Enumeration of intersection graphs of x-monotone curves

Andrew Suk (UC San Diego)

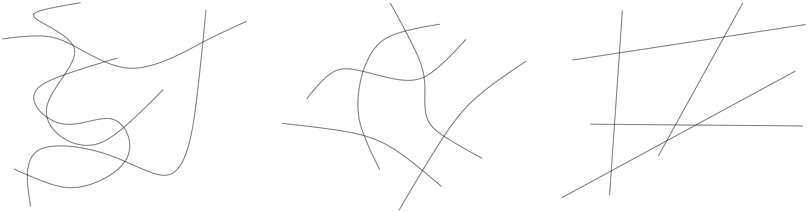
September 14, 2024

Coauthors

Jacob Fox and János Pach

Crossing patterns of curves

General curves vs. Pseudo-segments vs. Segments



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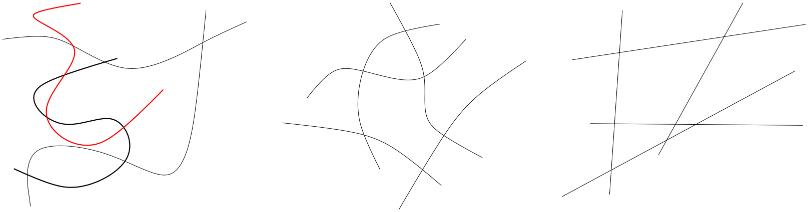

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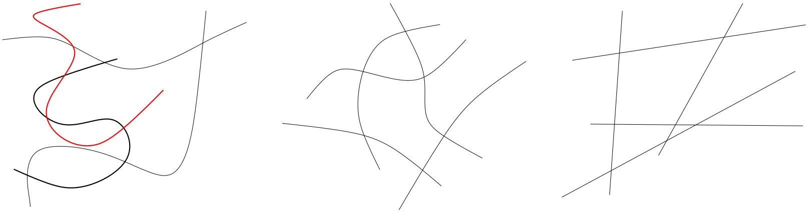

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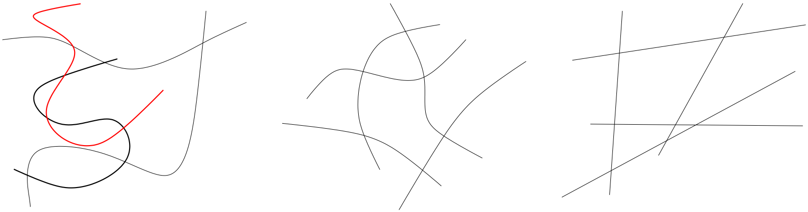

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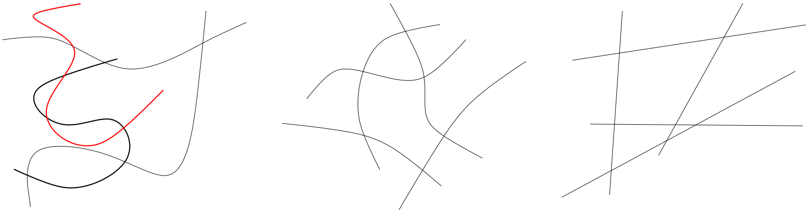

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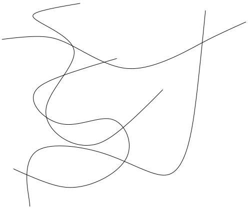
General curves (string graphs)



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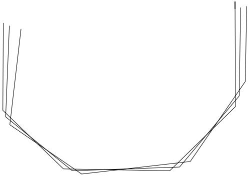

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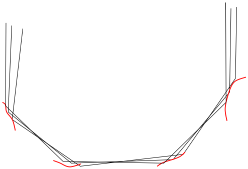

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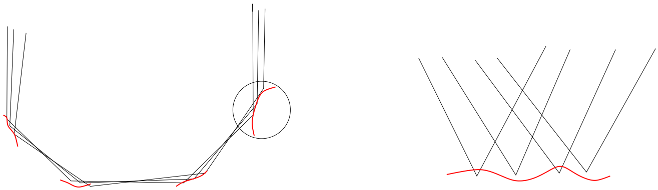

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Application of the Milnor-Thom theorem

Theorem (Pach-Solymosi, 2001)

$|\mathcal{S}_n| = 2^{O(n \log n)}$.

