On k-planar Graphs without Short Cycles

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uOttawa

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FernUniversität in Hagen

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drawable s.t. each edge
involved
in $\leq k$ crossings

On k-planar Graphs

Question.

How many edges can a k-planar graph have?

On k-planar Graphs

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How many edges can a k-planar graph have?

k	lower	upper
0	$3n$	$3n$
1	$4n$ [2]	$4n$ [2]
2	$5n$ [4]	$5n$ [4]
3	$5.5n$ [3]	$5.5n$ [3]
k	$\Omega(\sqrt{k})n$ [4]	$3.81\sqrt{k}n$[1]

[1] Ackerman 2019
[2] Bodendiek, Schumacher, Wagner 1983
[3] Pach, Radoičić, Tardos, Tóth 2006
[4] Pach, Tóth 1997

On k-planar Graphs without Short Cycles

Question.

How many edges can a k-planar graph have if there are...

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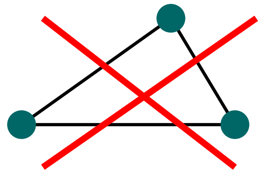
How many edges can a k-planar graph have if there are...

- No cycles of length three (C_3-free) 

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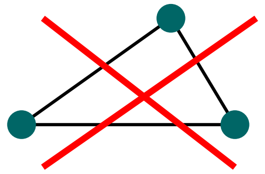
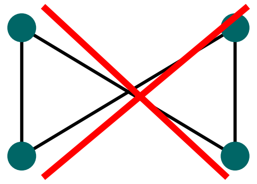
- No cycles of length three (C_3-free) 
- No cycle of length four (C_4-free) 

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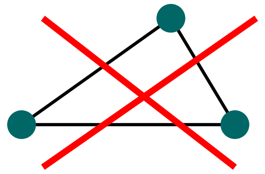
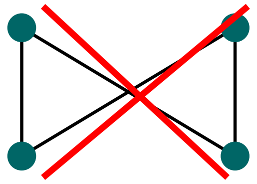
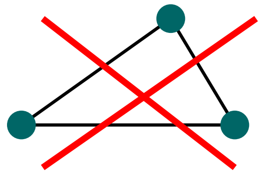
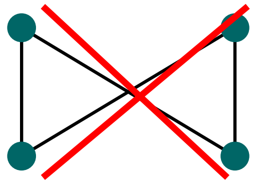
- No cycles of length three (C_3-free) 
- No cycle of length four (C_4-free) 
- No cycles of length three or four (girth 5)  

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	C_3-free		C_4-free		Girth 5	
k	lower	upper	lower	upper	lower	upper
0	$2n$	$2n$	$\frac{15n}{7}$ [3]	$\frac{15n}{7}$ [3]	$\frac{5n}{3}$	$\frac{5n}{3}$
1	$3n$ [2]	$3n$	$2.4n$	$2.5n$	$\frac{13n}{6}$	$2.4n$
2	$3.5n$ [1]	$4n$	$2.5n$	$3.93n$	$\frac{16n}{7}$	$3.597n$
3	$4n$ [1]	$5.12n$	-	$4.933n$	$2.5n$	$4.516n$
k		$3.19\sqrt{k}n$		$3.016\sqrt{k}n$ $O(\sqrt[3]{k})n$ [4]		$2.642\sqrt{k}n$ $O(\sqrt[3]{k})n$ [4]

[1] Angelini, Bekos, Kaufmann, Pfister, Ueckerdt 2018
[2] Czap, Przybylo, Skrabul'áková 2016
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k		$3.19\sqrt{kn}$		$3.016\sqrt{kn}$		$2.642\sqrt{kn}$
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Techniques

On k-planar Graphs without Short Cycles

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Techniques upper bound

- Density Formula

Technique lower bound

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Techniques upper bound

- Density Formula
- Discharging Method

Technique lower bound

Discharching Method

Ackerman Tardos 2007
Ackerman 2019

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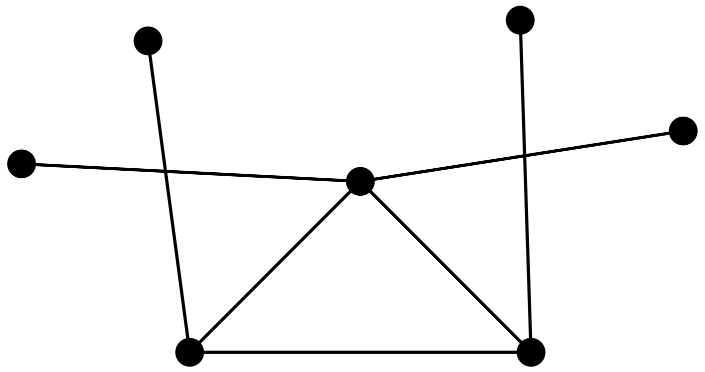
Graph (V,E), Planarization (V',E')



