Rectilinear Crossing Number of ...

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ON THE NUMBER OF CROSSINGS IN A COMPLETE GRAPH

by FRANK HARARY †‡ and ANTHONY HILL
(Received 12th August 1963)

1. Introduction

The purpose of this article is to describe two problems which involve drawing graphs in the plane. We will discuss both complete graphs and complete bicoloured graphs. The *complete graph* K_n with n points or vertices has a line or edge joining every pair of distinct points, as shown in fig. 1 for $n = 2, 3, 4, 5, 6$.

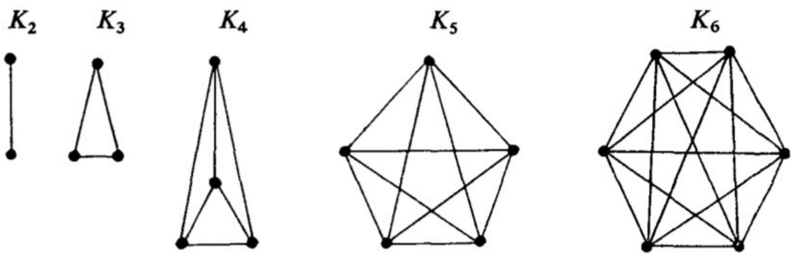

FIG. 1

In each of these complete graphs every edge is a straight line segment. In K_2, K_3 and K_4, no two distinct edges intersect. As anyone can plainly see, the number of intersections or crossings in K_5 as drawn in 5 and in K_6 is 15. It is stipulated that such an intersection involves only two edges and not more.

For any graph G, we say that the *crossing number* $c(G)$ is the minimum number of crossings with which it is possible to draw G in the plane. We note that the edges of G need not be straight line segments, and also that the result is the same whether G is drawn in the plane or on the surface of a sphere. Another invariant of G is the *rectilinear crossing number*, $\bar{c}(G)$, which is the minimum number of crossings when G is drawn in the plane in such a way that every edge is a straight line segment. We will find by an example that

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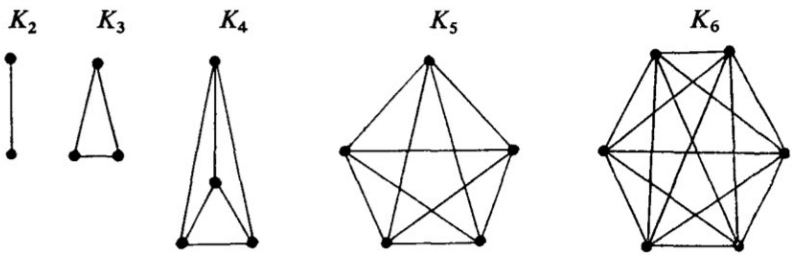

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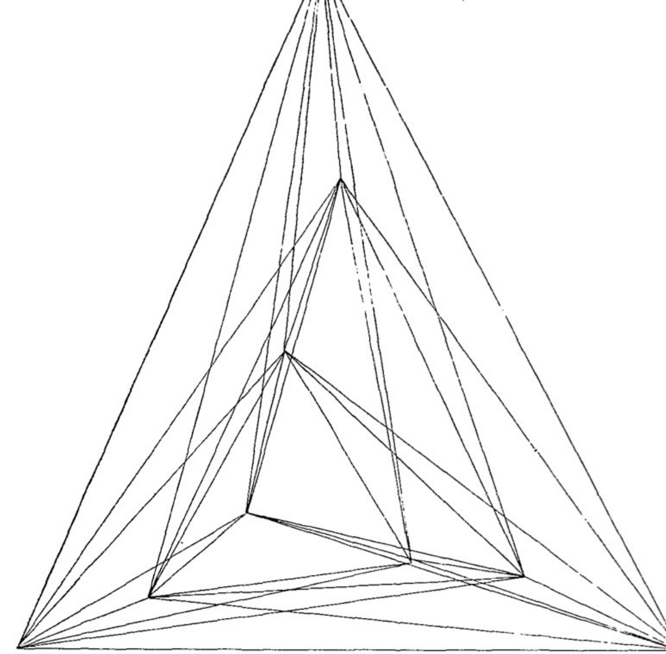


FIG. 7

an upper bound, followed by assertions rather than proofs that they are also a lower bound.

In general, it would be interesting to express for any graph G the numbers $c(G)$ and $\bar{c}(G)$ in terms of other invariants of G.

REFERENCES

(1) I. FÁRY, On straight line representation of planar graphs, *Acta Univ. Szeged*, **11** (1948), 229-233.

(2) R. GUY, A combinatorial problem, *Bull. Malayan Math. Soc.*, **7** (1960), 68-72.

(3) C. KURATOWSKI, Sur le problème des courbes gauches en Topologie, *Fund. Math.*, **16** (1930), 271-283.

(4) K. ZARANKIEWICZ, On a problem of P. Turán concerning graphs, *Fund. Math.*, **41** (1954), 137-145.

graph invariants

th: [Wagner'48, Fáry'48, Stein'51]

For every planar graph G

$$cr(G) = \overline{cr}(G) = \emptyset$$

graph invariants

Th: [Wagner'48, Fáry'48, Stein'51]
For every planar graph G
$$\mathrm{cr}(G) = \overline{\mathrm{cr}}(G) = \emptyset$$

- What about other graph classes?

graph invariants

th: For every planar graph G

$$cr(G) = \overline{cr}(G) = \emptyset$$

obvious: $\Theta(m^2)$

- What about other graph classes?
- What are "good" classes, with "small" cr and $\overline{cr}$?

good graph classes?

$\Omega\left(\frac{m^3}{n^2}\right)$

Crossing lemma $\Longrightarrow$ $\Theta(n)$ best we can hope for.