Pseudo-Segments: Old results

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x-monotone pseudo-segments

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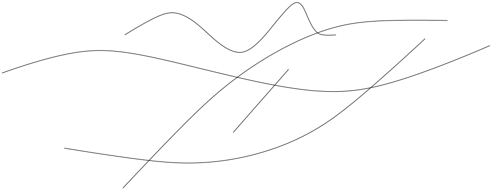

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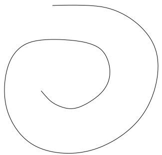
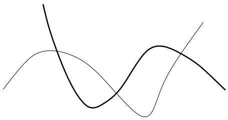
Not allowed





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Theorem (Fox-Pach-S., 2024+)

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Point-line incidences

Kyncl: $|\mathcal{P}_n| \leq 2^{O(n^{3/2}\log n)}$

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$P = n^{1/3} \times n^{2/3}$ grid $\qquad L = n$ lines

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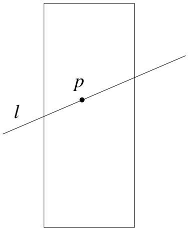

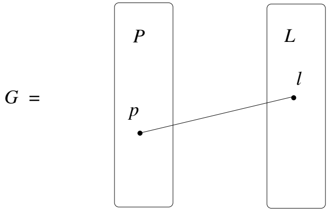

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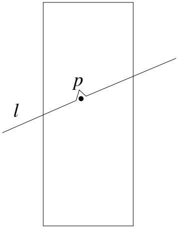

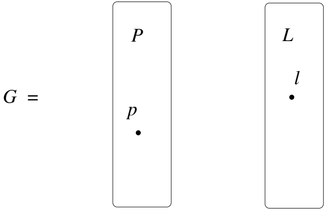

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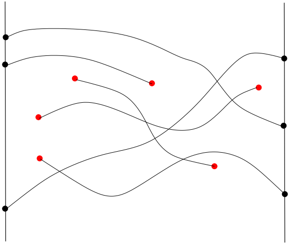
Sketch Proof: $f(n,p)$ be the number of labeled intersection graphs of at most n x-monotone pseudo-segments in a vertical strip, such that there are at most p endpoints inside the strip.



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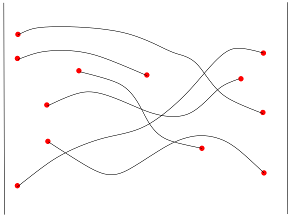
Goal: $f(n, 2n) \leq 2^{n^{3/2-\varepsilon}}$



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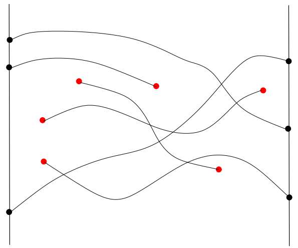

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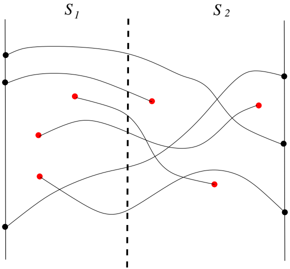

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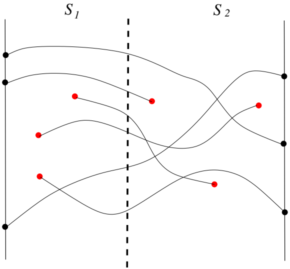

A_i be the curves in S_i that goes entirely through S_i.

B_i be the curves with at least 1 endpoint inside S_i.

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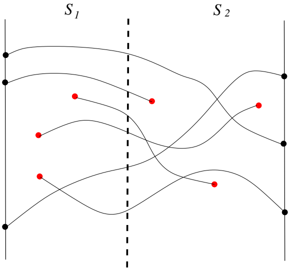

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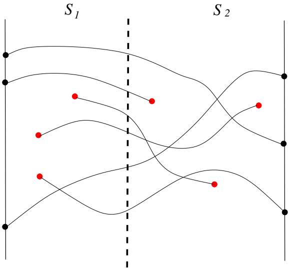

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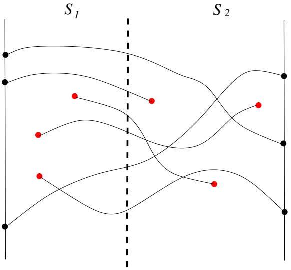
$$f(n,p) \leq n!$$