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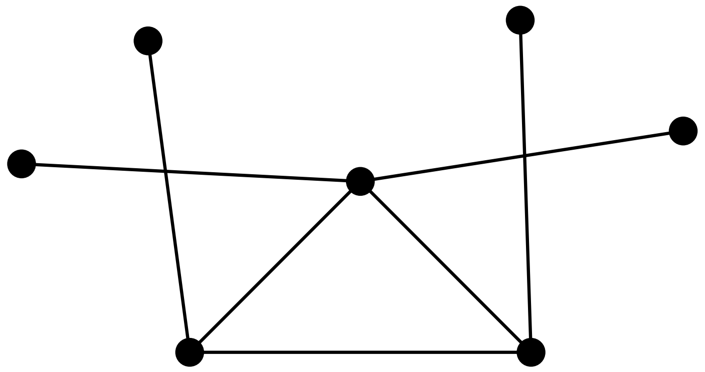

Charge faces of planarization

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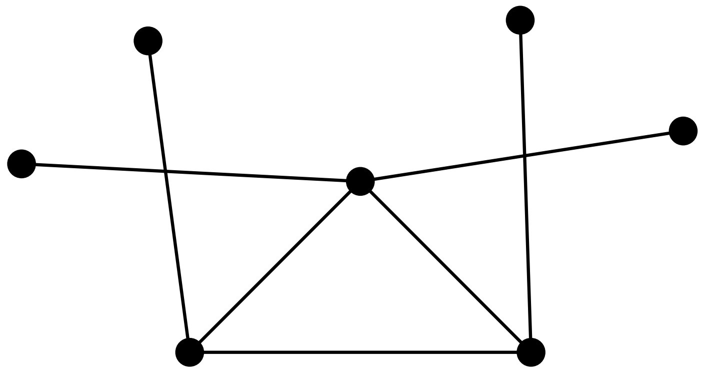

Charge faces of planarization

Charge of face f = ch(f) =
|Vertices of **graph** at face| + |Vertices of **planarization** at face| - 4

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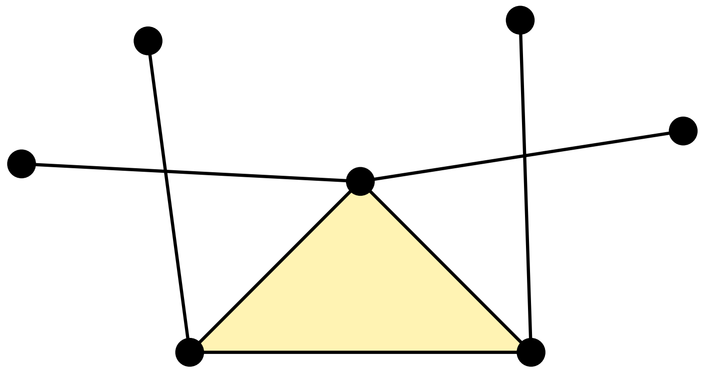

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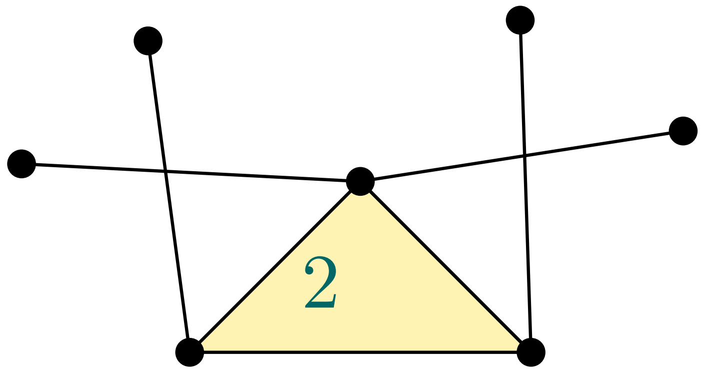