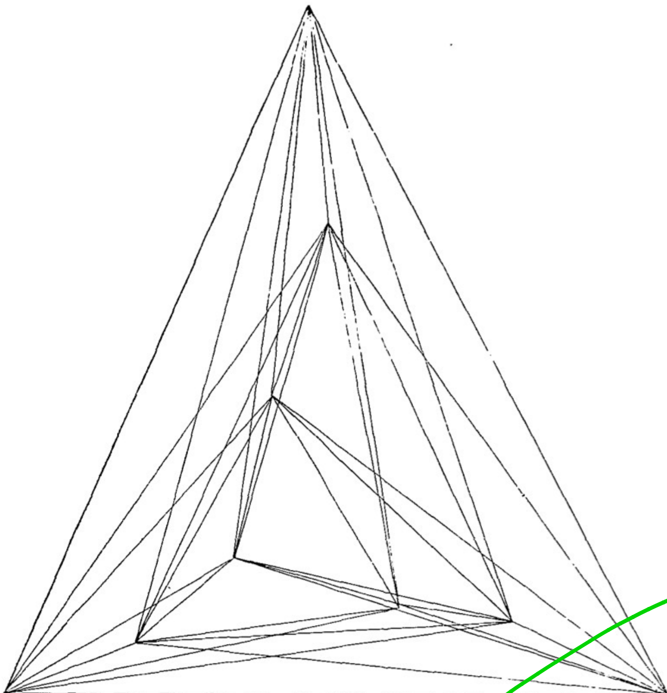
good graph classes

→ those with $\Theta(n)$ (rectilinear) crossing number

CANDIDATES: only classes with $\Theta(n)$ EDGES

- bounded degree → random cubic → $\Omega(n^2)$
- minor closed classes → K_5 → $\Omega(n^2)$

  → n-vertex subdivision



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In general, it would be interesting to express for any graph G the numbers $c(G)$ and $\bar{c}(G)$ in terms of other invariants of G.

CANDIDATE INVARIANT:
minor closed
and bounded degree

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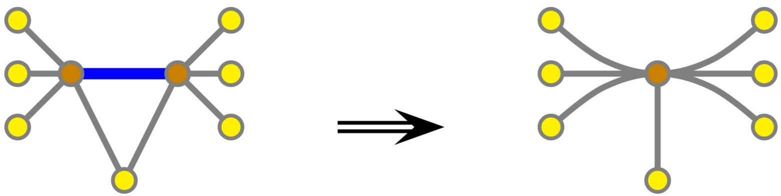
graph minors

EDGE CONTRACTIONS :



H is a minor of **G**

$\stackrel{\text{def}}{=}$ **H** can be obtained from a subgraph of **G** by contracting edges

minor closed properties:

planarity, having genus γ, being knotless, being linkless, having treewidth **k** . . .

good graph classes

those with $\Theta(n)$ (rectilinear) crossing number

CANDIDATES: minor-closed bounded-degree classes?

good graph classes

those with $\Theta(n)$ (rectilinear) crossing number

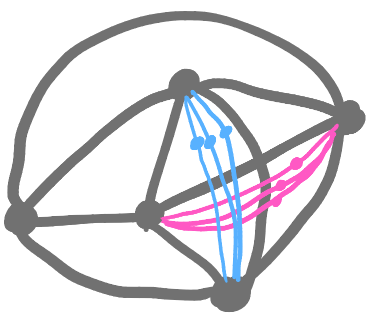
CANDIDATES: minor-closed bounded-degree classes?

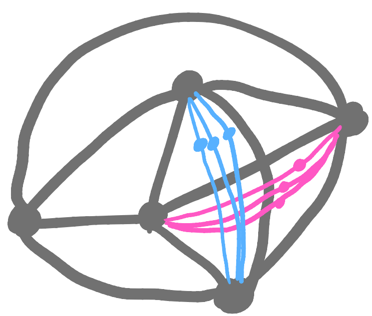
Yes, for CR

Good graph classes

Th: [D., Kawarabayashi, Mohar, Wood 2008]

For every graph G from a proper minor-closed class of graphs $\mathrm{cr}(G) \in \Theta(\Delta \cdot n)$.





...

Good graph classes

KNOW:

Proper minor closed classes have $\Theta(n)$ cross. num.

Good graph classes

KNOW: Proper minor closed classes have $\Theta(n)$ cross. num.

What about rectilinear crossing number?

good graph classes

KNOW: Proper minor closed classes have $\Theta(n)$ cross. num.

what about rectilinear crossing number?

[Bienstock & Dean '93]

$CR(G) = 4 \quad \overline{CR}(G) \nearrow$

good graph classes

KNOW: Proper minor closed classes have $O(n)$ cross. num.

What about rectilinear crossing number?

[Bhatt & Leighton '84]

$$\overline{cr}(G) \leq c \cdot \lg n \left(cr(G) + \sum d_i^2 \right)$$

good graph classes

KNOW: Proper minor closed classes have $O(n)$ cross. num.

What about rectilinear crossing number?

KNOW: $O(n \cdot \lg n)$

QUESTION:

KNOW:
Proper minor closed classes have $\Theta(n)$ cross. num.

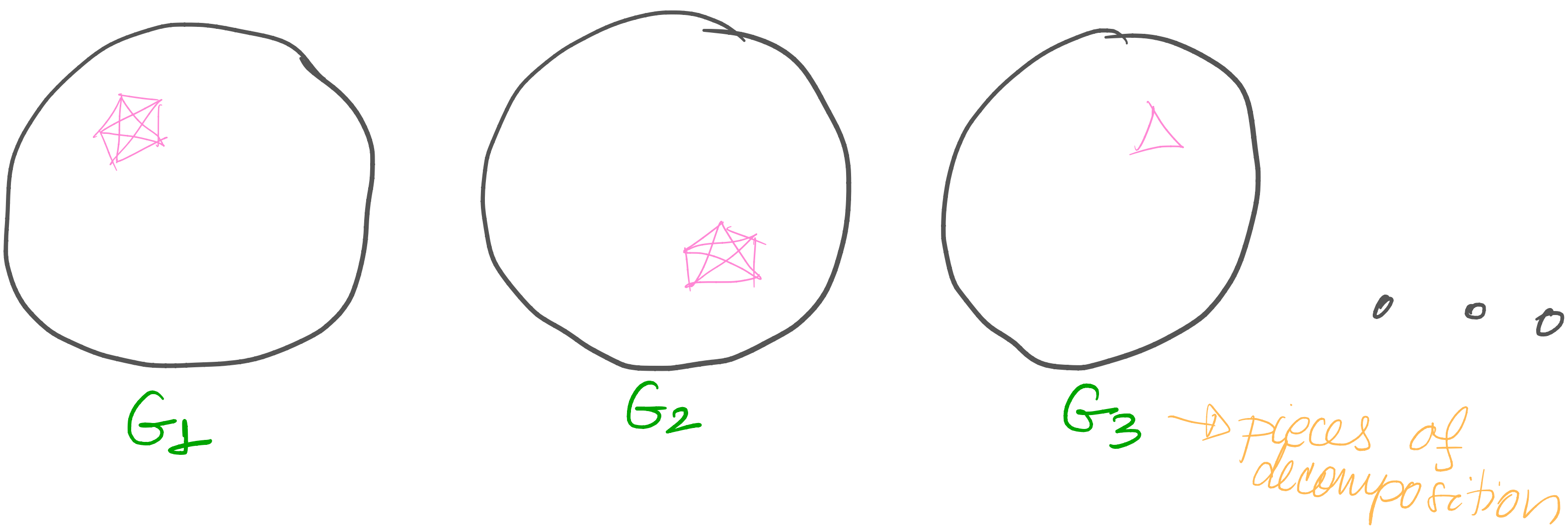
WANT:
For every graph G from a proper minor-closed class of graphs $\overline{cr}(G) \in f(\Delta) \cdot n$

not known for much simpler graph classes

What do these graphs look like?

