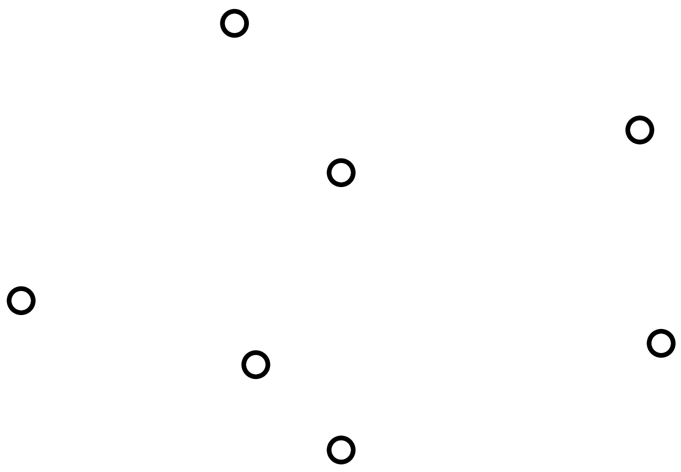
# Holes in (G-)Convex and Simple Drawings

**Helena Bergold**, **Joachim Orthaber**, **Manfred Scheucher**, and **Felix Schröder**

GD 2024: September 19

# Introduction: Holes



$n$ points in general position

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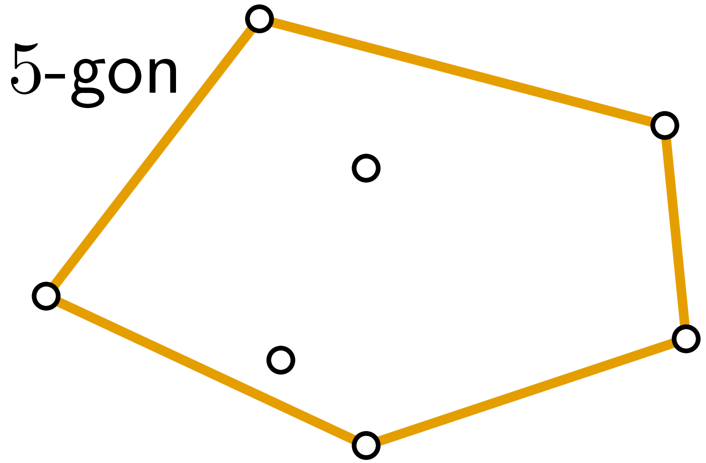


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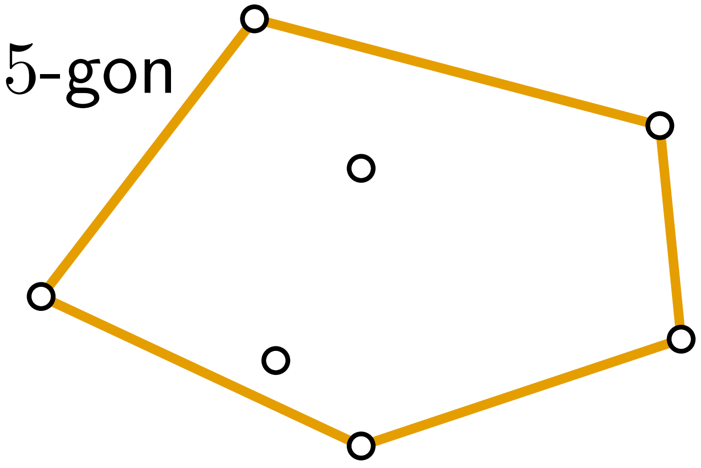


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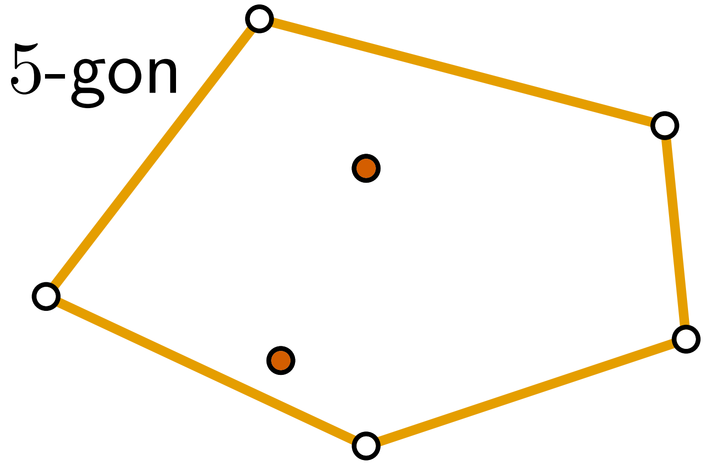


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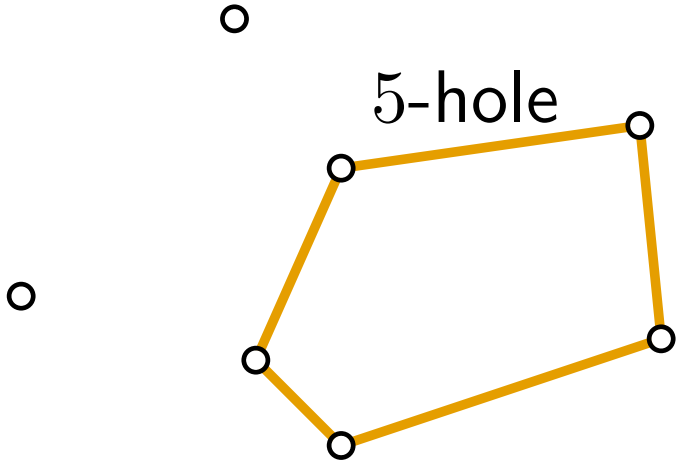


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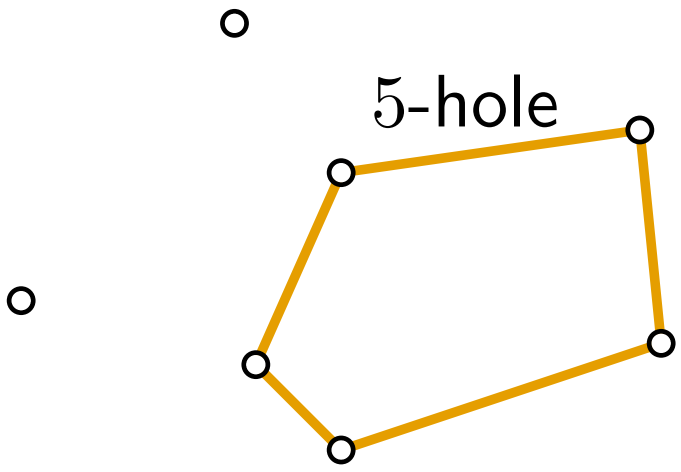


**Existence?**

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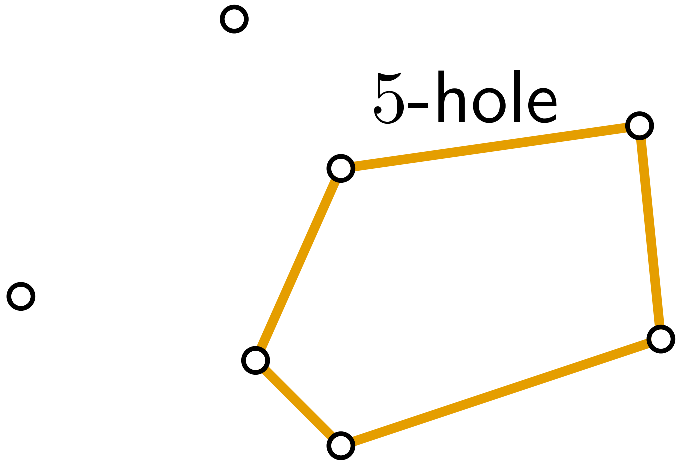


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4-holes

✓

($\geq 5$ points)

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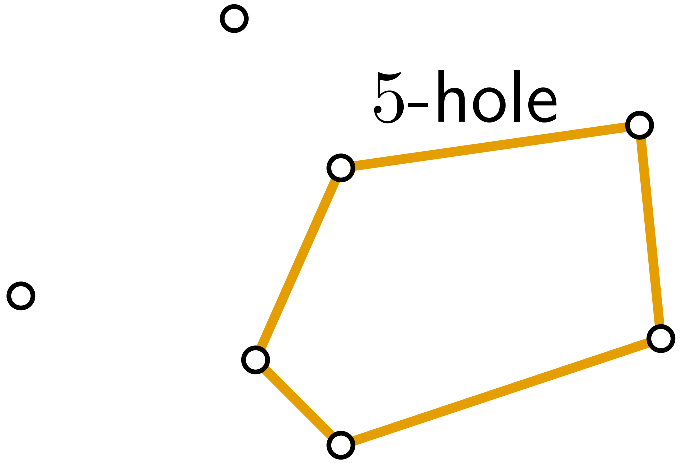


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|---|---|
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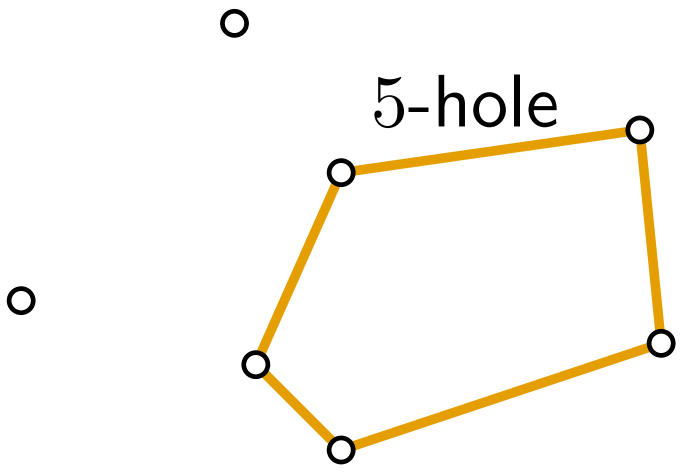


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7-holes ✗ [Horton, 1983] Horton sets

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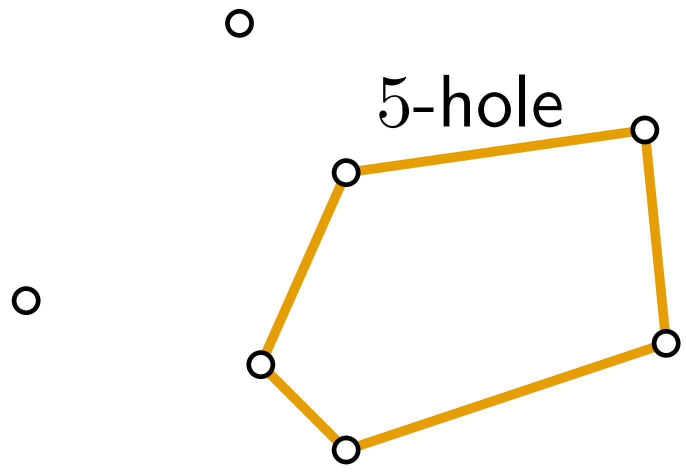


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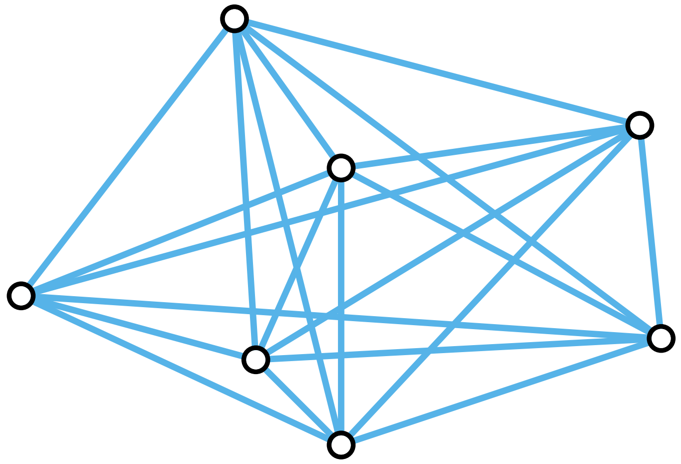
geometric drawing of $K_n$