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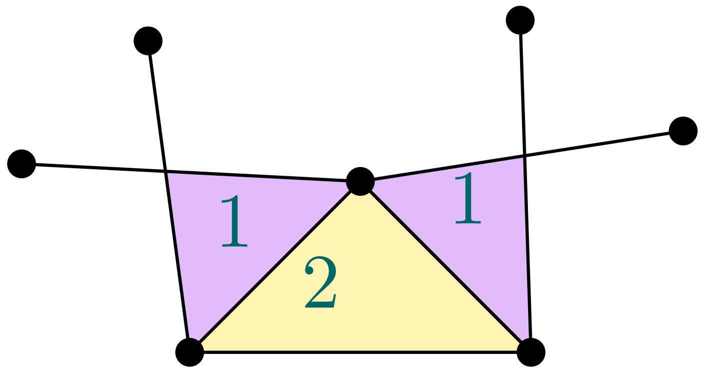

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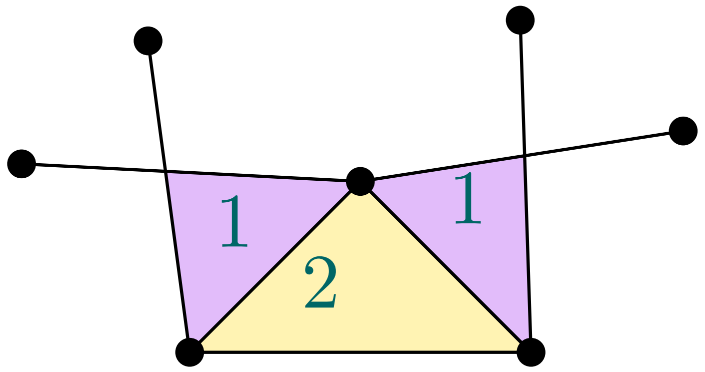

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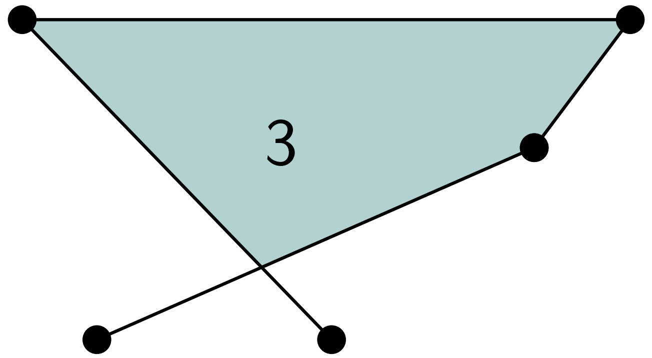

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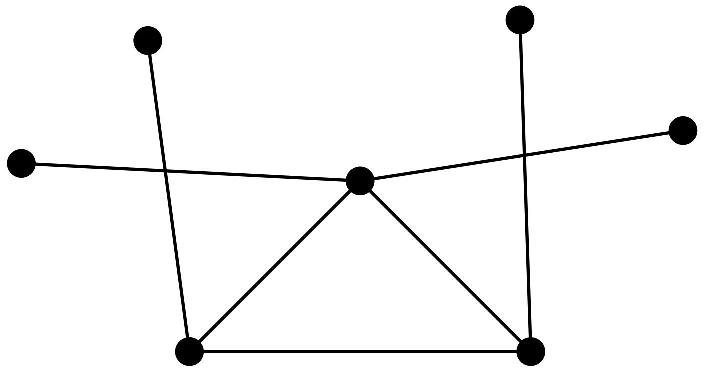

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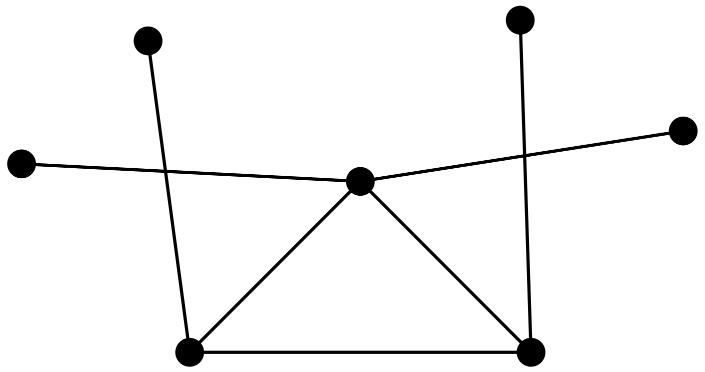