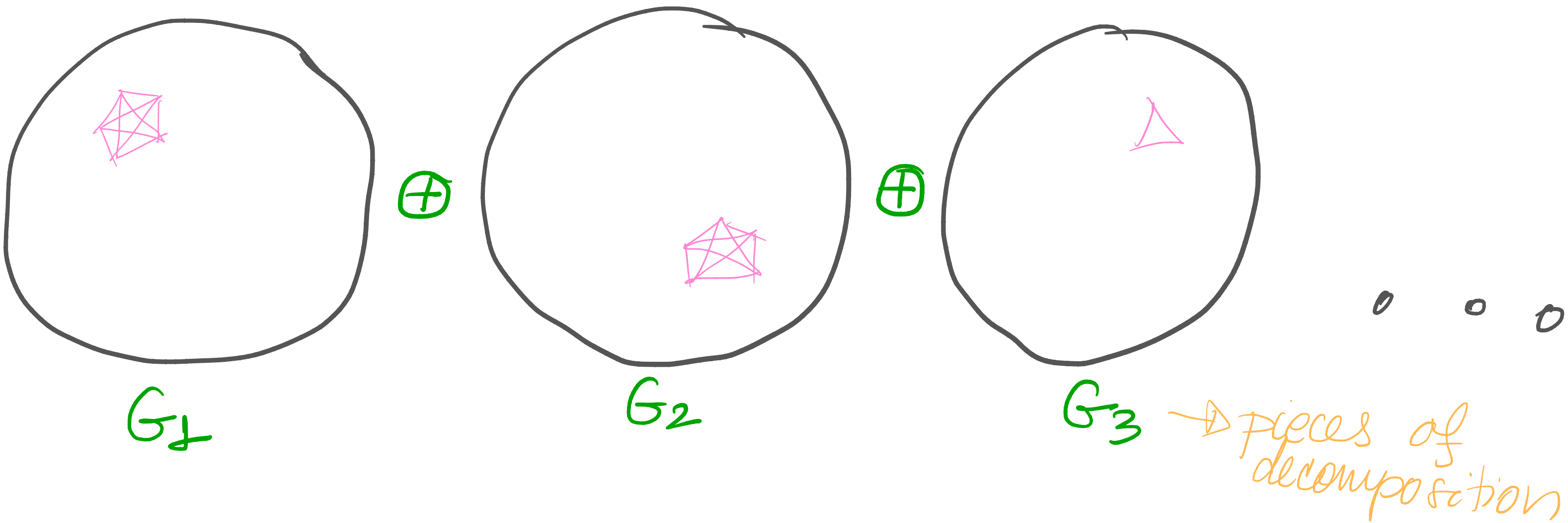
Proper minor-closed classes

Structure theorem [Robertson & Seymour]



Proper minor-closed classes

Structure theorem [Robertson & Seymour]



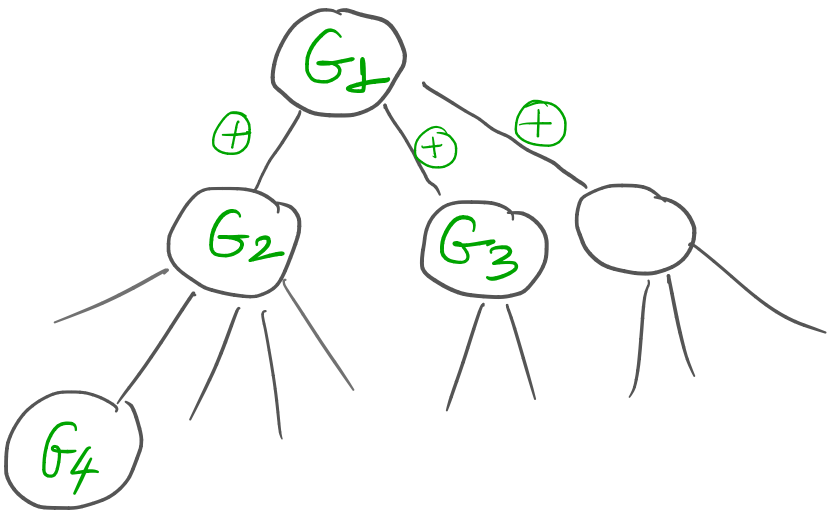
G can be obtained by $\oplus$ of $G_1, G_2, G_3, \dots$

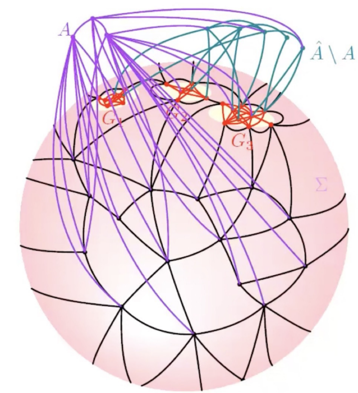
↳ K_n-minor free

↳ clique sums

Proper minor-closed classes

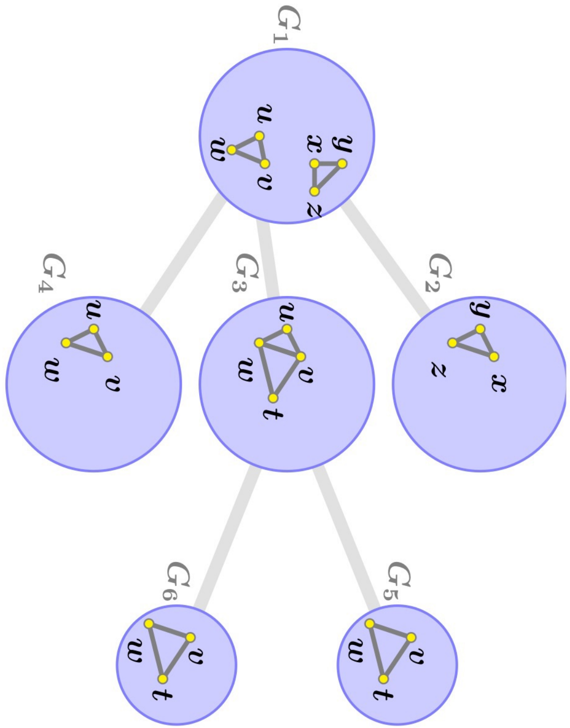
Structure theorem [Robertson & Seymour]