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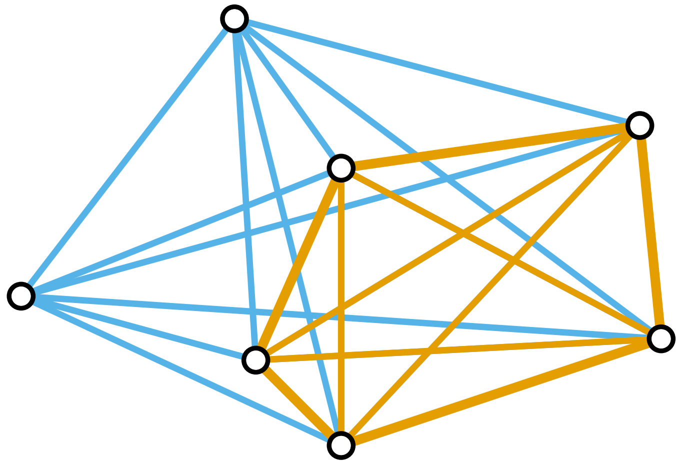

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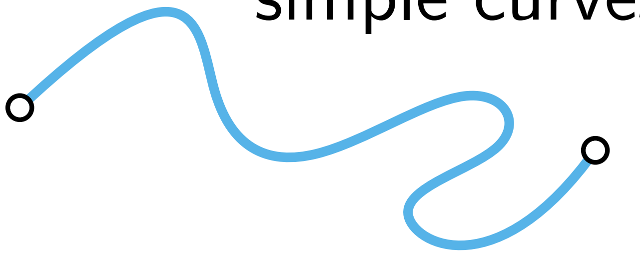

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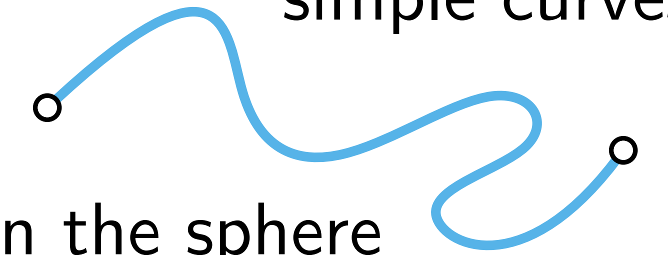

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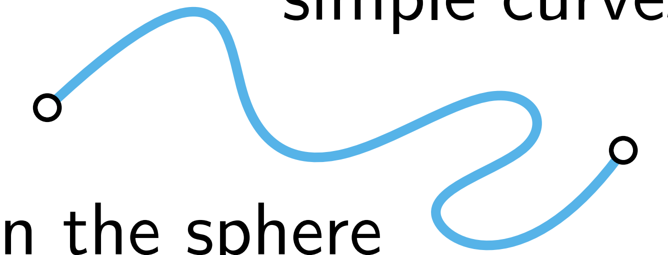
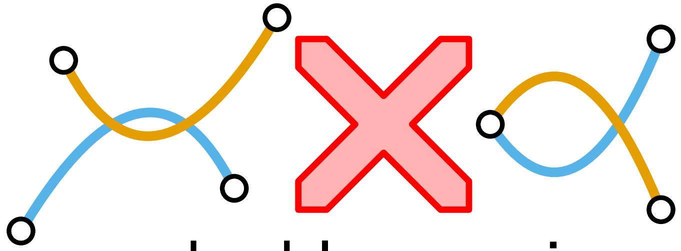
simple curves



on the sphere



no double-crossings

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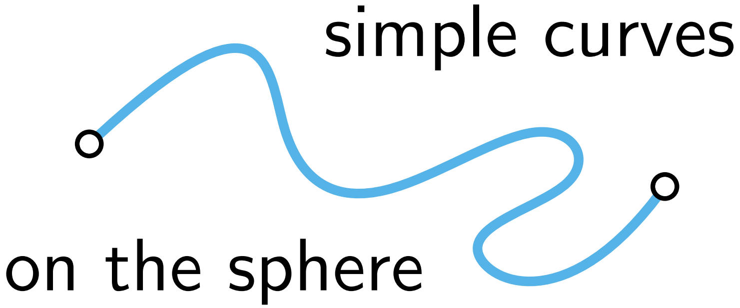
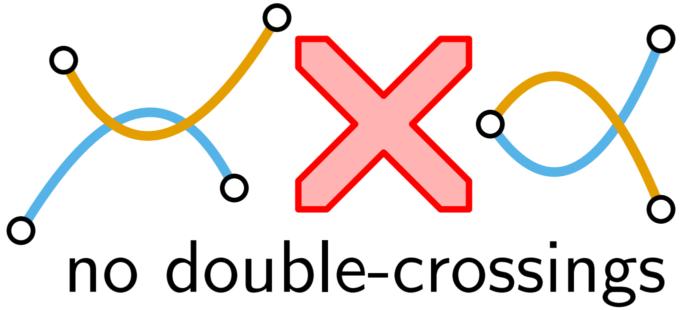
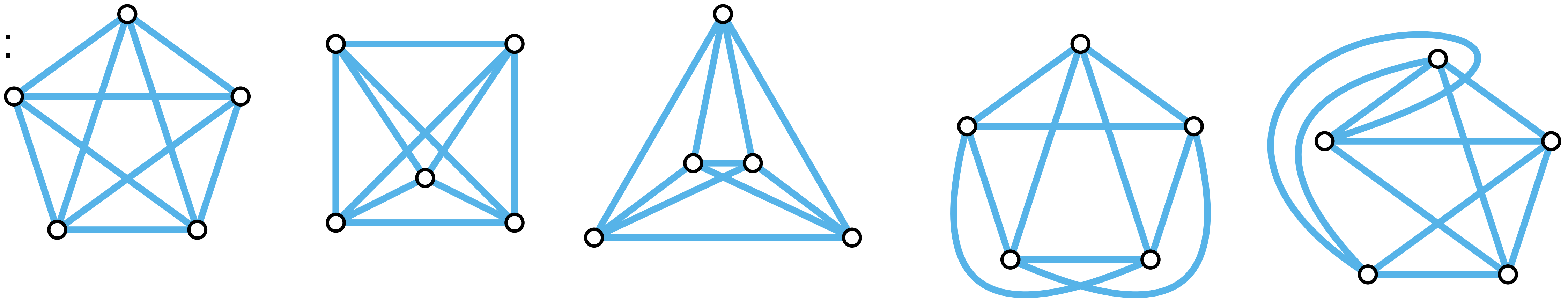



$K_5$:



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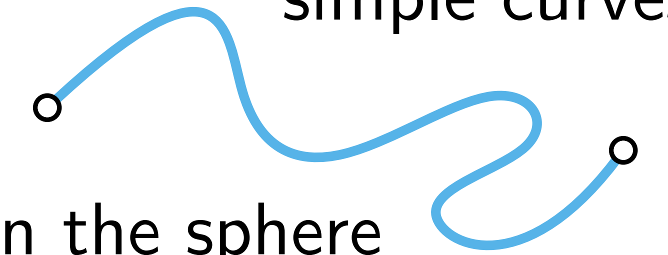


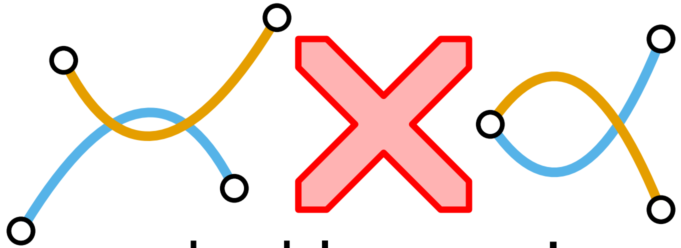


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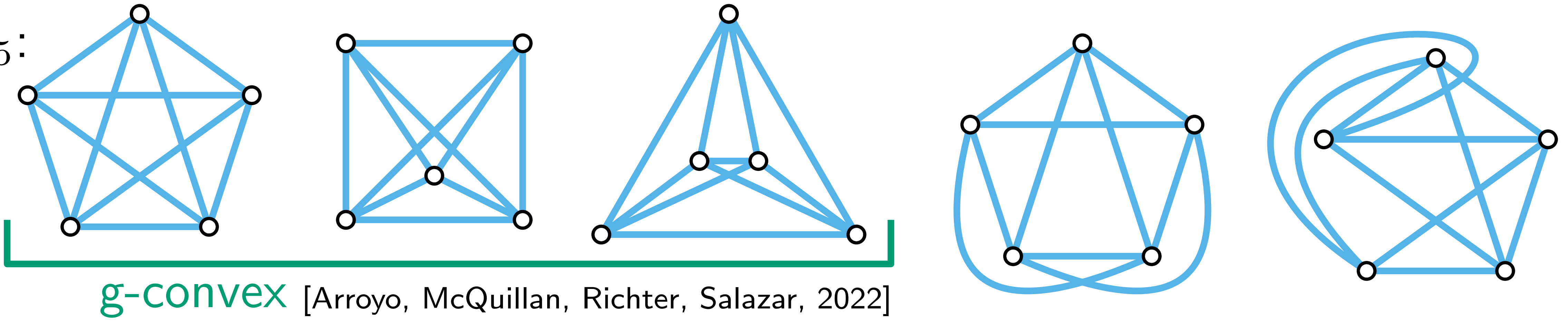

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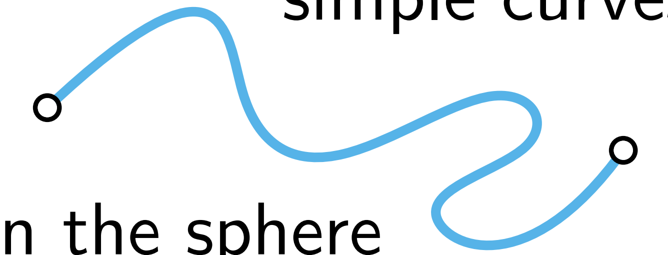

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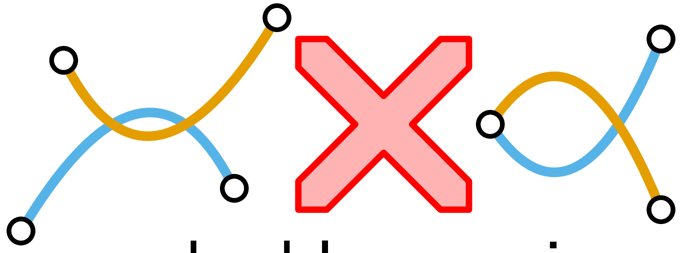

no double-crossings

$k$-gon

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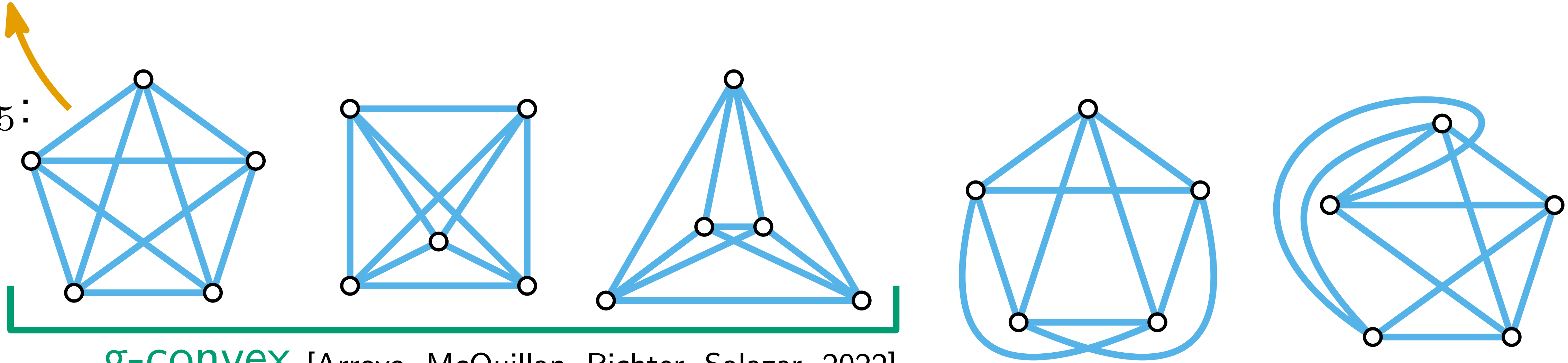

g-convex [Arroyo, McQuillan, Richter, Salazar, 2022]

