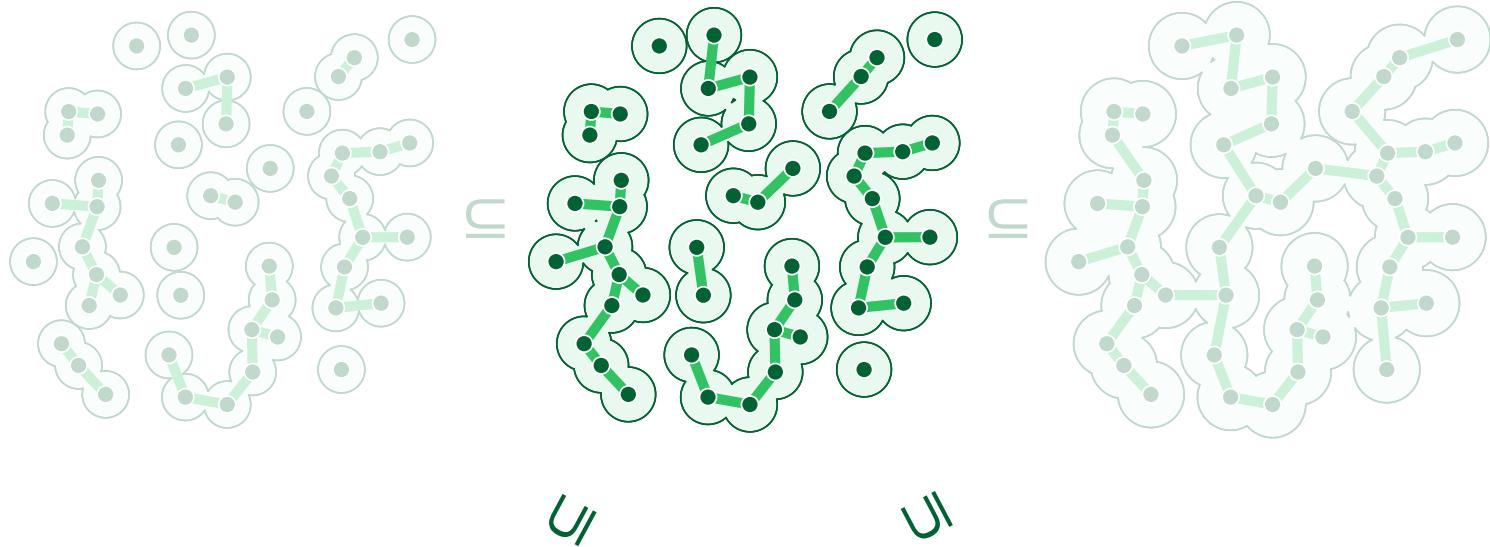
components  
of  
connectivity



components of connectivity  $\leftrightarrow$  minimal spanning trees

# Sequence of topological spaces

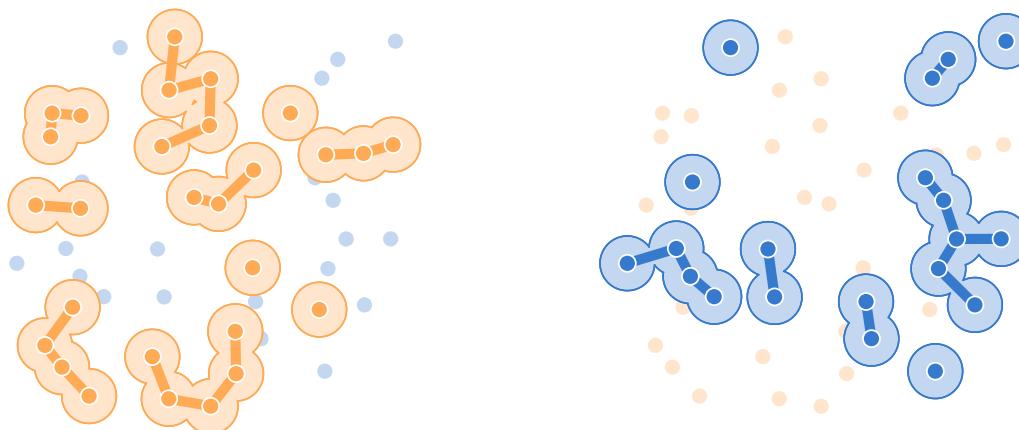
components  
of  
connectivity



$\sqcup$

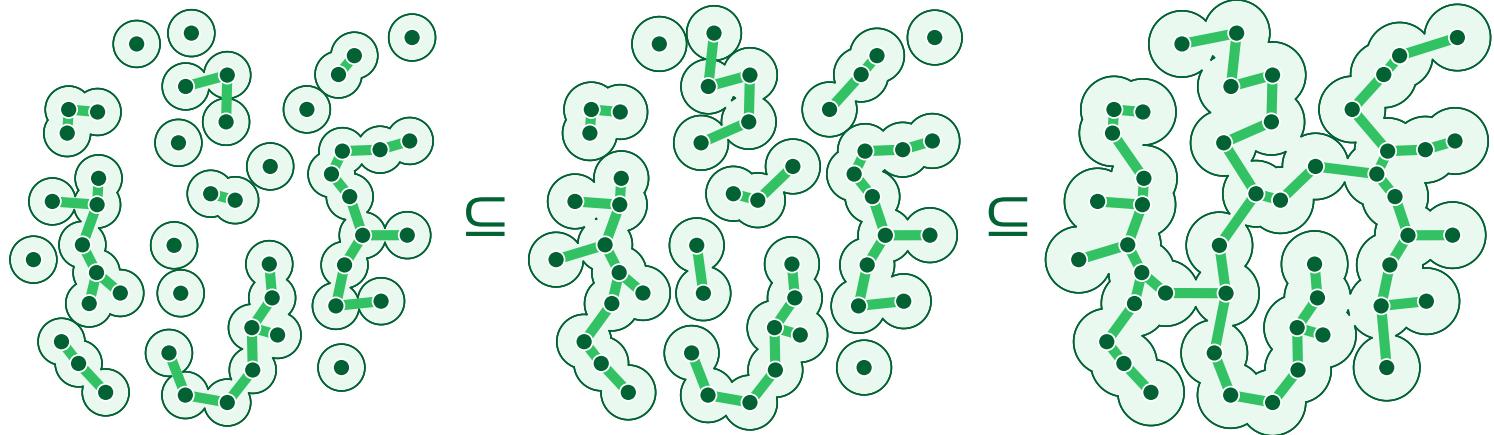
$\sqcup$

How do colors  
spatially interact?

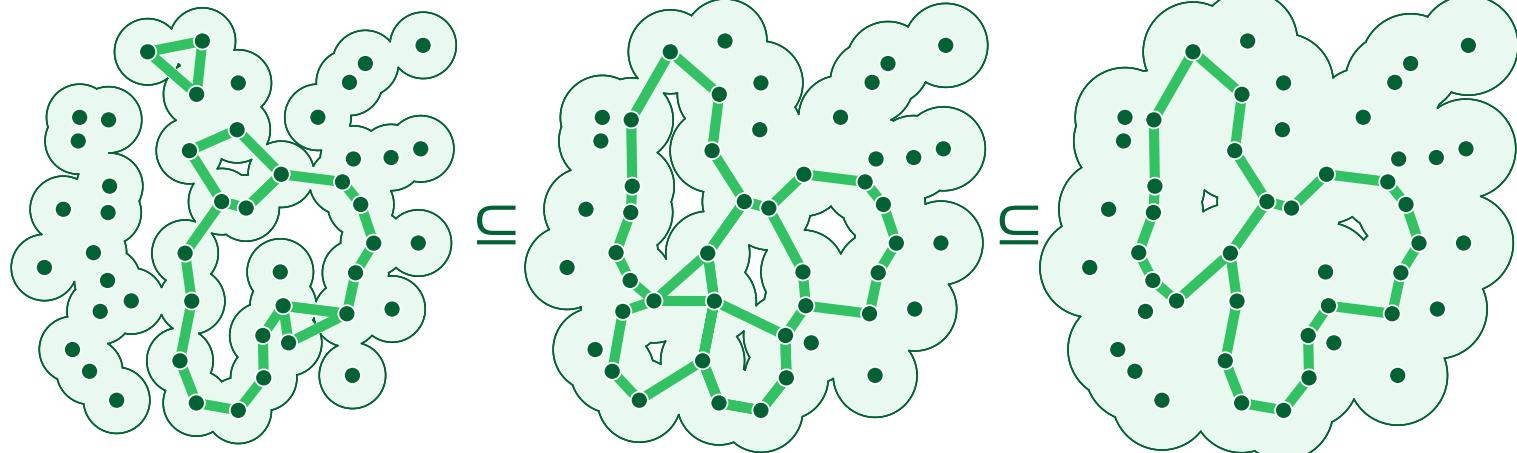


# Sequence of topological spaces

components  
of  
connectivity



loops



# Generalizations, open questions

---

- more colors, more dimensions
  - finite sets → straightforward generalization of presented results
- lattices
- average MST-ratio over colorings
  - for  $n$  random points:  $\sqrt{2}$  with high probability as  $n \rightarrow \infty$
  - bounds for arbitrary sets, variance (also for maximum MST-ratio), ...
- other summaries from topological data analysis
  - loops, ...
  - nice geometric reformulations
    - → discrete geometry, stochastic geometry

Follow up research:

- Ameli, Motiei, Saghafian: *The Complexity of Maximizing the MST-ratio* [arXiv:2409.11079]
- Dumitrescu, Pach, Tóth: *Two trees are better than one* [arXiv:2312.09916]
- D., Edelsbrunner, Rosenmeier, Saghafian:  
Expected 1-norms of various TDA summaries for two colors [preprint soon to appear]

# Thank you for your attention!

---

- more colors, more dimensions
  - finite sets → straightforward generalization of presented results
- lattices
- average MST-ratio over colorings
  - for  $n$  random points:  $\sqrt{2}$  with high probability as  $n \rightarrow \infty$
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