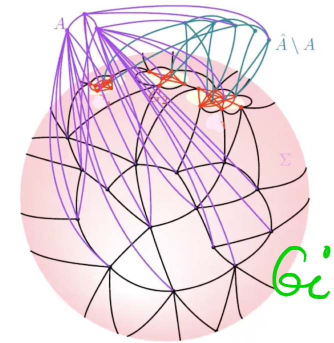


G_i : almost embeddable

$\oplus$: clique sums



approach



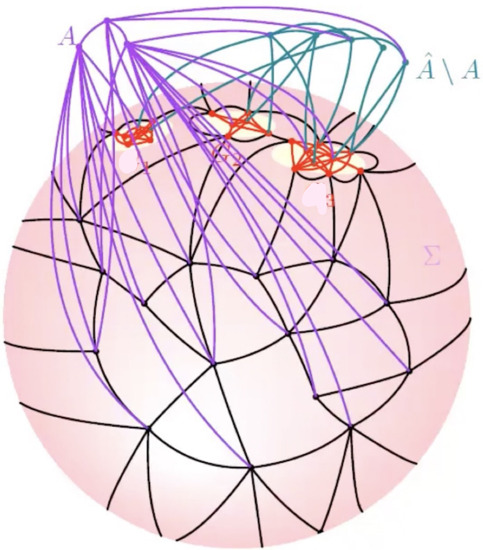
NEED:

1. $\overline{cr}(G_i) \le f(\Delta) \cdot n_i$

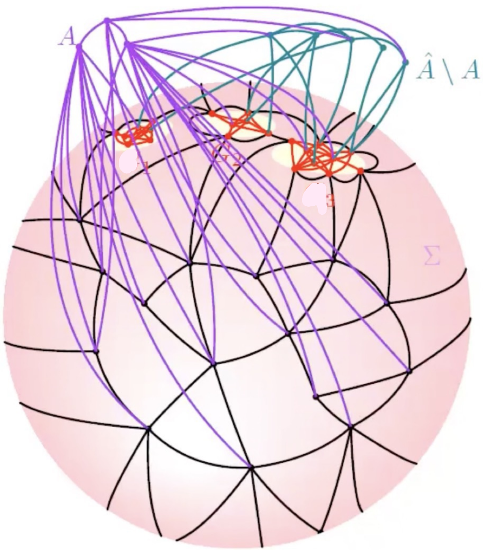
   $\hookrightarrow \sum \overline{cr}(G_i) = f(\Delta) \cdot n$

2. clique sums not the sum much

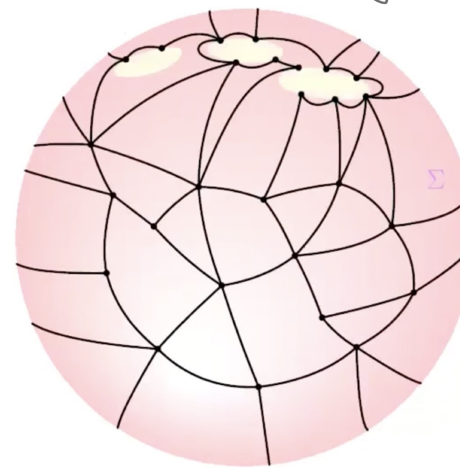
G: almost embeddable graphs



G_i: almost embeddable graphs



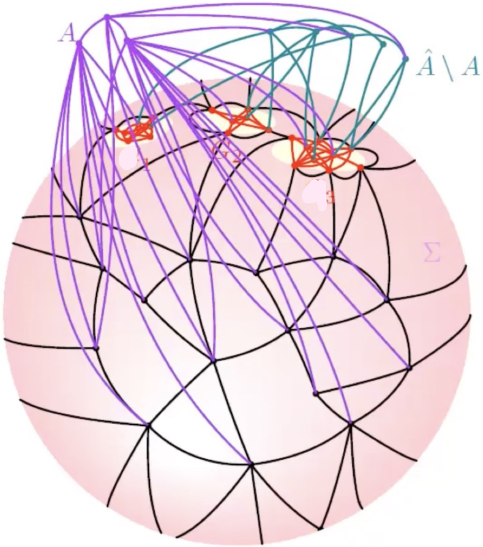
bounded genus



Böröczky, Pach, Tóth, 2006:

$$cr(G_i) \le c \cdot \Delta \cdot n$$

G_i: almost embeddable graphs



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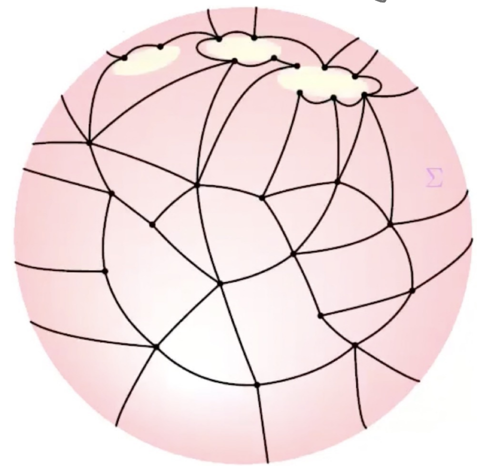

$$\overline{CR}(G_i) \leq ??$$

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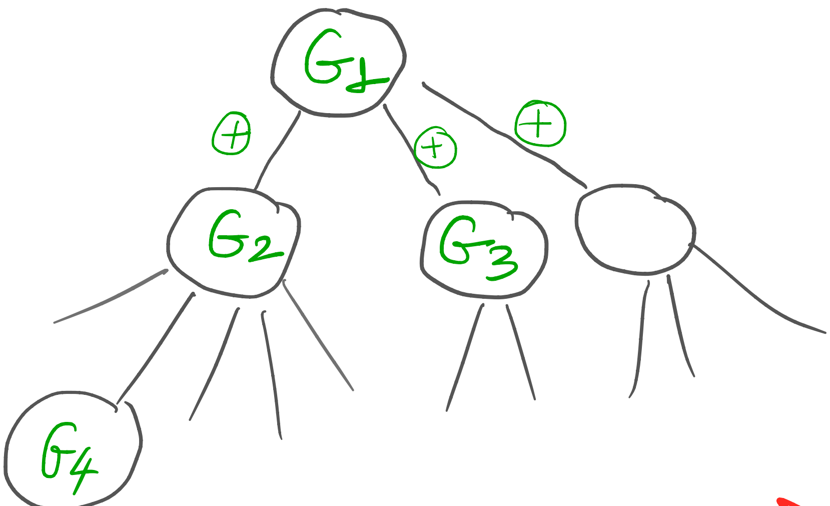
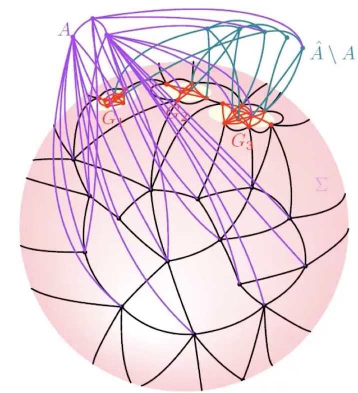
Proper minor-closed classes