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$4n - 8 = \sum_f ch(f)$

Discharching Method

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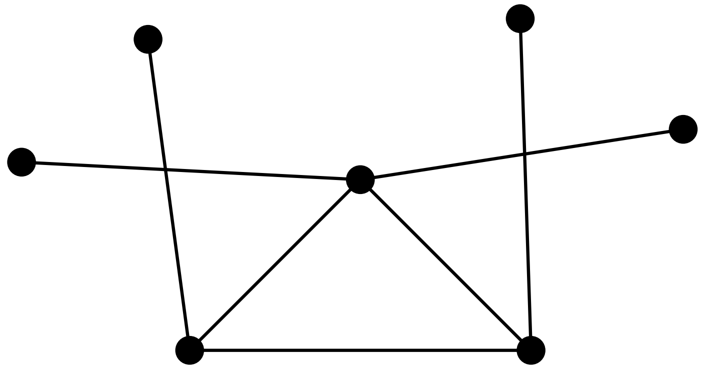

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(Re)distribute charges to vertices and collect discharges of $\geq \alpha$ at pair (vertex,face)

Discharching Method

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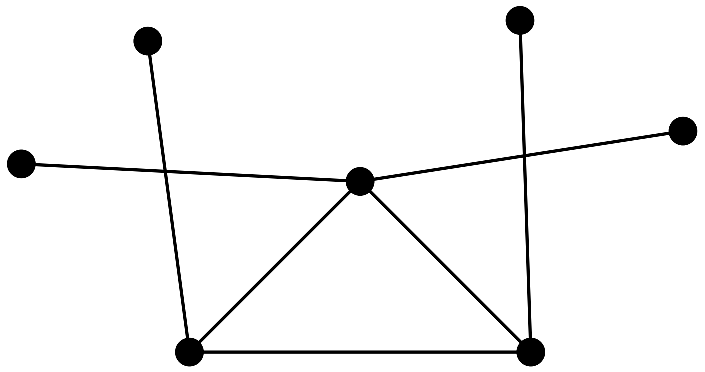

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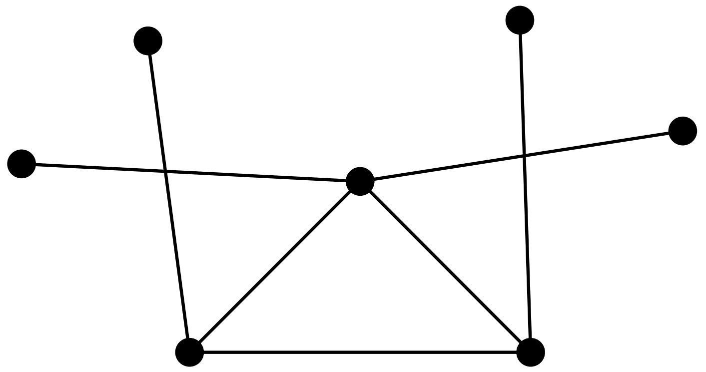

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$|\mathbf{E}| \leq \frac{\mathbf{2}}{\alpha}(\mathbf{n} - \mathbf{2})$

Discharching Method

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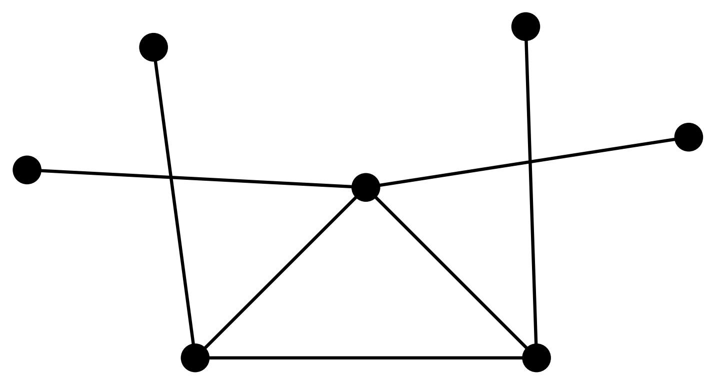