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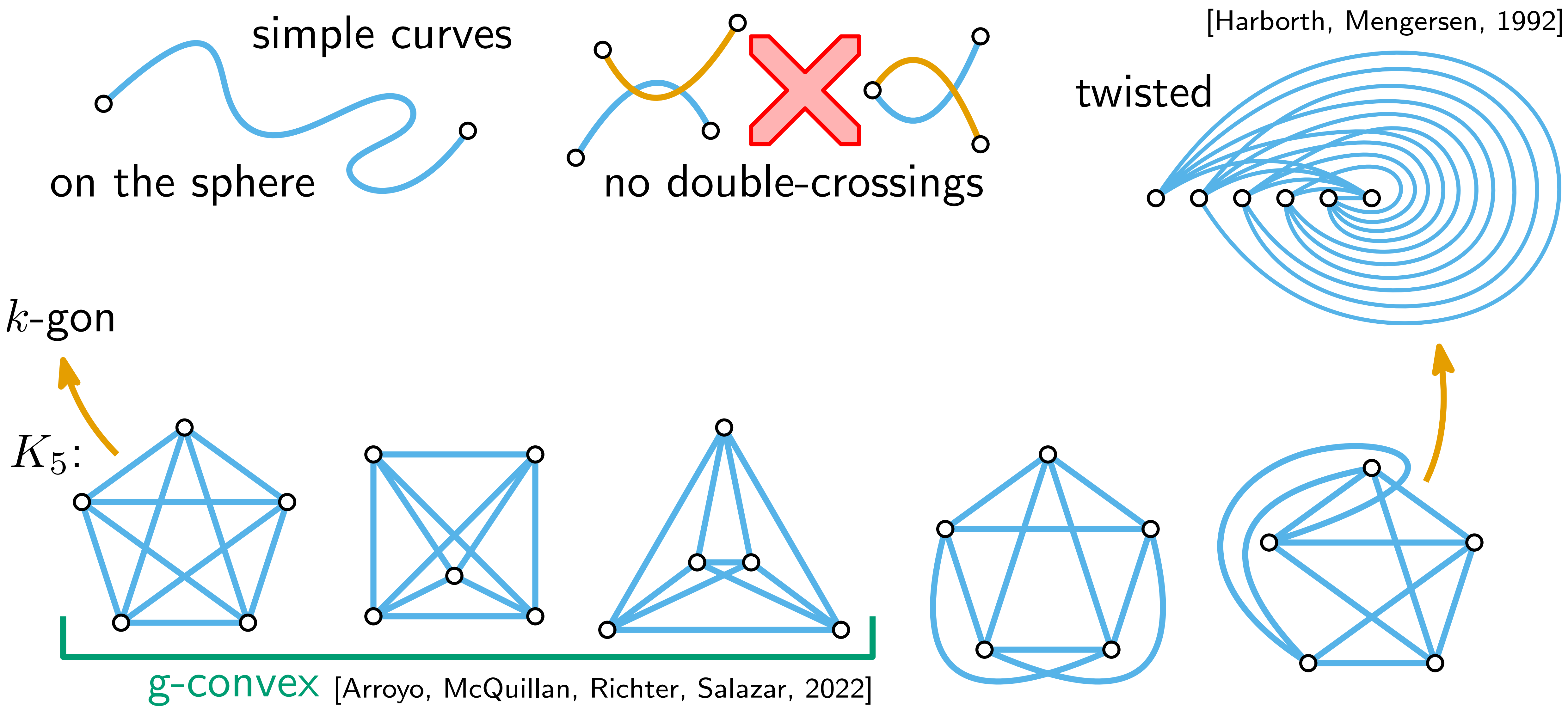

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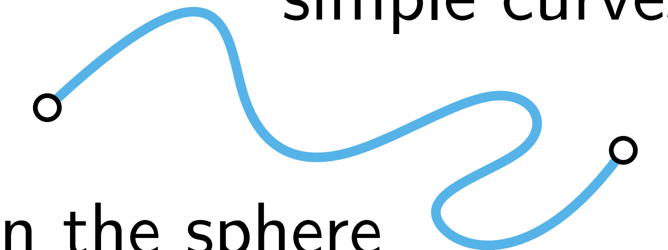

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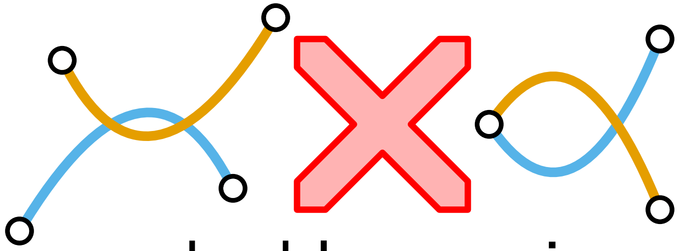
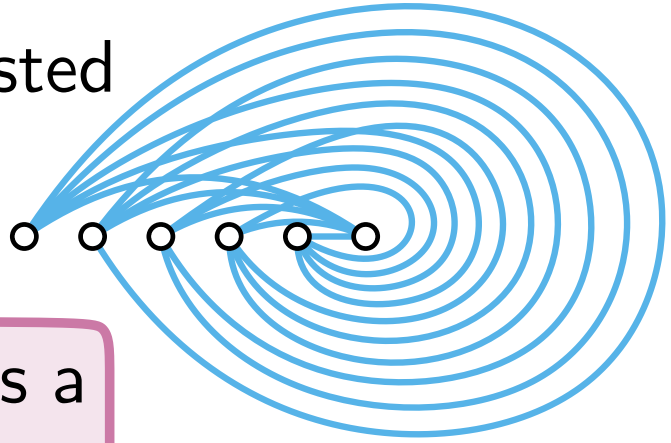


[Harborth, Mengersen, 1992]

twisted



$k$-gon

**Theorem [Suk, Zeng, 2022]** There always exists a $k$-gon or twisted subdrawing of size $\mathcal{O}(\log(n)^{1/4})$.

improves [Pach, Solymosi, Tóth, 2003]

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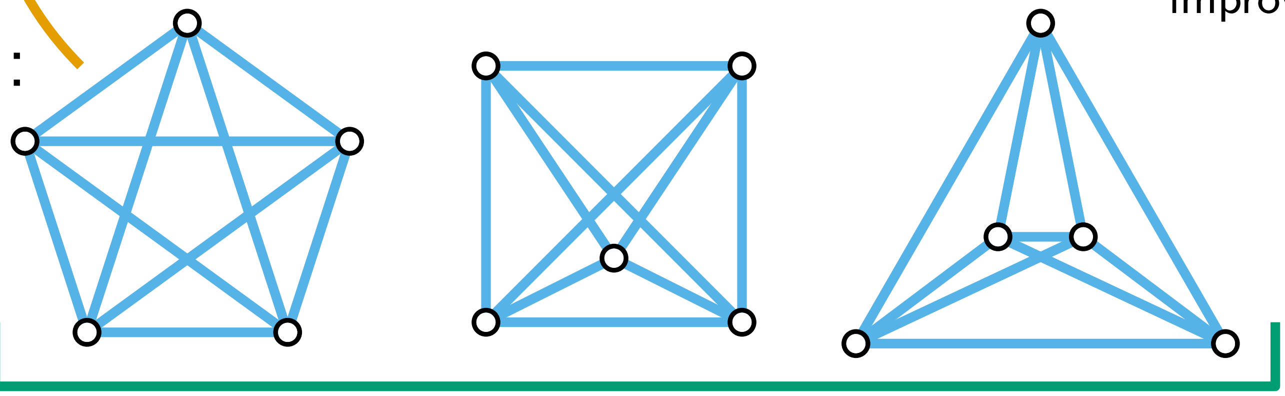

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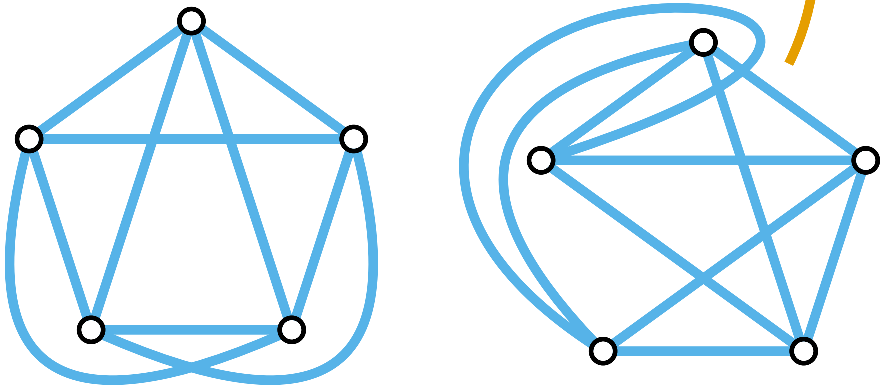

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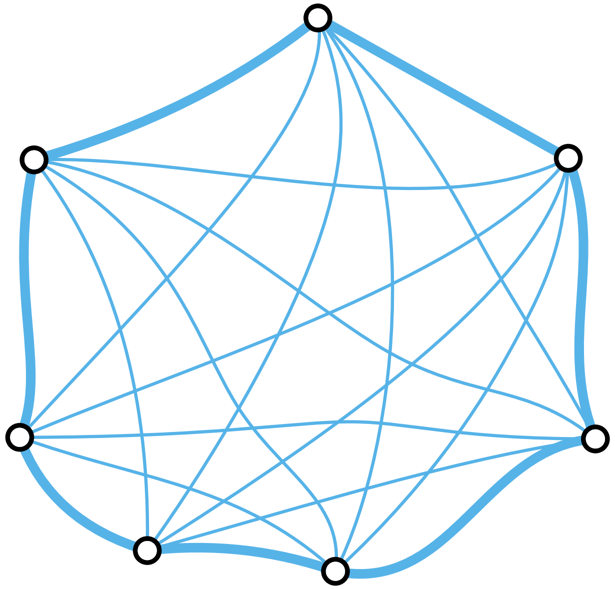

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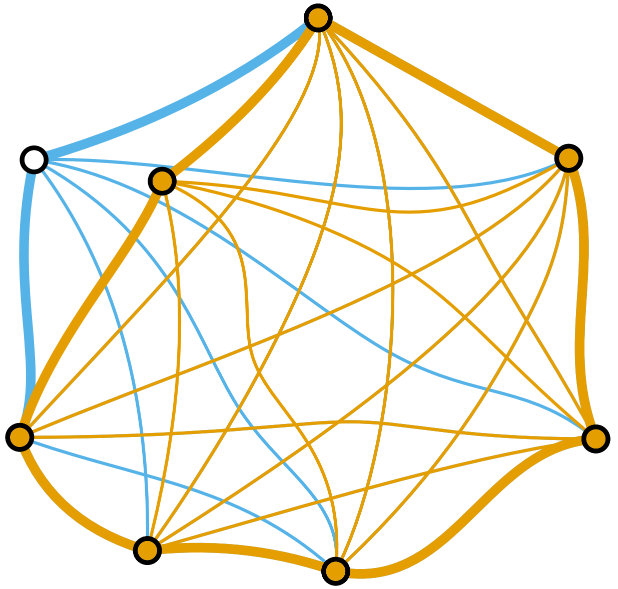