Structure theorem [Robertson & Seymour]





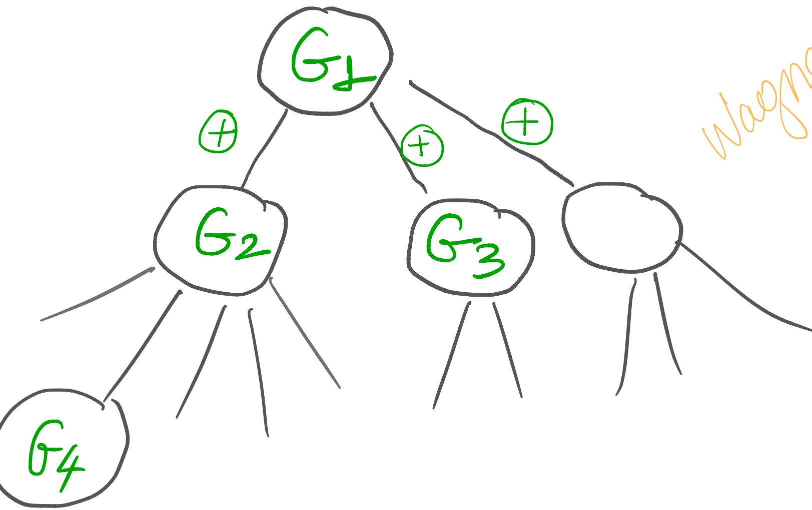
G_i : almost embeddable

$\oplus$: clique sums

$\overline{\mathrm{CR}}(G_i) \leq ??$

Simpler G_i-s

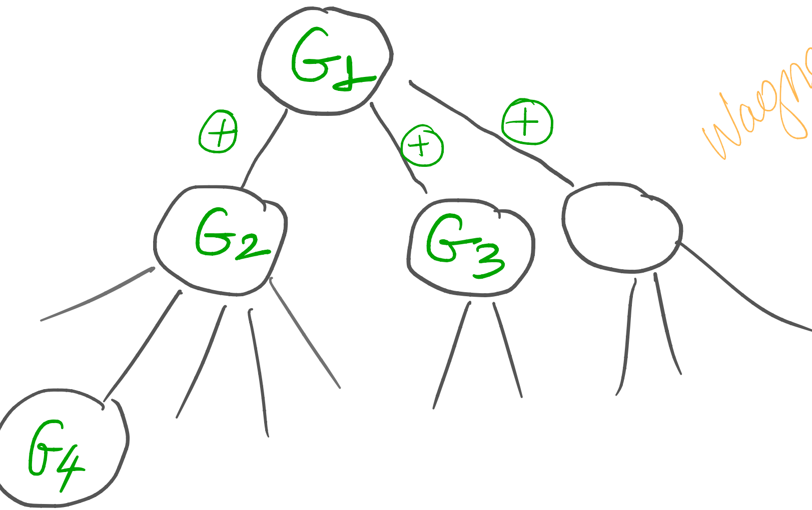
GRAPH CLASSES:



Wagner

- G_i planar or V_8
  $\Rightarrow$ K_5-minor free

Simpler G_i-s



GRAPH CLASSES:

Wagner

- G_i planar or V_8
  $\Rightarrow$ K_5-minor free
- G_i all of constant size
  $\Rightarrow$ bounded treewidth

Simpler G_i-s

GRAPH CLASSES:

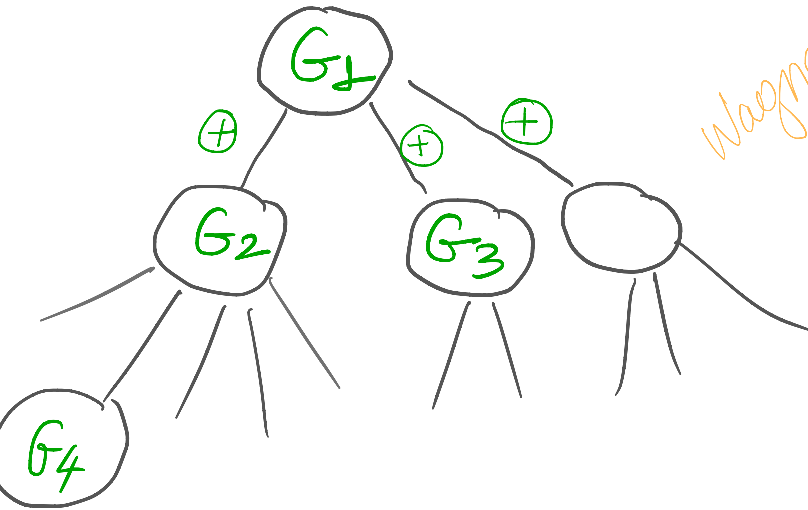
Wagner

- G_i planar or V_8
  $\Longrightarrow$ K_5-minor free ✓

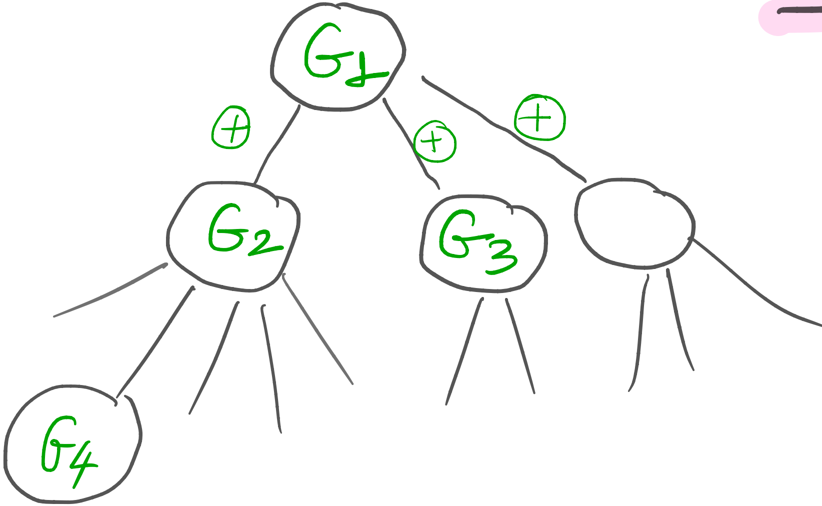
- G_i all of constant size
  $\Longrightarrow$ bounded treewidth ✓