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$|\mathbf{E}| \leq \frac{\mathbf{2}}{\alpha}(\mathbf{n} - \mathbf{2})$

bigger $\Longrightarrow$ better bound

Discharching Method

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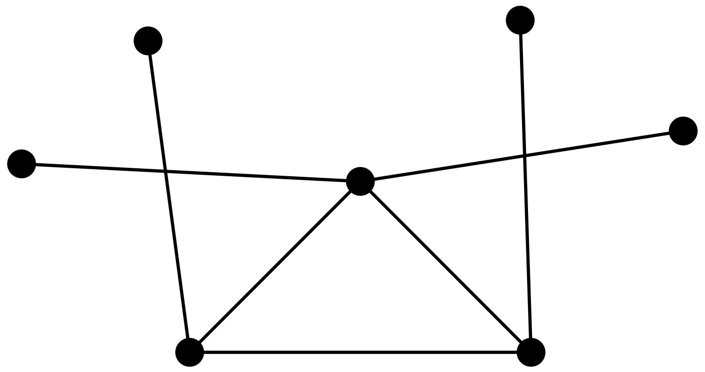
Graph (V,E), Planarization (V',E')



(Re)distribute charges to vertices and collect discharges of $\geq \alpha$ at pair (vertex,face)

Idea: Have each face discharge α to its vertices

Discharching Method

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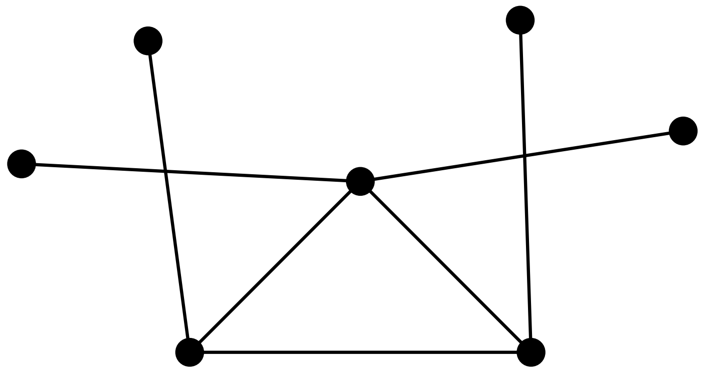

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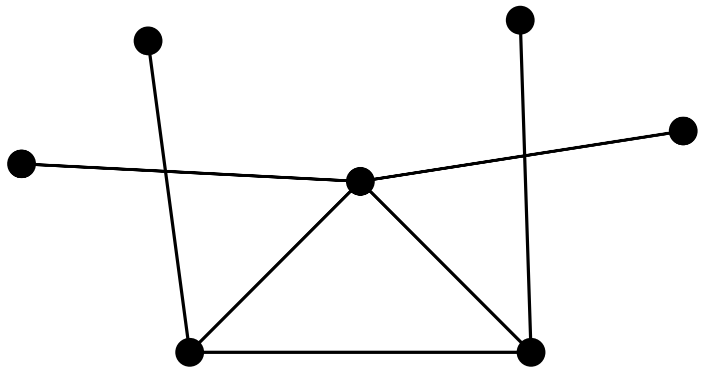

(Re)distribute charges to vertices and collect discharges of $\geq \alpha$ at pair (vertex,face)

Idea: Have each face discharge α to its vertices

But: If α is big, some faces don't have enough

Faces with lots of charge can send it to other faces

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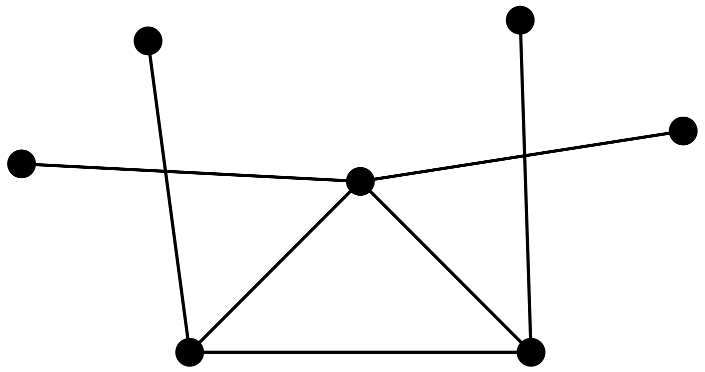