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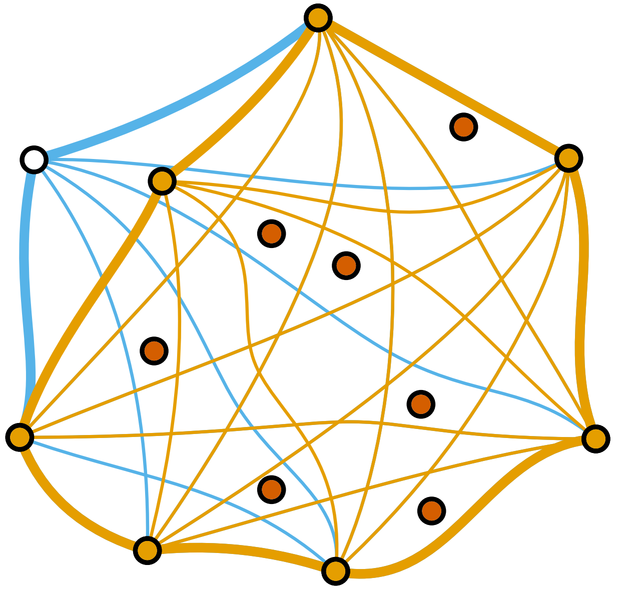

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$+$ all vertices inside

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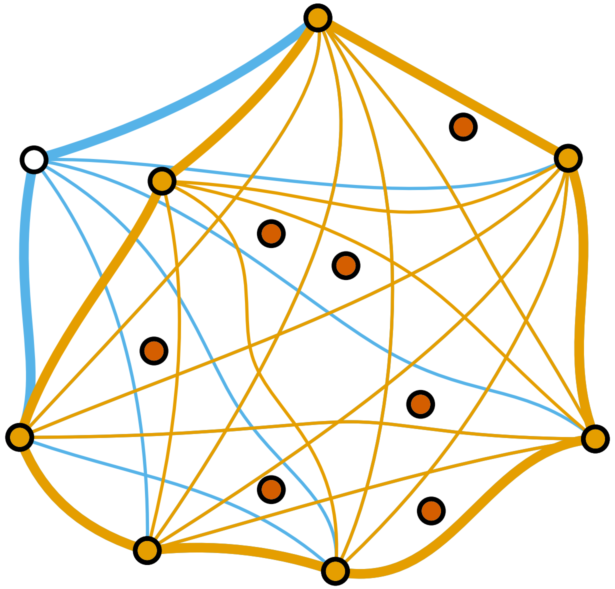

**Lemma:** The subdrawing induced by all vertices of and inside a minimal $k$-gon is pseudolinear.

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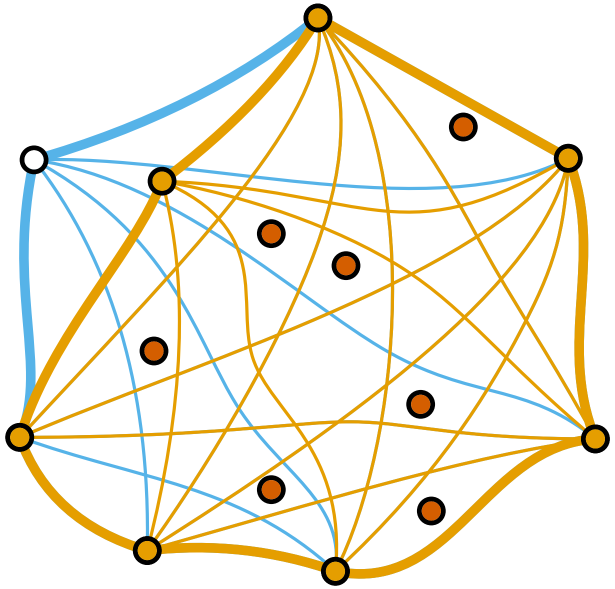

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use result on geometric drawings

[Heule, Scheucher, 2024] or [Gerken, 2008]

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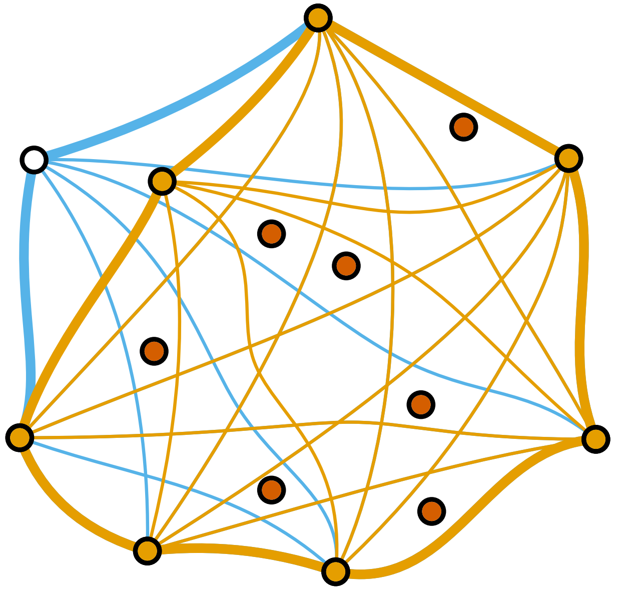

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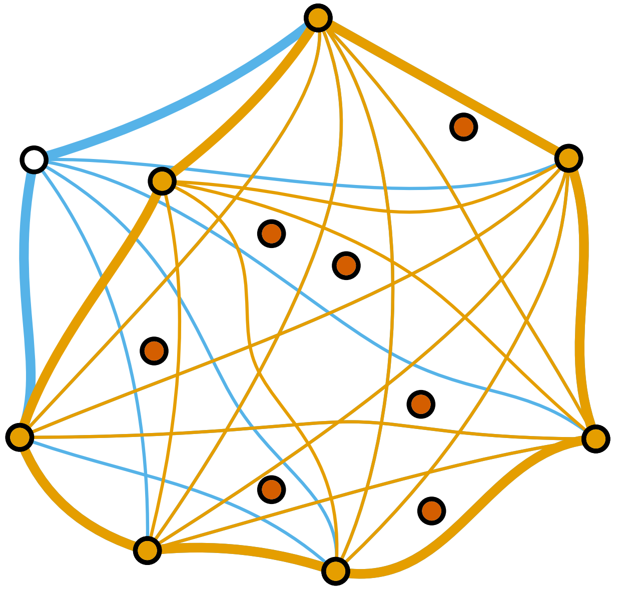

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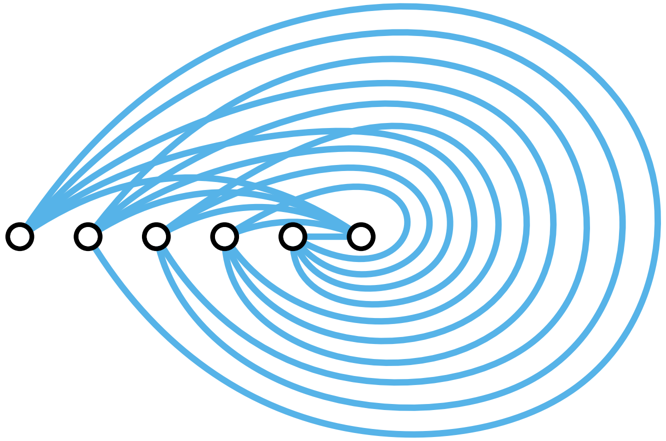
**Theorem:** Every g-convex drawing of $K_n$ (for large enough $n$) contains a 6-hole.

# In Simple Drawings

twisted drawings have no $5$-gons



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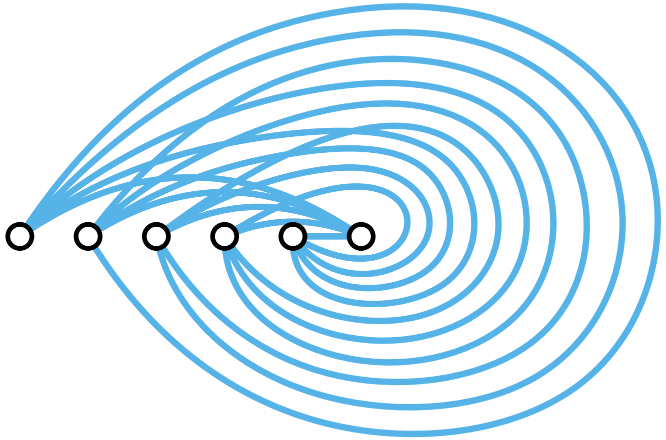
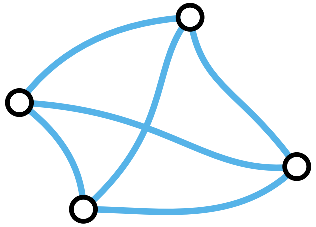
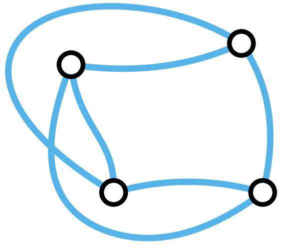


but every crossing is a $4$-gon





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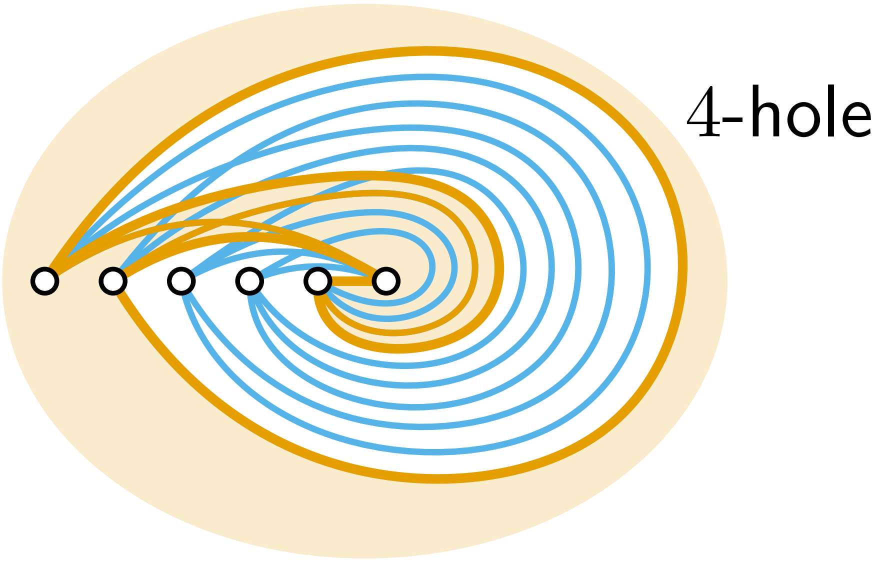


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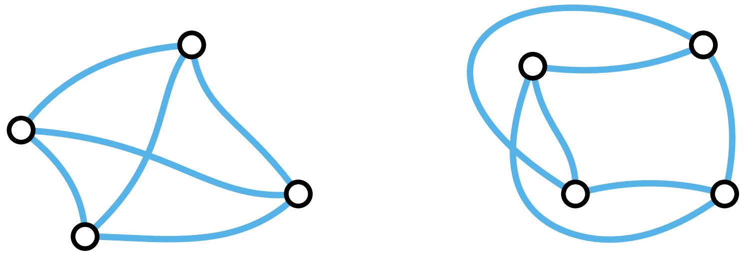