- G_i planar or bounded tw
  $\Longrightarrow$ single-crossing minor free ✓

RS



SCG minor free graphs



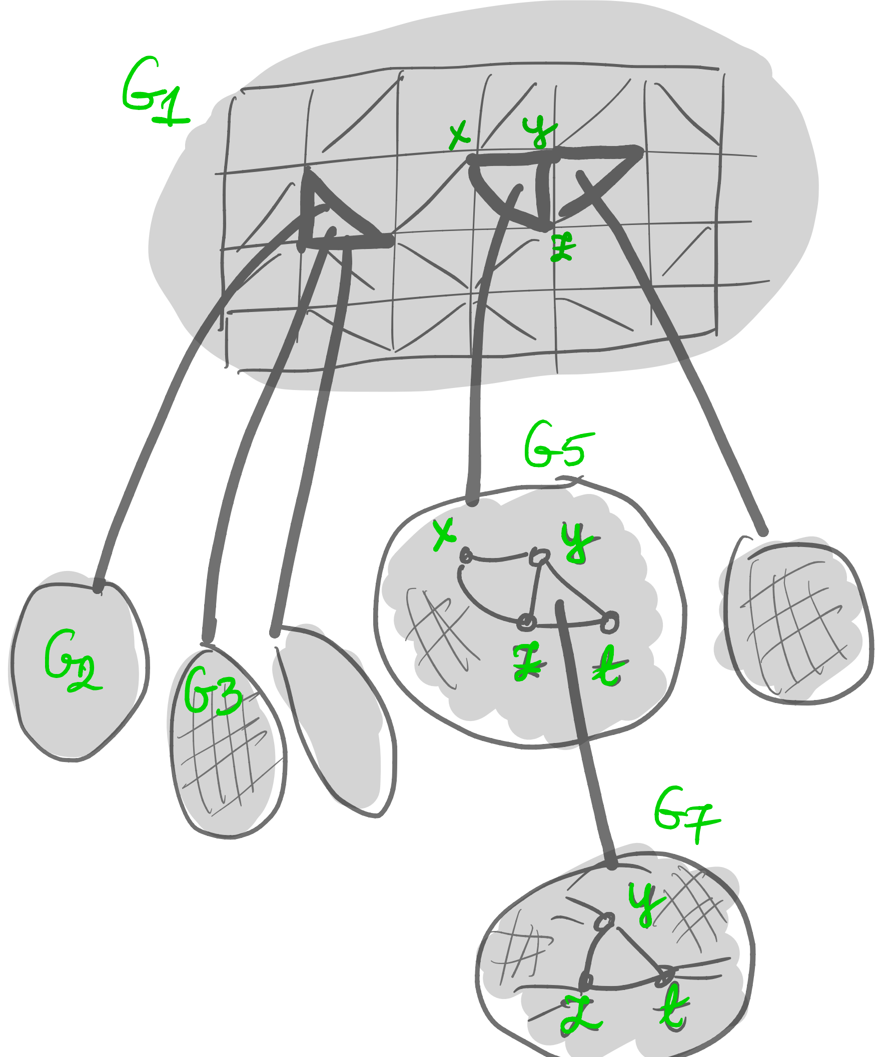
G_i suppose $\overline{cr}(G_i) \le f(\Delta) \cdot n$

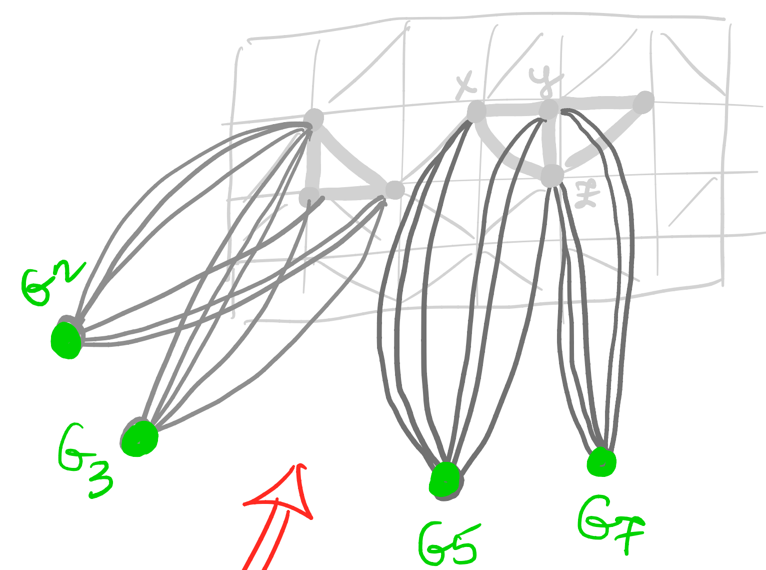
$\oplus$: clique sums

not the sum much ??