C_4-free, 1-planar

Discharching Method

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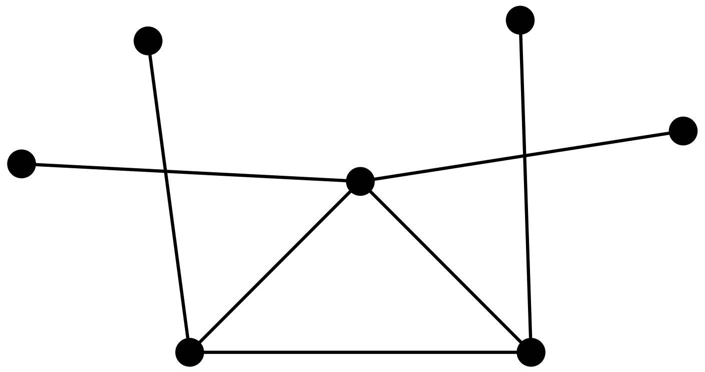

C_4-free, 1-planar

$\alpha = \frac{4}{5}$

Discharching Method

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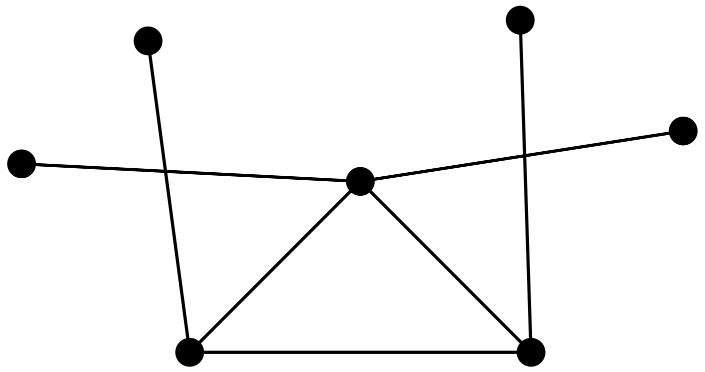

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triangular faces need 'help'

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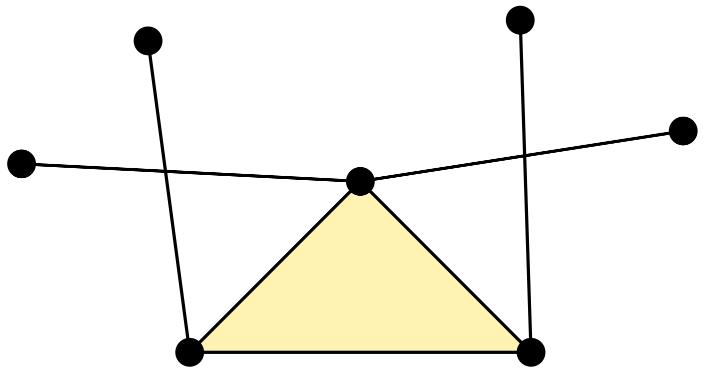

$\frac{2}{5}$ too little

C_4-free, 1-planar

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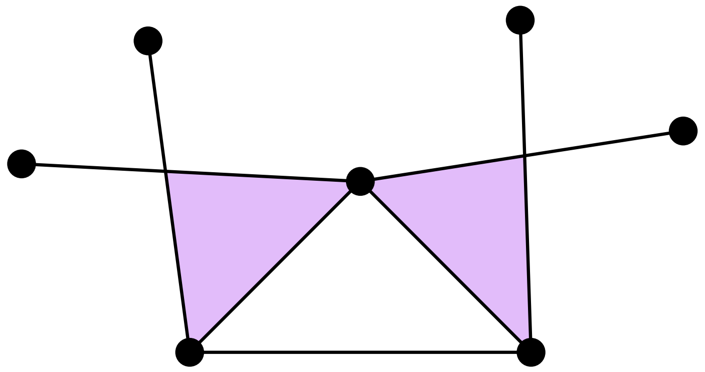