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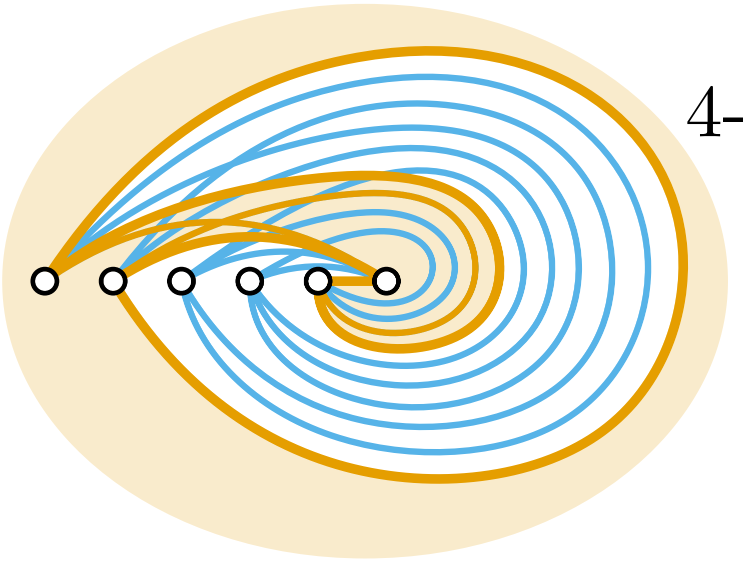
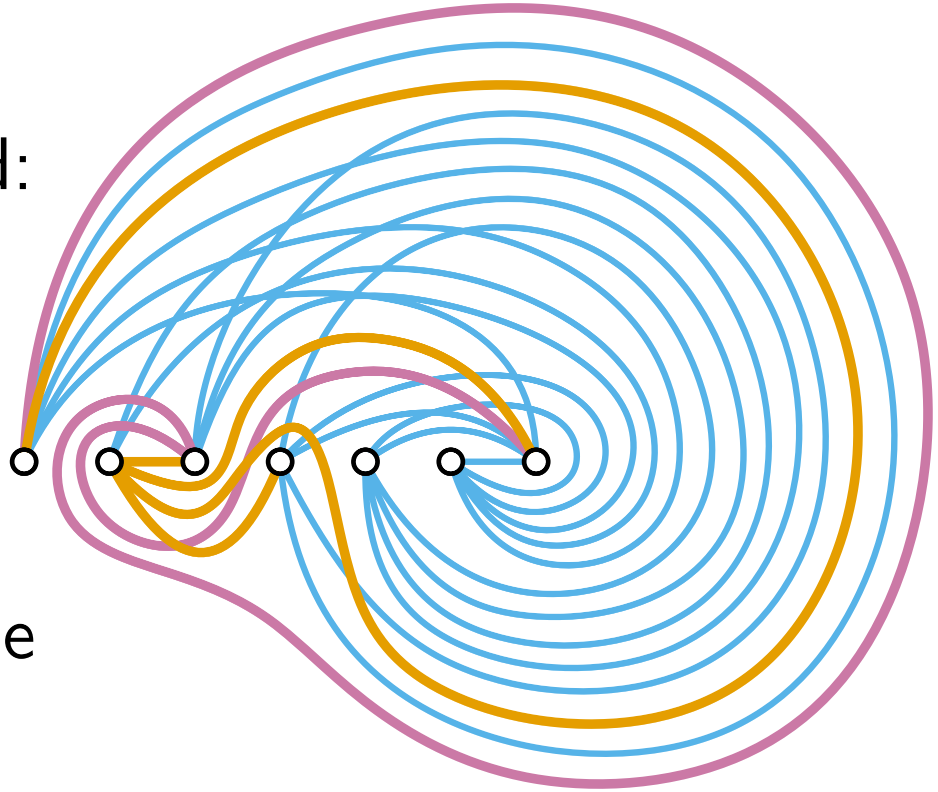


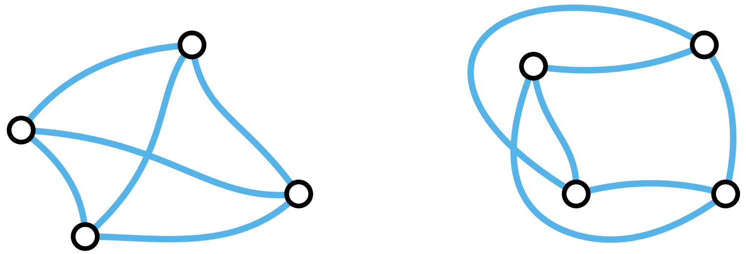
$4$-hole

modified:



no $4$-hole

but every crossing is a $4$-gon



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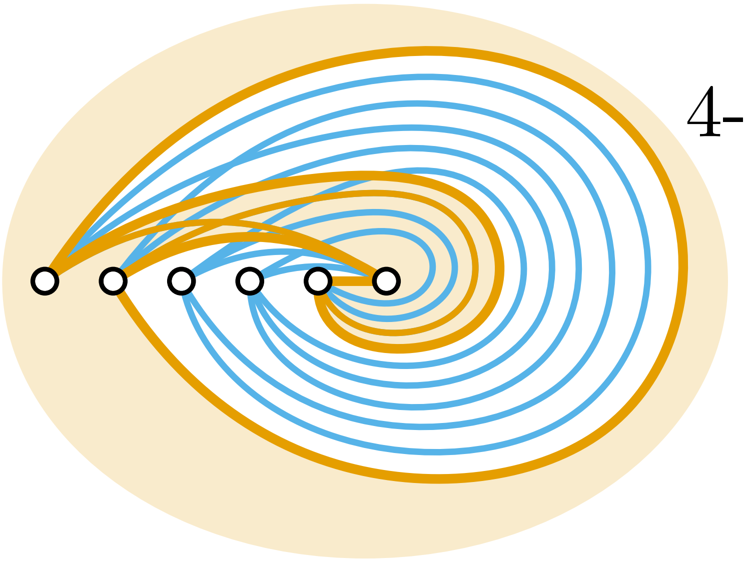

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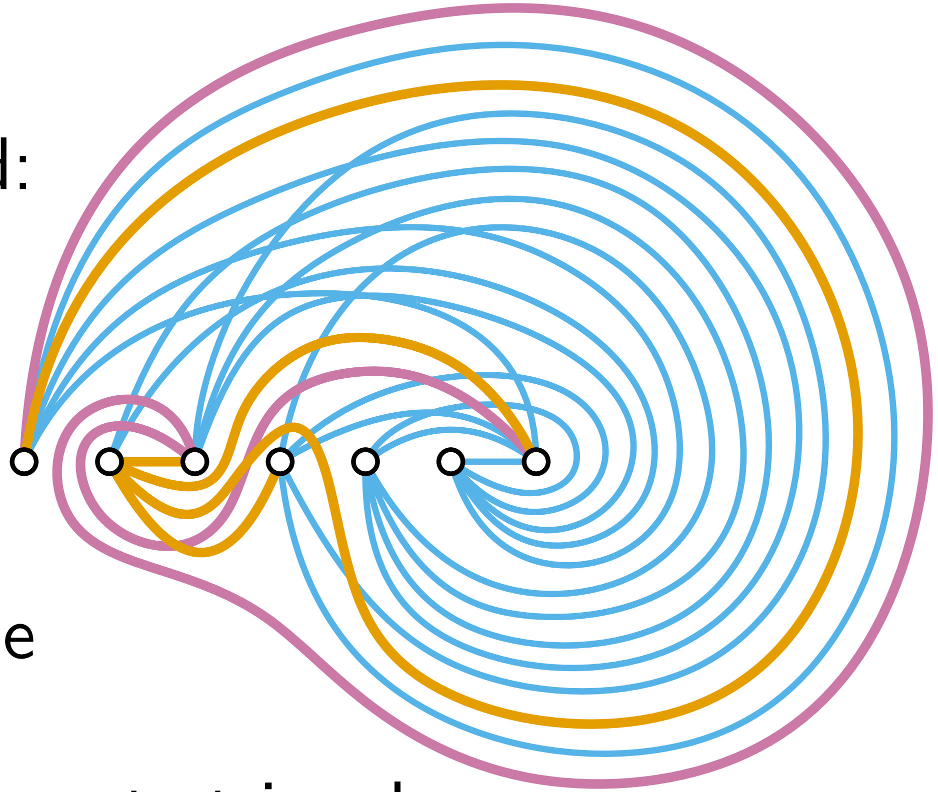

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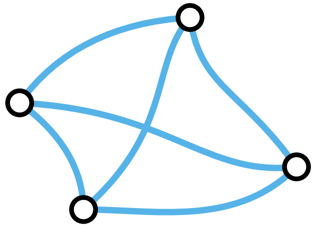
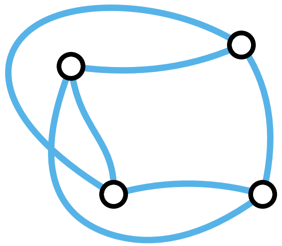
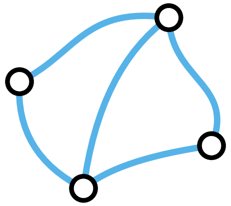




not even 2 empty triangles
glued together



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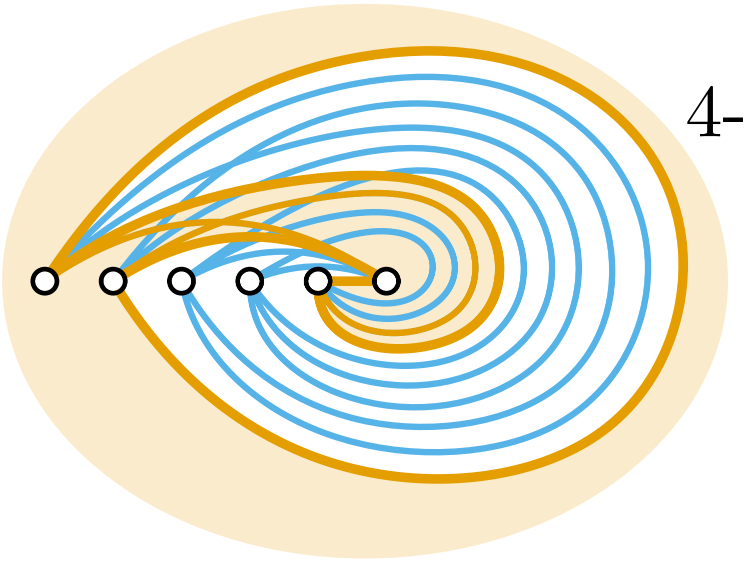

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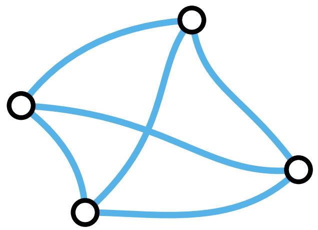
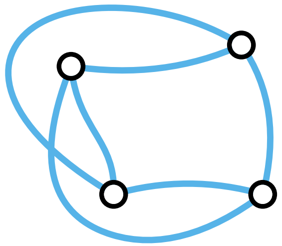
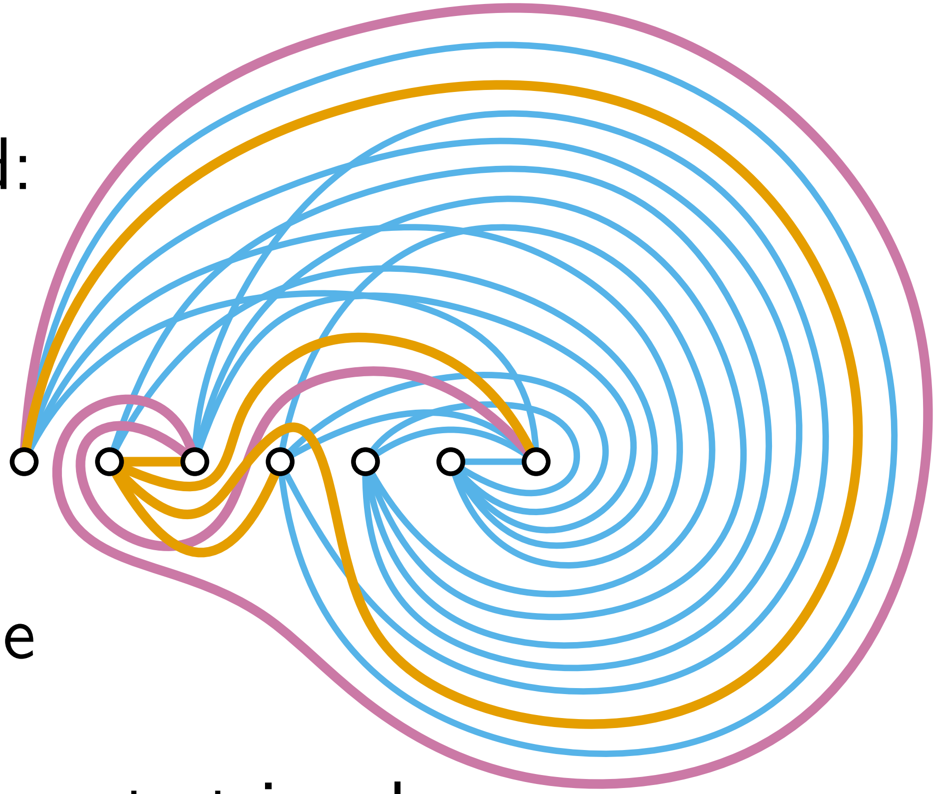