Sketch Proof



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$$f(n,p) \leq n!|A_1|! \cdot |A_2|!$$

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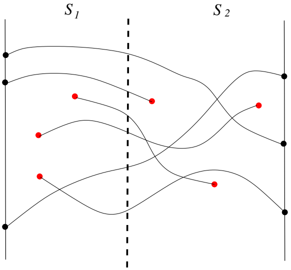

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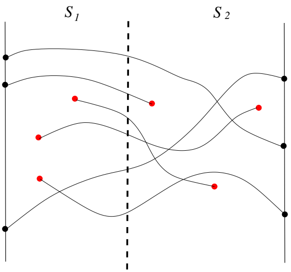

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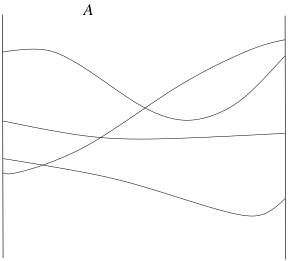

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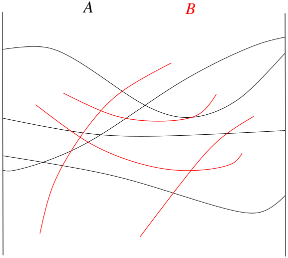

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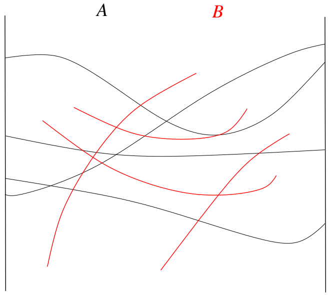

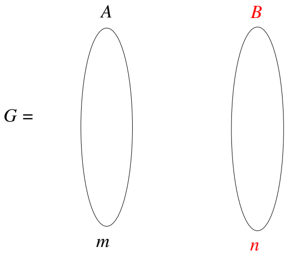

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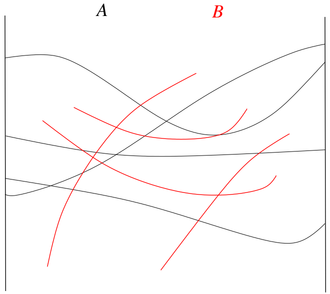
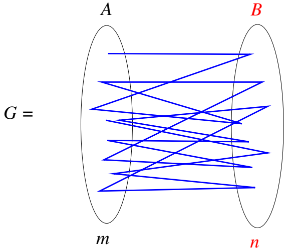
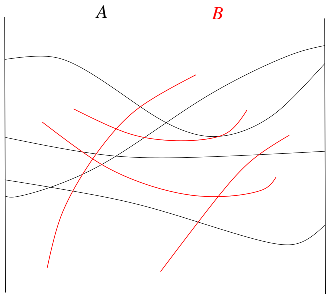


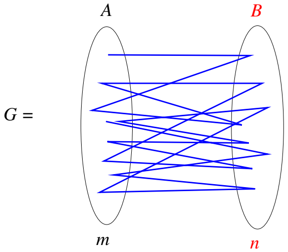


VC-dimension of intersection graphs of pseudo-segments

Theorem (Pach-Tóth, 2006)

Let G be the intersection graph of a family of n pseudo-segments in the plane. Then G has VC-dimension at most d, where d is an absolute constant. Here, d is at most a tower of 2's of height 8.

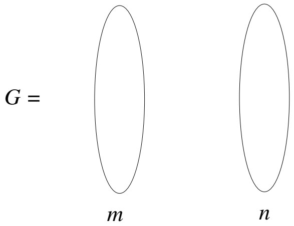




Lemma (Fox-Pach-S. 2024+, Alon-Moran-Yehudayoff 2017)

The number of bipartite graphs with parts of size m and n with VC-dimension at most d is at most

$$2^{O(m^{1-1/d} n \log m)}.$$



Bipartite graphs with bounded VC-dimension

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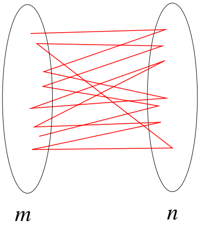
$G =$ 