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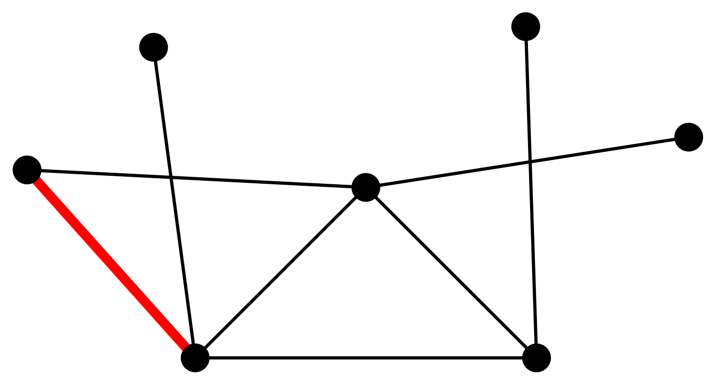

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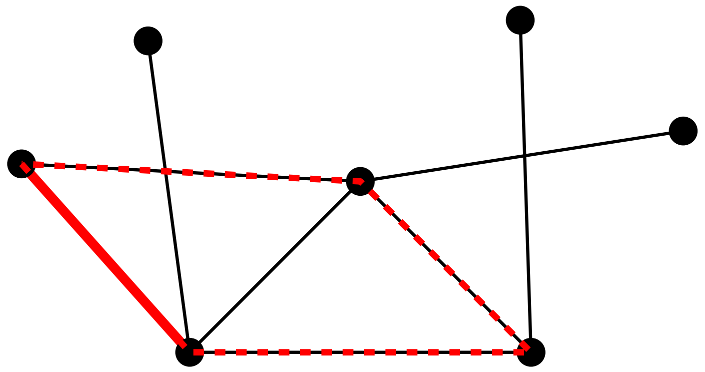

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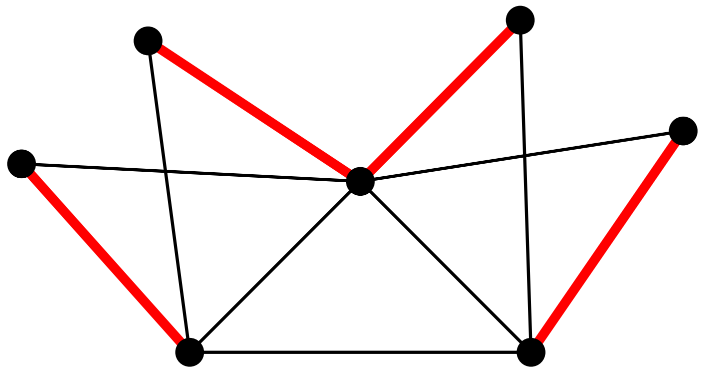

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Ackerman 2019

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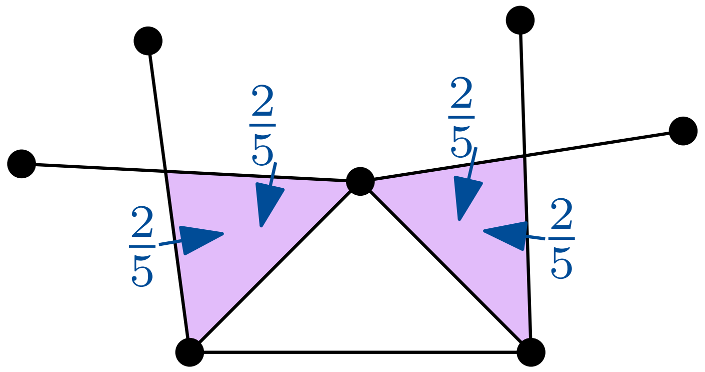

C_4-free, 1-planar

$\alpha = \frac{4}{5}$

triangular faces need 'help'

Discharching Method

Ackerman Tardos 2007
Ackerman 2019

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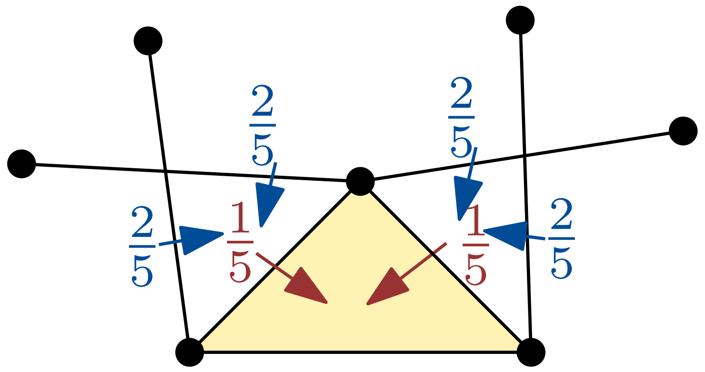