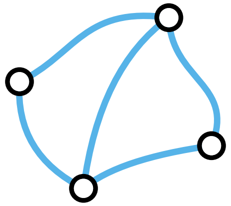


modified:



no $4$-hole

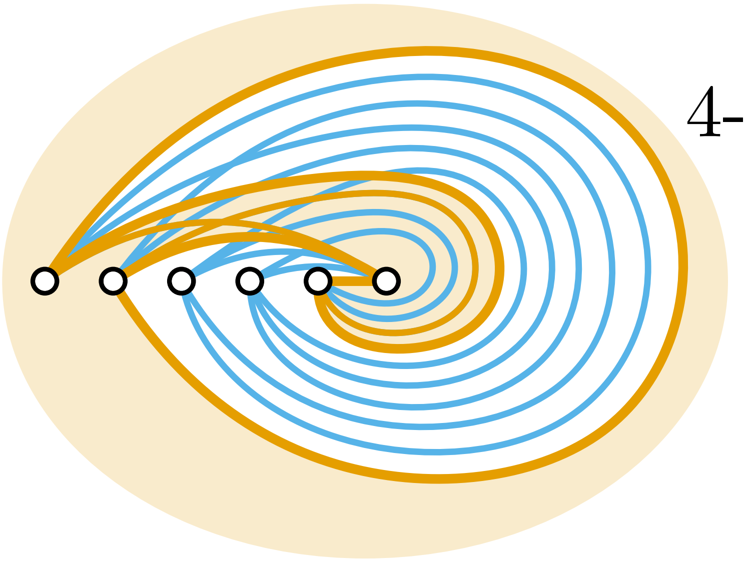
not even 2 empty triangles glued together



what now?

# In Simple Drawings

twisted drawings have no $5$-gons

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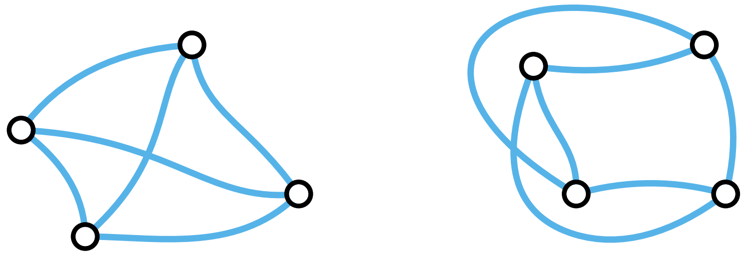
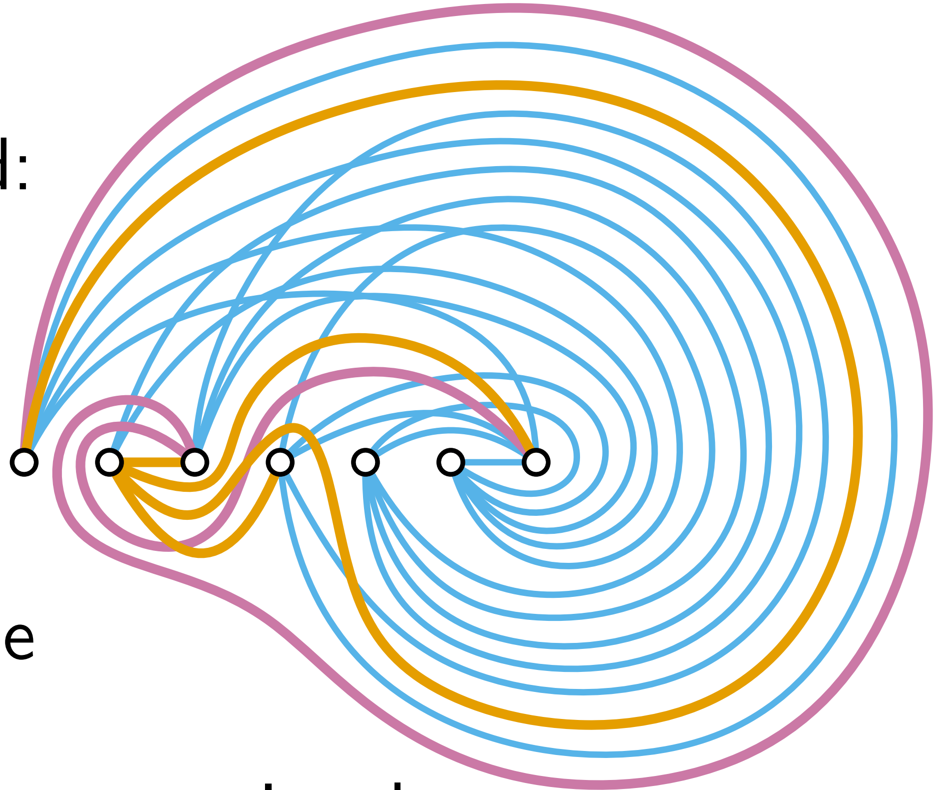


modified:



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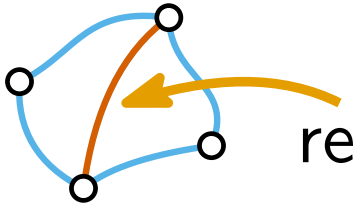
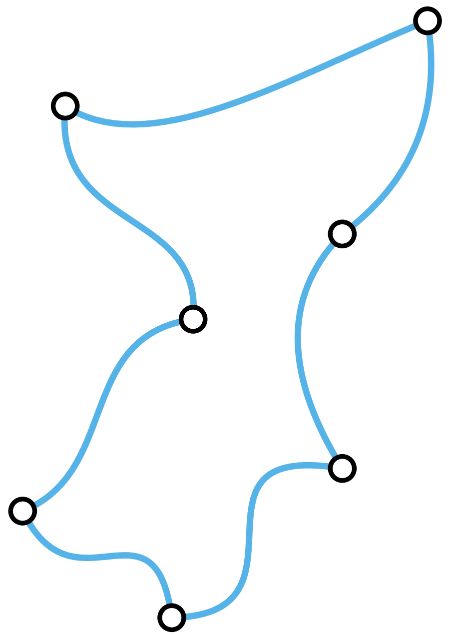



what now?

# Empty Cycles

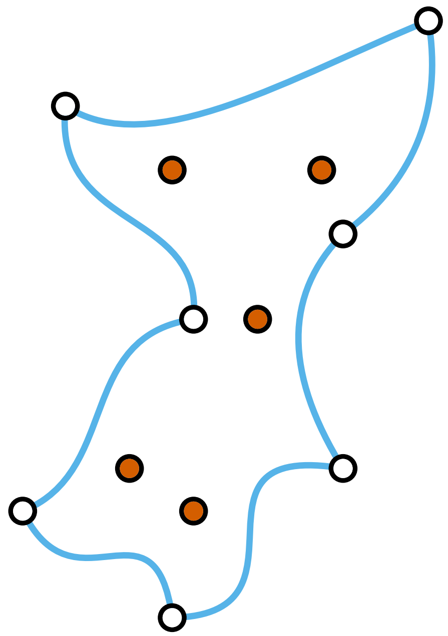
crossing-free cycle on $k$ vertices



# Empty Cycles

crossing-free cycle on $k$ vertices

no vertices on one side



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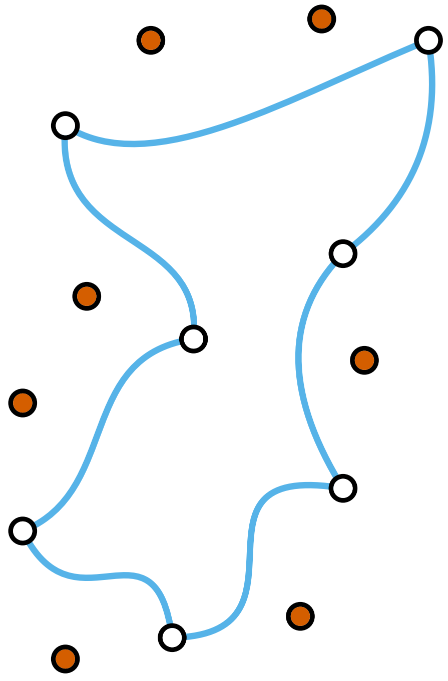

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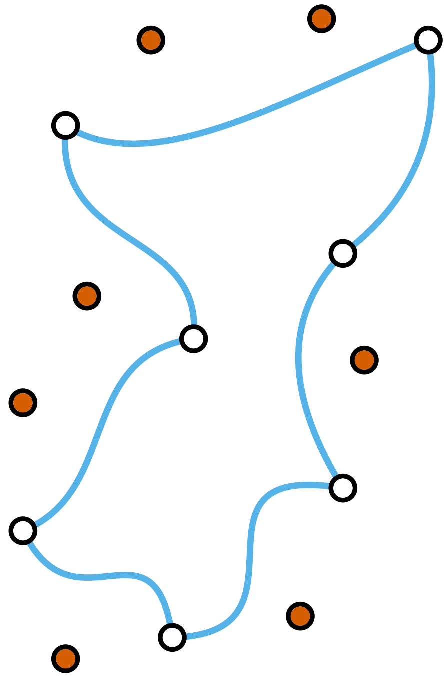

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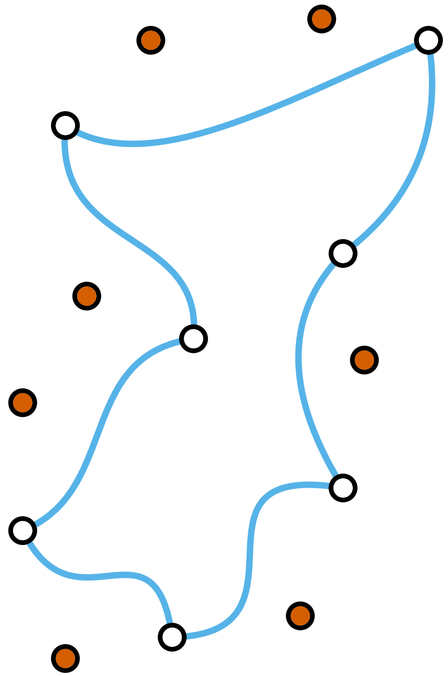
**Theorem:** Every vertex in a simple drawing of $K_n$ is incident to an empty $4$-cycle.