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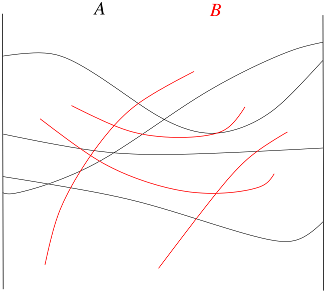

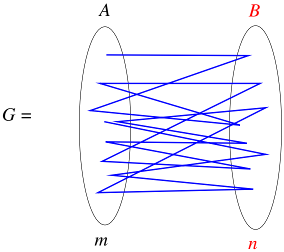

At most $2^{O(m^{1-1/d} n \log m)}$ intersection graphs between $\mathcal{A}$ and $\mathcal{B}$

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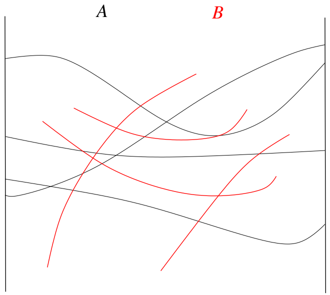

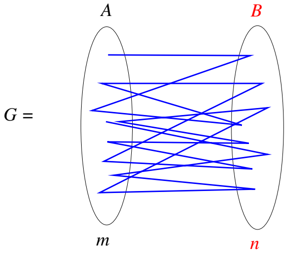

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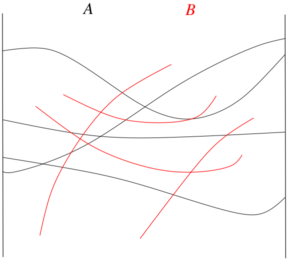
$$|\mathcal{A}| = |\mathcal{B}| = n \qquad 2^{O(n^{2-1/d} \log m)}$$

Cutting lemma structure

$|\mathcal{A}| = m$, $|\mathcal{B}| = n$



Cutting Lemma for $\mathcal{A}$ based on $r = \frac{m^{1/2+\varepsilon}}{n^{2\varepsilon}}$ curves.

Cutting lemma structure

$|\mathcal{A}| = m$

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$$(m!)^2$$

Cutting lemma structure

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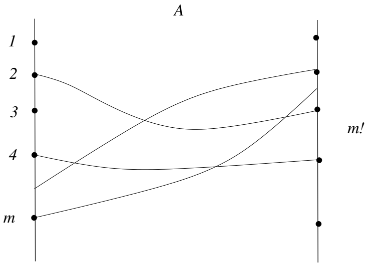

Cutting Lemma for $\mathcal{A}$ based on $r = \frac{m^{1/2+\varepsilon}}{n^{2\varepsilon}}$ curves.

$$(m!)^2 \cdot m^r \cdot 2^{r^2 \log r}$$

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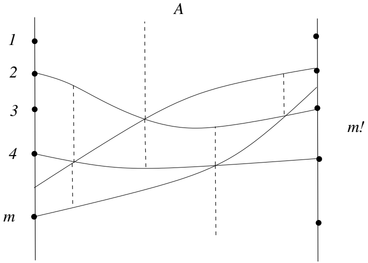

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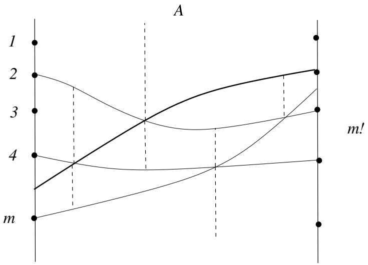

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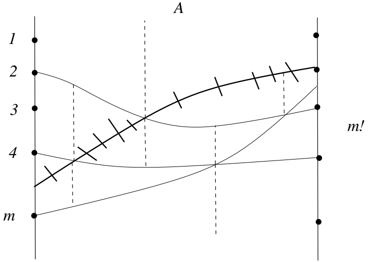

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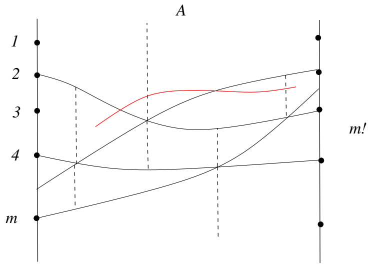
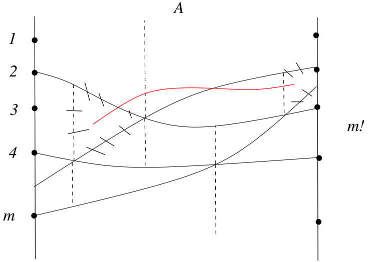