previous solution many bends

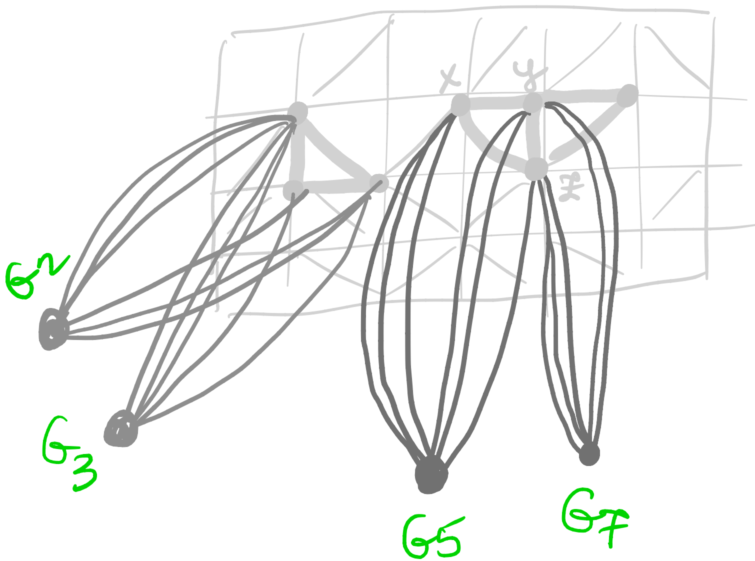
construction





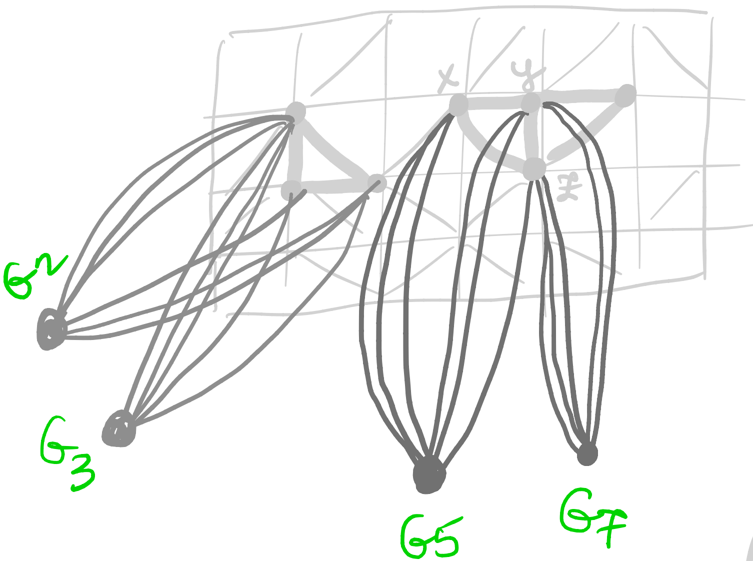
SIMPLICIAL BLOW-UP of G1

construction



Find a good rectilinear drawing blow-ups

construction



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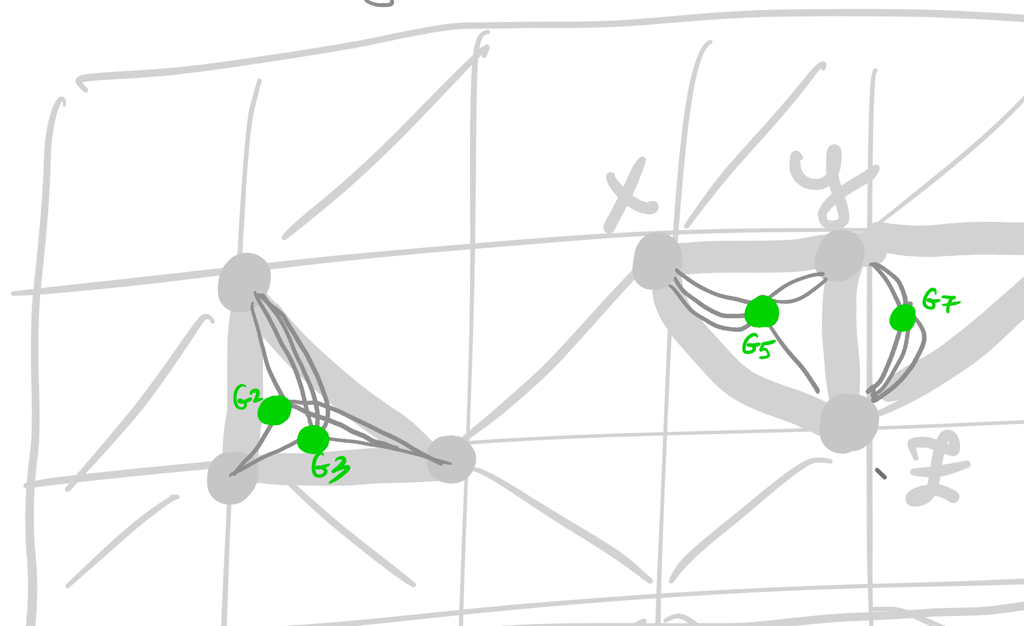
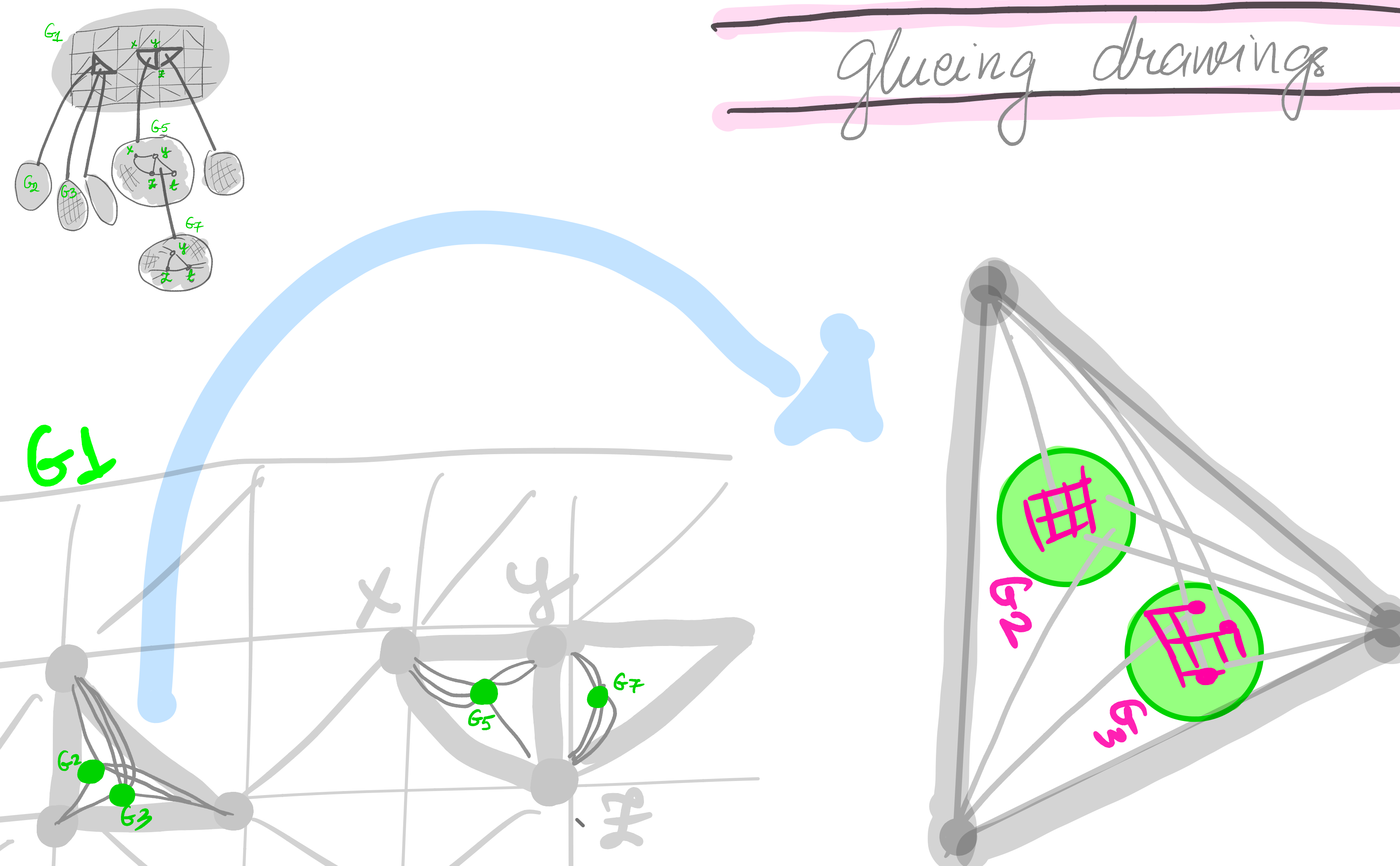
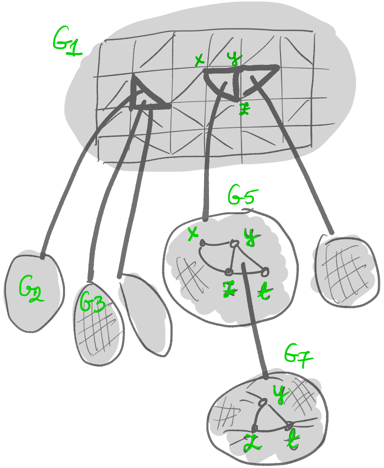