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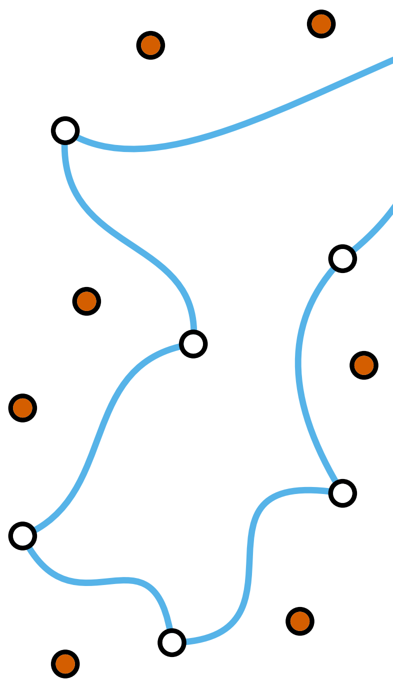

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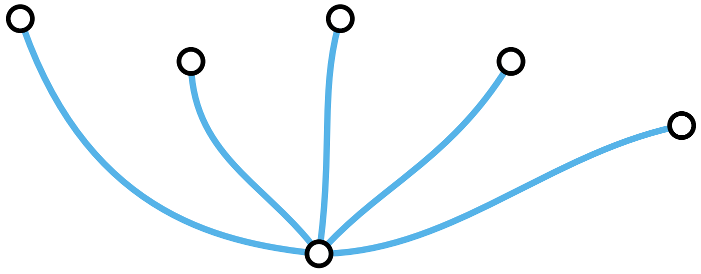
*proof idea:* start with spanning star



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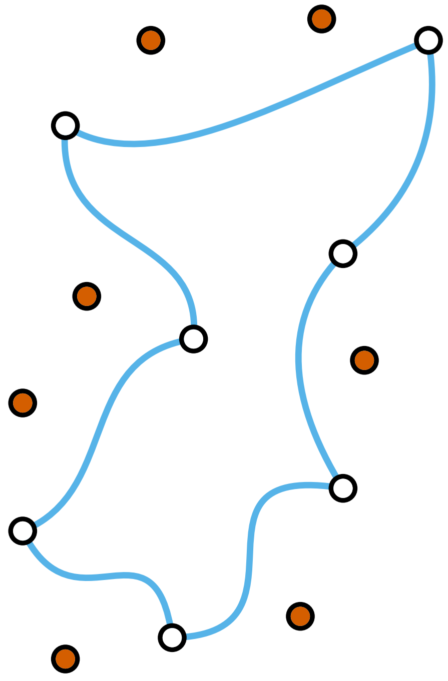

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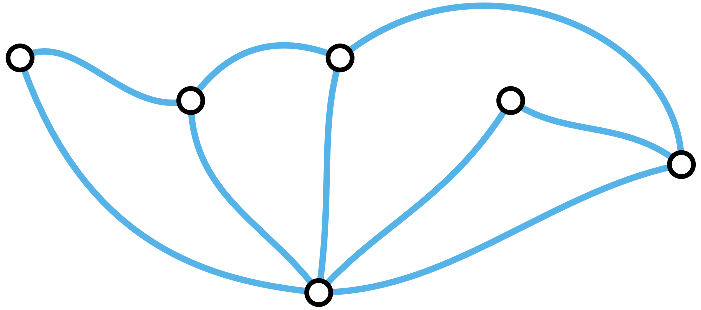
extend to maximal plane subdrawing

[García, Pilz, Tejel, 2021]



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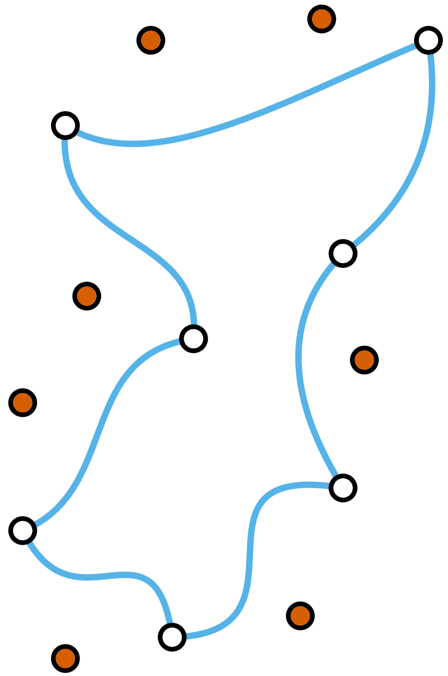

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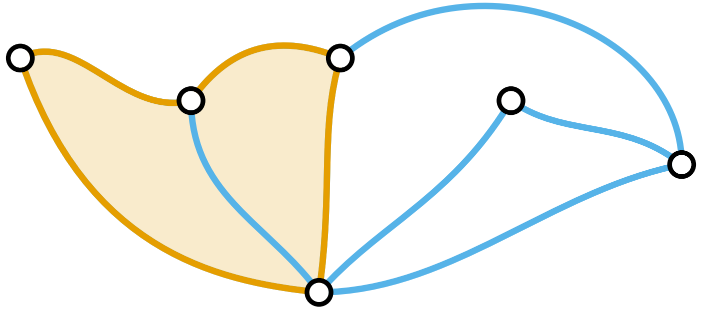
"Euler's formula" $\rightarrow$ empty $4$-cycle



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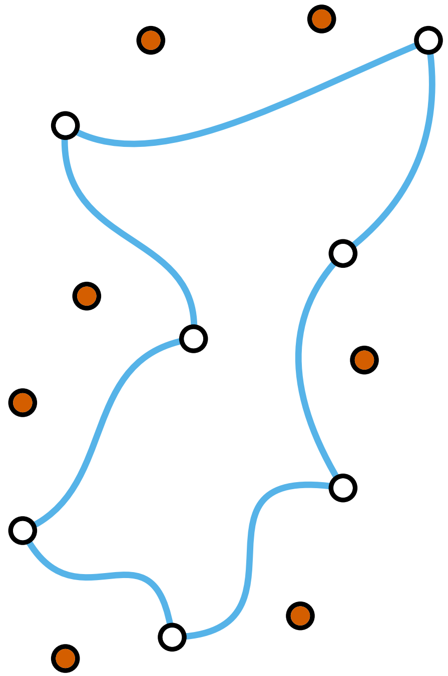

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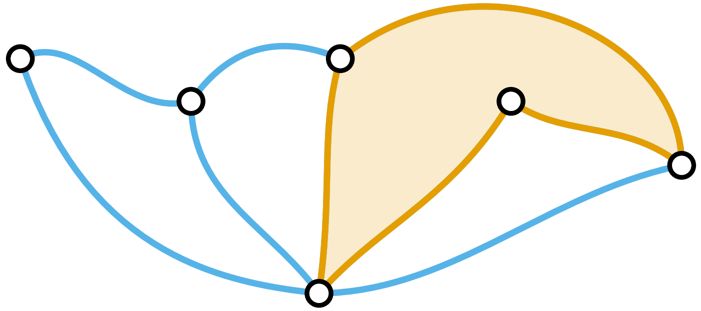

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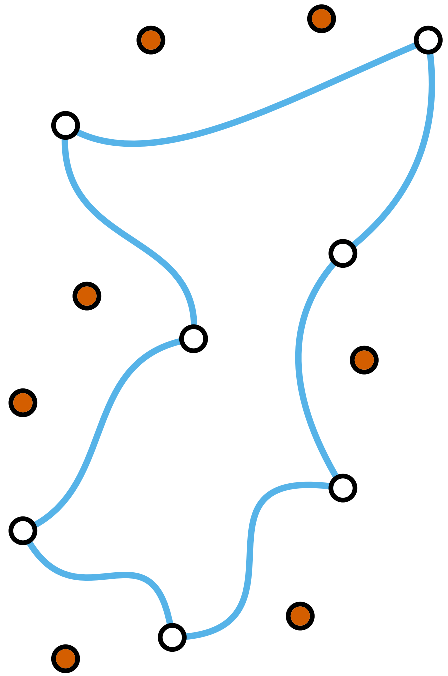
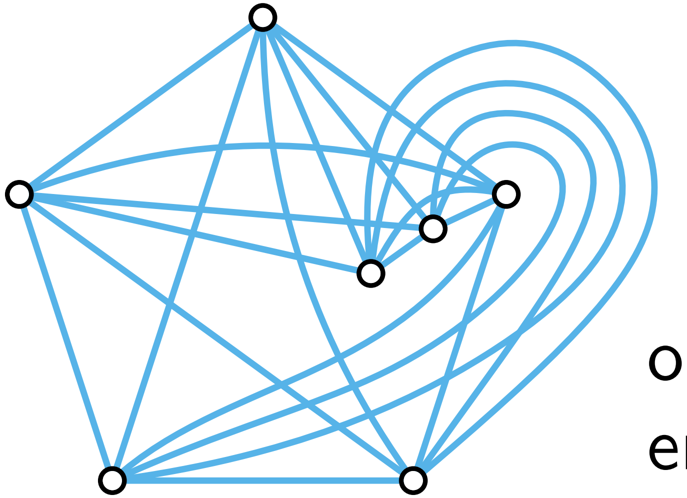
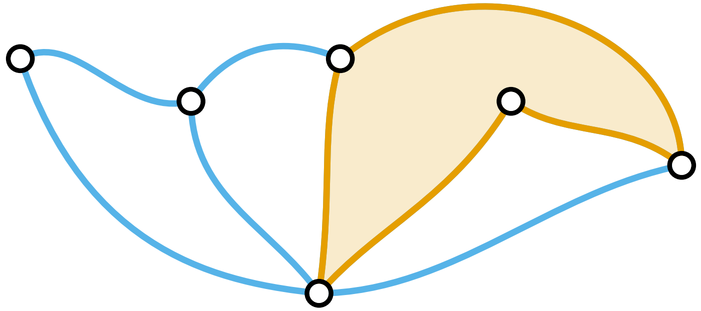




only $\mathcal{O}(n^2)$ empty $4$-cycles



# Conclusion

G-convex drawings: $6$-holes ✓

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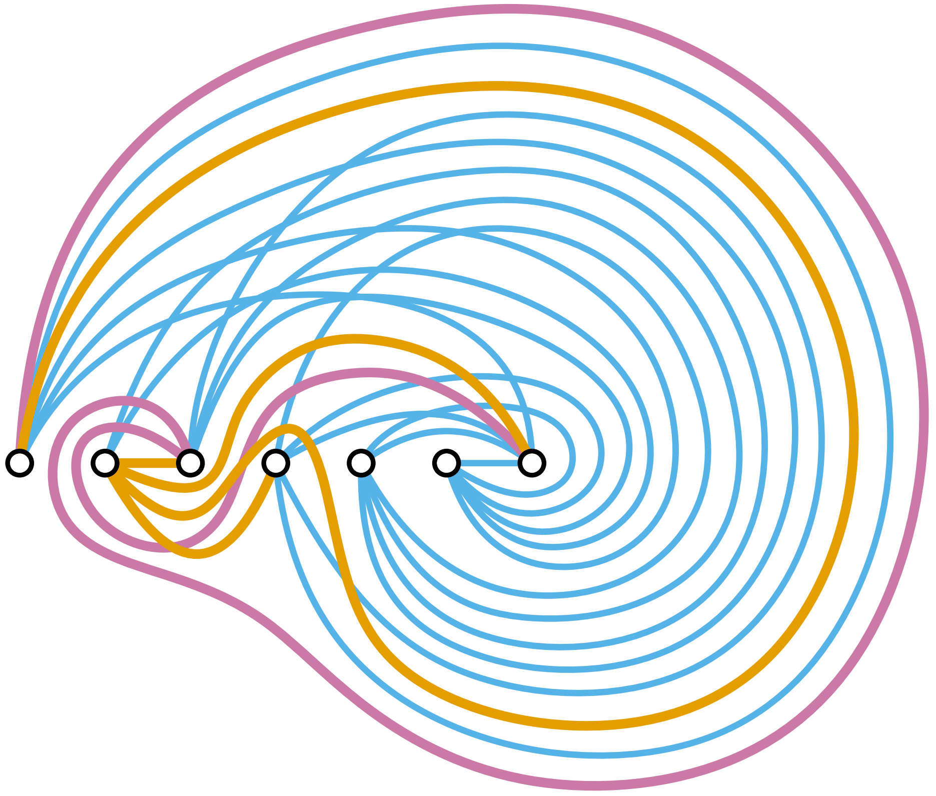
Simple drawings: $4$-holes ✗



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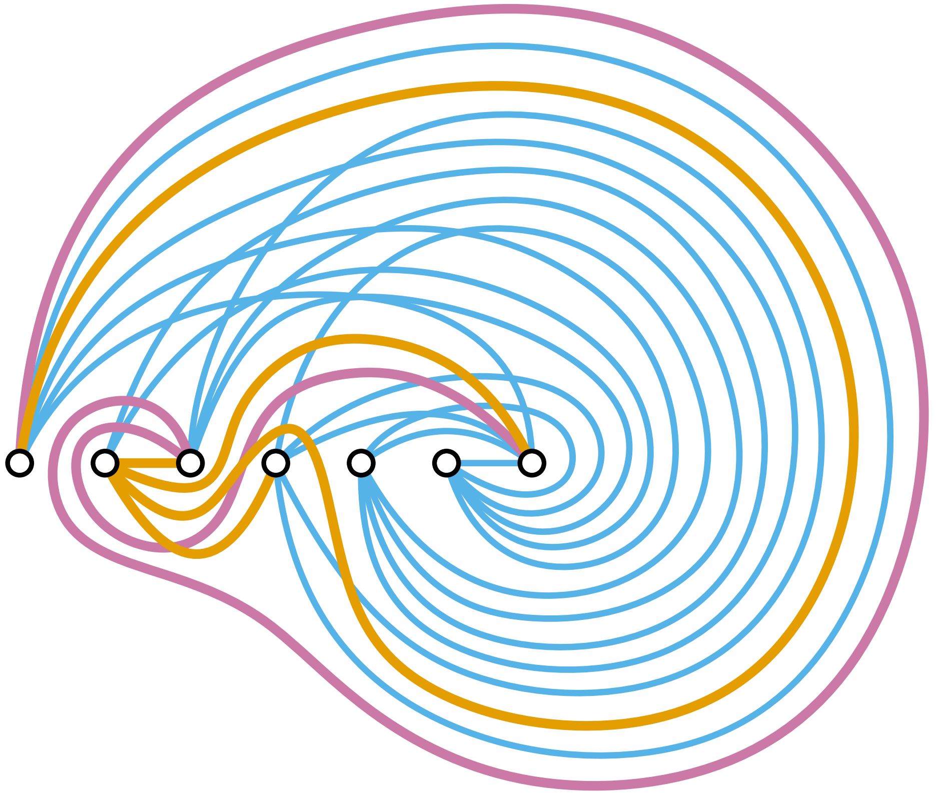