Cutting Lemma for $\mathcal{A}$ based on $r = \frac{m^{1/2+\varepsilon}}{n^{2\varepsilon}}$ curves.

$$(m!)^2 \cdot m^r \cdot 2^{r^2 \log r} \cdot m^{O(rm)} \cdot (r^4)$$

Cutting lemma structure

$|\mathcal{A}| = m$, $|\mathcal{B}| = n$,



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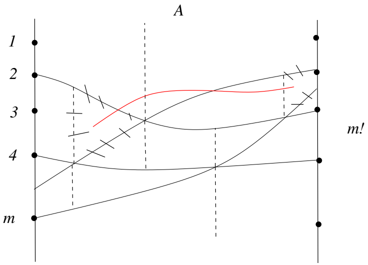

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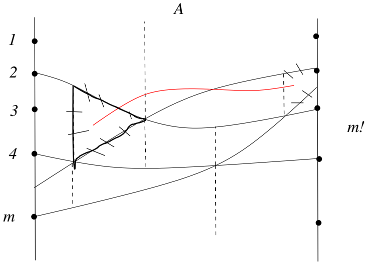

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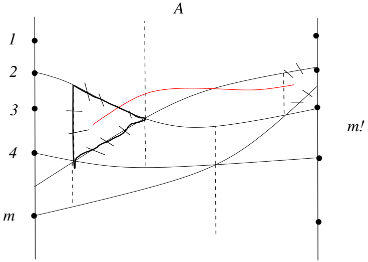

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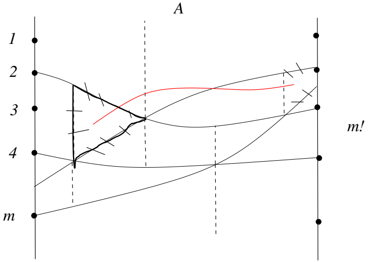
$$(m!)^2 \cdot m^r \cdot 2^{r^2 \log r} \cdot m^{O(rm)} \cdot (r^4)^n \prod 2^{O\left(\left(\frac{m}{r}\right)^{1-1/d} n_i \log m\right)}$$

Cutting lemma structure

$|\mathcal{A}| = m$, $|\mathcal{B}| = n$,



Cutting Lemma for $\mathcal{A}$ based on $r = \frac{m^{1/2+\varepsilon}}{n^{2\varepsilon}}$ curves.

$$2^{O(n^{d/(2d-1)} m^{(2d-2)/(2d-1)} \log^2 m)} + 2^{O(n^{3/2-1/d} \log n)} + 2^{O(m \log^3 m)}$$

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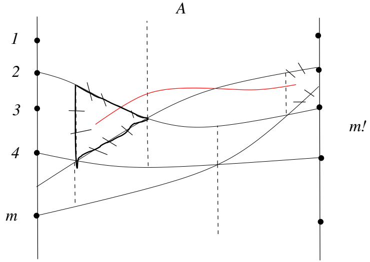

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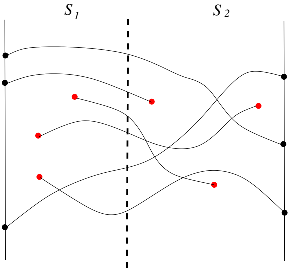
$$2^{O(n^{3/2-1/(4d-2)}\log^2 m)}$$

Sketch Proof

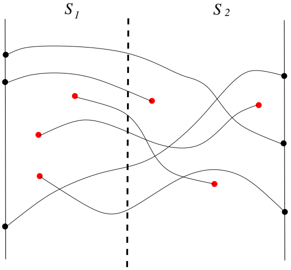


A_i be the curves in S_i that goes entirely through S_i.

B_i be the curves with at least 1 endpoint inside S_i.

$$f(n,p) \leq n!|A_1|! \cdot |A_2|! f(p/2, p/2) \cdot f(p/2, p/2) \cdot (????)^2.$$

Sketch Proof



A_i be the curves in S_i that goes entirely through S_i.