glueing drawings

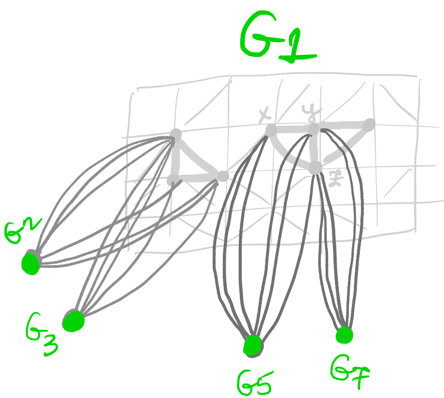


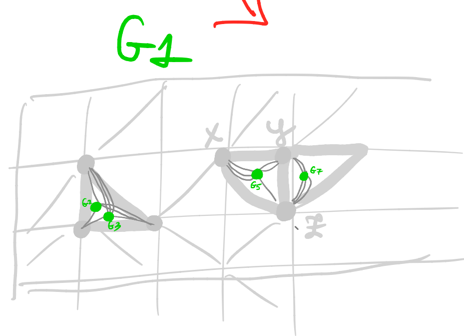
finish line



NEED:

1. $\overline{cr}(G_i^*) \leq f(\Delta)\cdot n$
   ↳ blow-up





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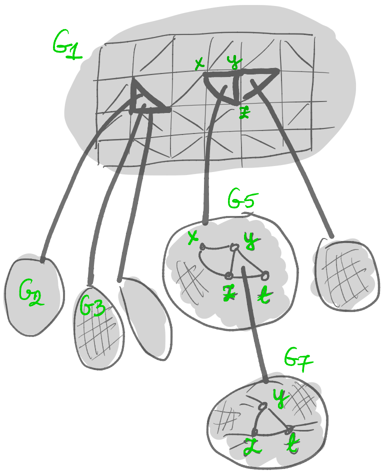

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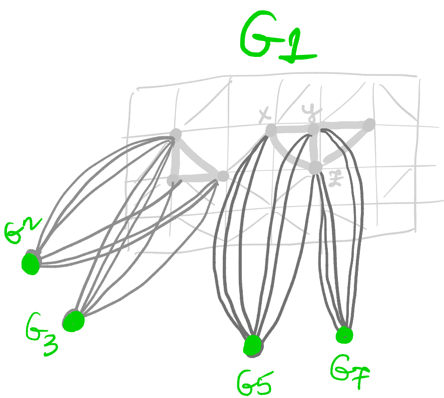
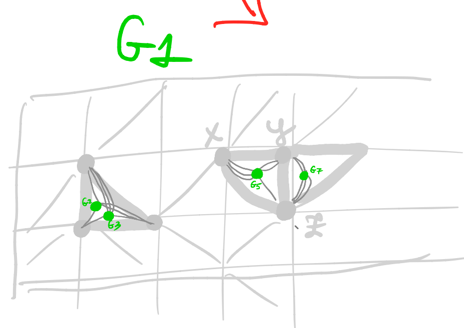
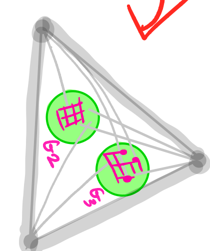




2. price of glueing
$g(\Delta)\cdot n$



Result

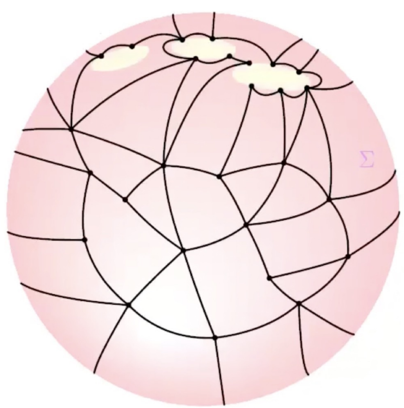
Th H = single-crossing graph

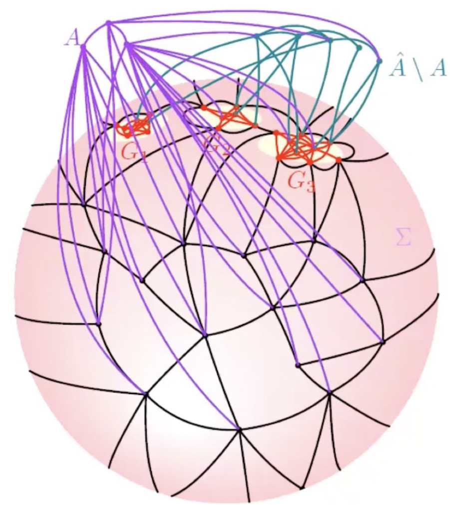
For every H-minor free graph G, $\overline{cr} \in \Theta(\Delta \cdot n)$ tight

- K_5-minor free
- bounded treewidth

Open Problems:

Is rectilinear crossing number of all proper minor closed classes at most $f(A)\cdot n$?





the end