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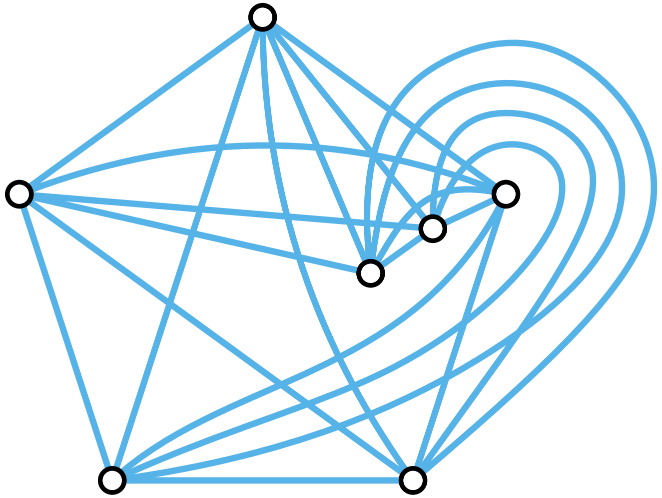
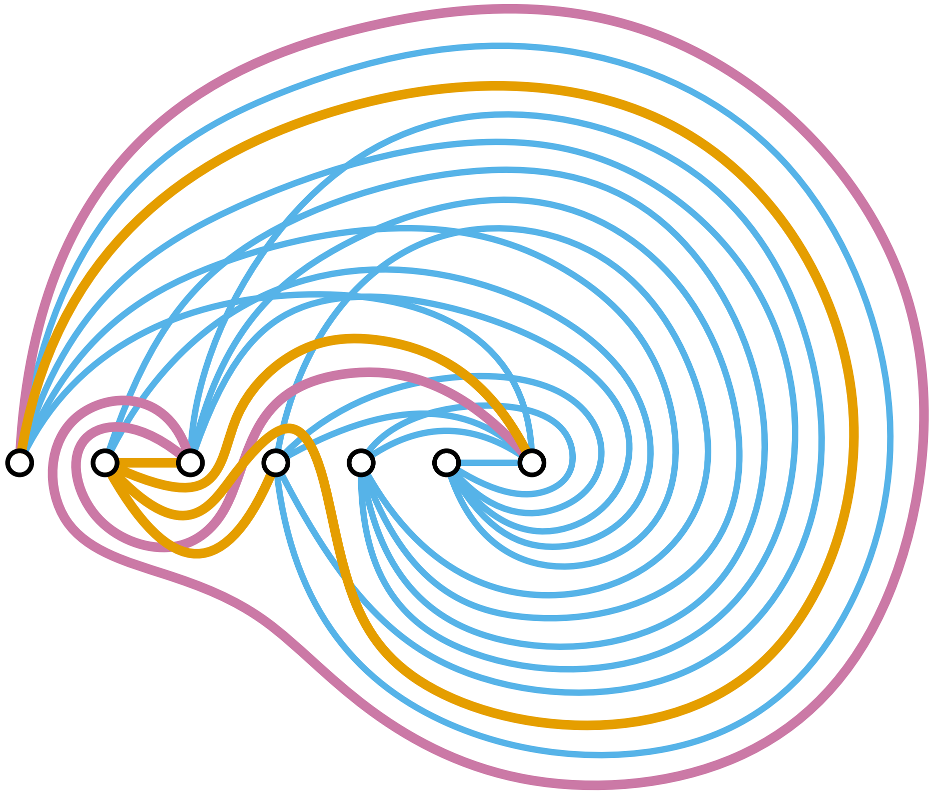


empty $4$-cycles ✓

**Question:** Is there a simple drawing with $\Theta(n)$ empty $4$-cycles?



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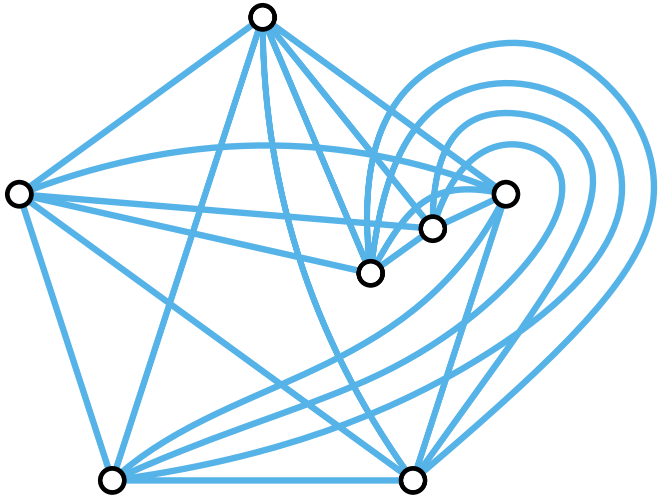

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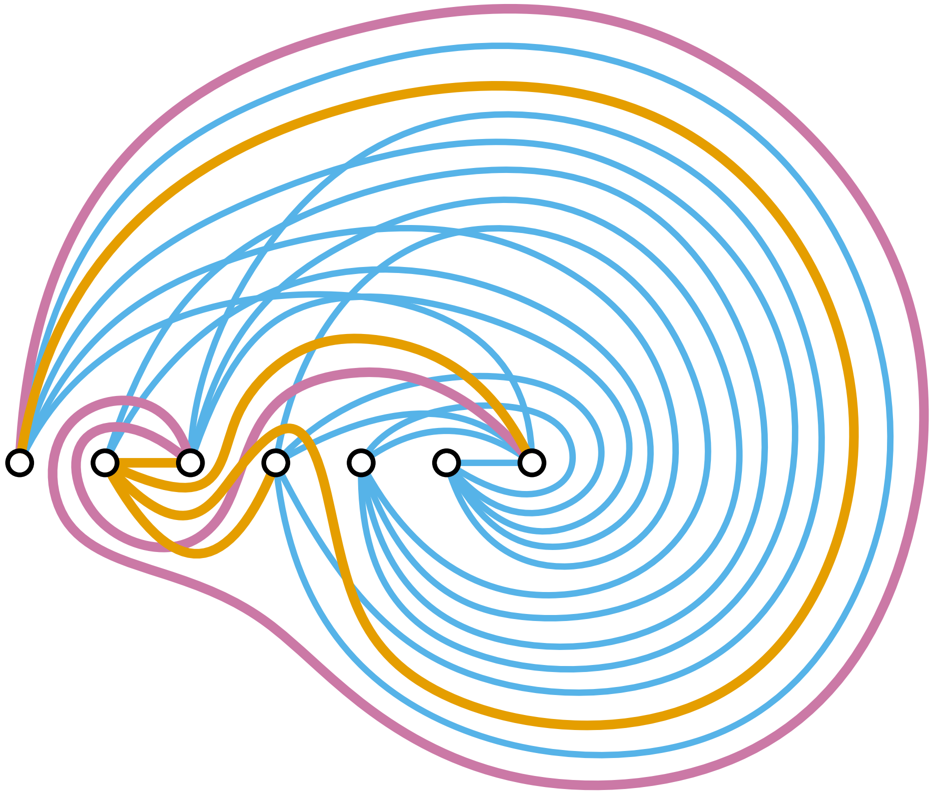


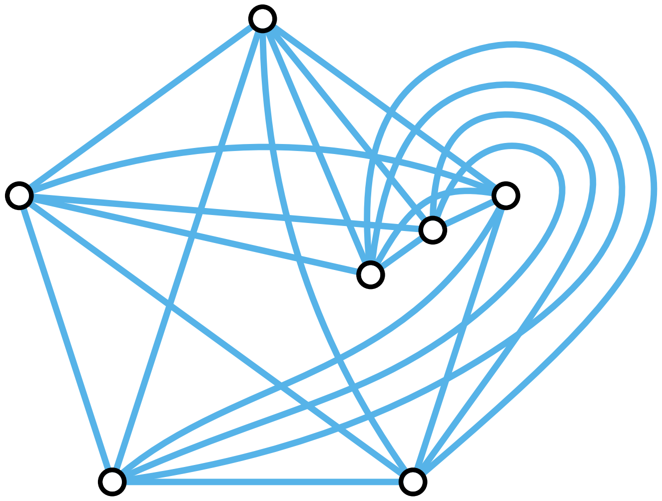
**Conjecture:** Every simple drawing of $K_n$ contains an empty $k$-cycle for each $3 \leq k \leq n$.

# Conclusion

G-convex drawings: $6$-holes ✓

Simple drawings: $4$-holes ✗

empty $4$-cycles ✓



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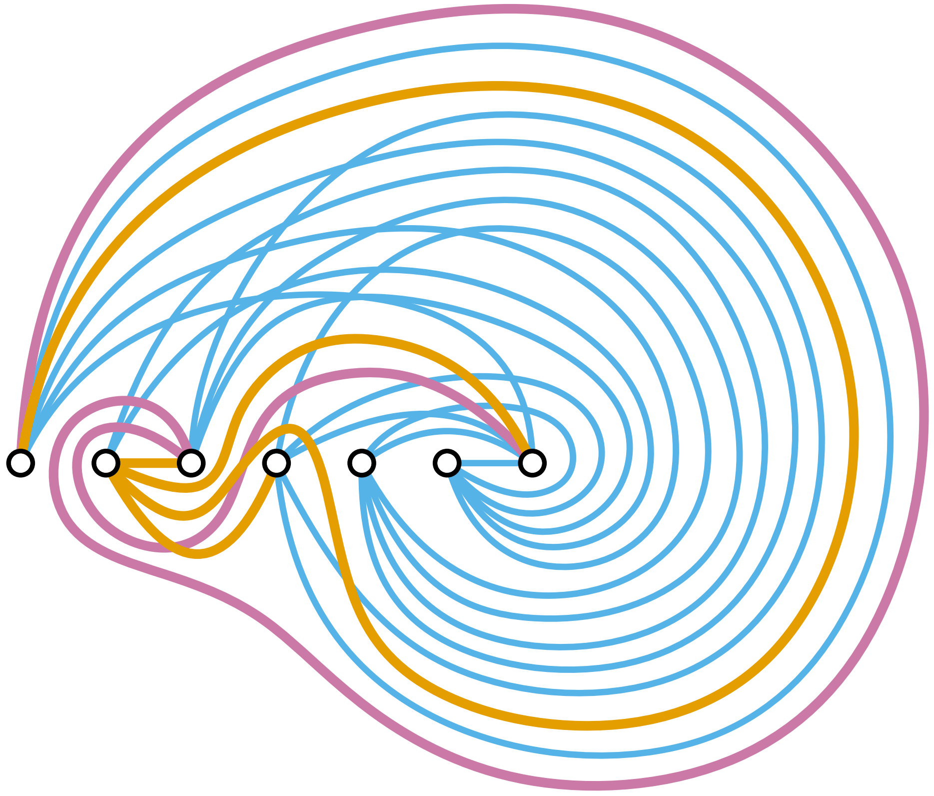

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**Thank you!**