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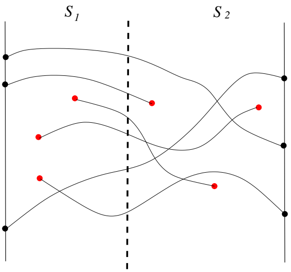
$$f(n,p) \leq (n!)^3 f^2(p/2, p/2) \cdot 2^{O(n^{3/2-1/(4d-2)}\log^2 m)}.$$

Sketch Proof



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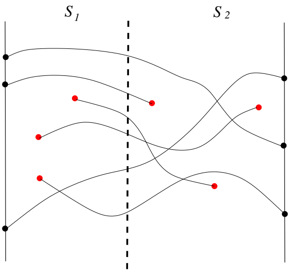

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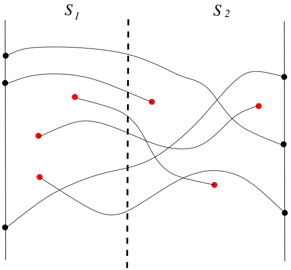
$$f(n, 2n) \leq 2^{O\left(n^{3/2-1/(4d-2)} \log^2 n\right)}.$$

Sketch Proof



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$$|\mathcal{P}_n^{mono}| \leq f(n, 2n) \leq 2^{O\left(n^{3/2-1/(4d-2)} \log^2 n\right)}.$$

$\square$

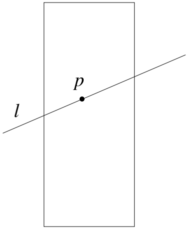
Open problems: Incidences between points and 2-intersecting curves

Theorem (Fox-Pach-S., 2024+)

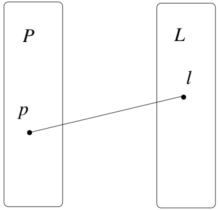
$$2^{\Omega(n^{4/3})} < |\mathcal{P}_n^{mono}| \leq |\mathcal{P}_n| \leq 2^{O(n^{3/2}\log n)}.$$

$P = n^{1/3} \times n^{2/3}$ grid $\qquad L = n$ lines

$|I(P, L)| = \Theta(n^{4/3})$



$G =$



Open problems: Incidences between points and 2-intersecting curves

Problem

Can we improve

$$2^{\Omega(n^{4/3})} \leq \mathcal{P}_n^{(2)}$$

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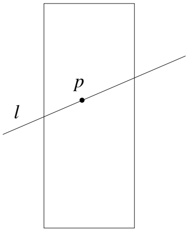

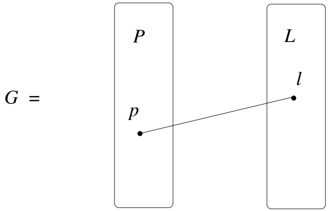

Open problems: Incidences between points and 2-intersecting curves

Problem

What is the maximum number of incidences between n points and n 2-intersecting curves in the plane?

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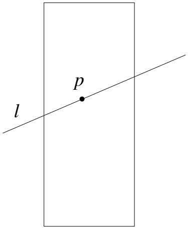

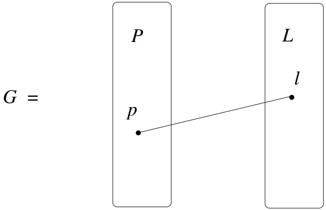

Open problems: Incidences between points and 2-intersecting curves

Problem

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$P = n$ points $\qquad$ $L = n$ 2-intersecting curves

Pach-Sharir, 1998

$$\Omega(n^{4/3}) \leq |I(P, L)| = O(n^{7/5})$$

Open problems: Incidences between points and k-intersecting curves

Problem

What is the maximum number of incidences between n points and n k-intersecting curves in the plane?

$P = n$ points $\qquad$ $L = n$ k-intersecting curves

Pach-Sharir, 1998

$\Omega(n^{4/3}) \leq |I(P, L)| = O(n^{\frac{3k-2}{2k-1}})$

Open problems: Incidences between points and k-intersecting curves

Problem

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Application:

$$2^{\Omega(n^{4/3})} < |\mathcal{P}_n^{(k)}| < 2^{O(n^{2-\varepsilon})}.$$

Thank you!

VC-dimension of graphs

Set system $\mathcal{F} \subset 2^V$, $|V| = n$.

Definition

A set $S \subset V$ is **shattered** by $\mathcal{F}$ if for all $X \subset S$, there is an $A \in \mathcal{F}$ such that $S \cap A = X$.

$\mathcal{F}$ = $\boxed{}$

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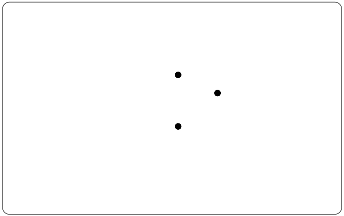
$\mathcal{F} =$ 

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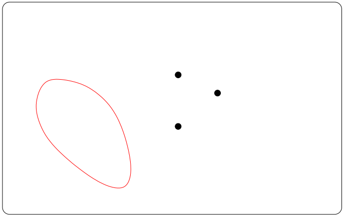

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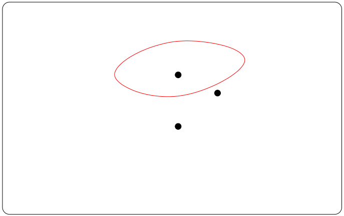
$\mathcal{F} =$ 

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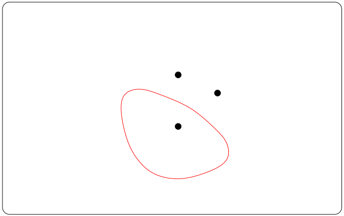

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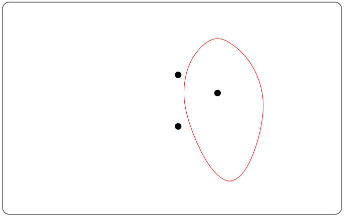
$\mathcal{F} =$ 

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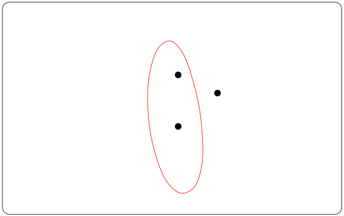
$\mathcal{F} =$ 

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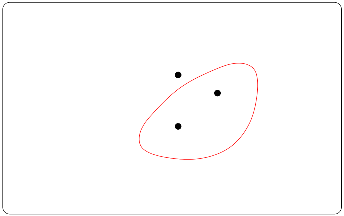
$\mathcal{F} =$ 

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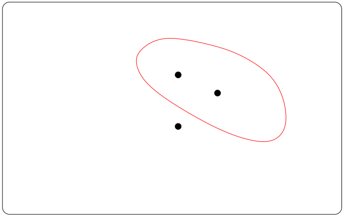
$\mathcal{F} =$ 

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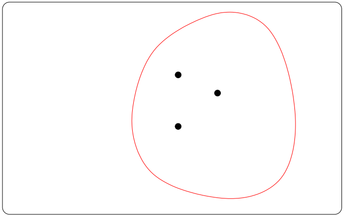
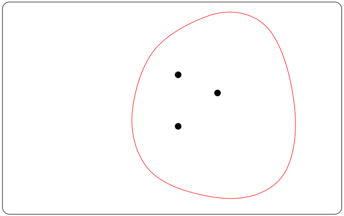


Definition: VC-dimension

Set system $\mathcal{F} \subset 2^V$, $|V| = n$.

Definition

The **VC-dimension of** $\mathcal{F}$ is the size of the largest subset $S \subset V$ that is shattered by $\mathcal{F}$.

$\mathcal{F} =$ 

$G = (V, E)$, let $\mathcal{F} \subset 2^V$ such that $\mathcal{F} = \{N(v) : v \in V\}$.
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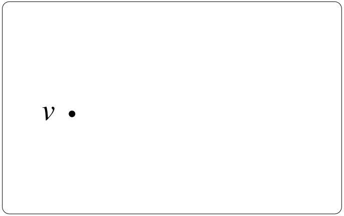
$G =$ 

VC-dimension of a graph

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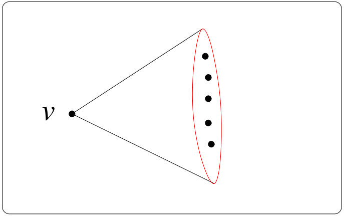

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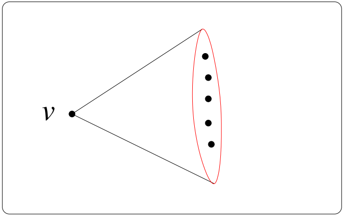
$G =$ 

Definition

The VC-dimension of G is the VC-dimension of $\mathcal{F}$.