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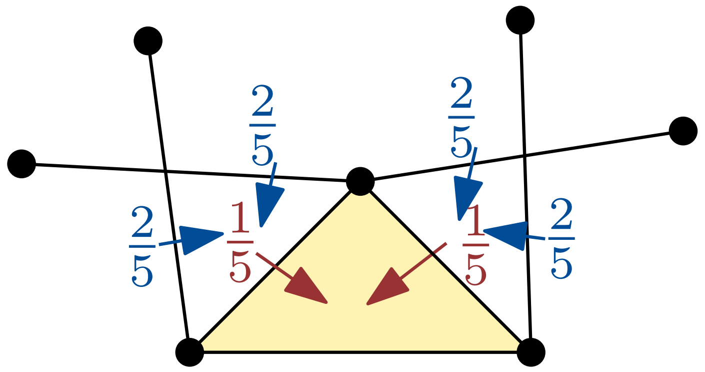

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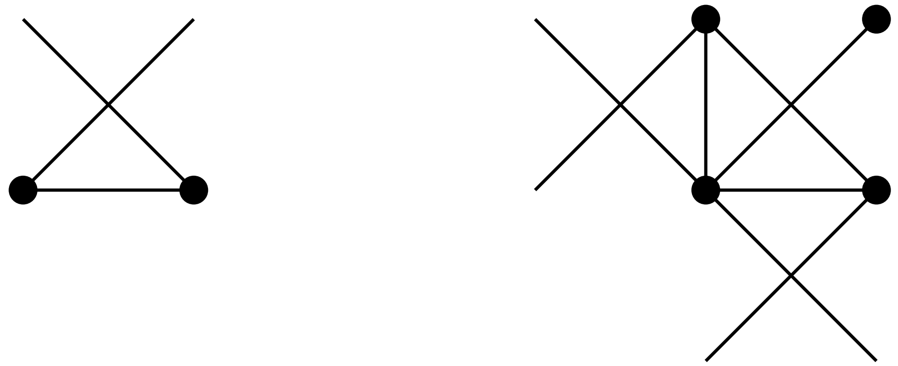

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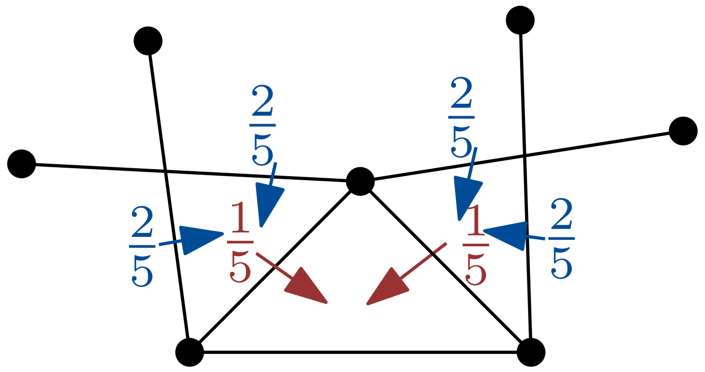

C_4-free, 1-planar

$\alpha = \frac{4}{5}$

triangular faces need 'help'

On k-planar Graphs without Short Cycles

Question.

How many edges can a k-planar graph without short cycles have?

	C_3-free		C_4-free		Girth 5	
k	lower	upper	lower	upper	lower	upper
0						
1		$\boxed{3n}$	$2.4n$	$\boxed{2.5n}$	$\frac{13n}{6}$	$\boxed{2.4n}$
2		$\boxed{4n}$	$2.5n$	$3.93n$	$\frac{16n}{7}$	$3.597n$
3		$5.12n$		$4.933n$	$2.5n$	$4.516n$
k		$3.19\sqrt{k}n$		$3.016\sqrt{k}n$		$2.642\sqrt{k}n$

Techniques upper bound

- Density Formula
- Discharging Method

Technique lower bound

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Techniques upper bound

- Density Formula
- Discharging Method
- Crossing Lemma (for the graph class)

Technique lower bound

Crossing Lemma

Upper bound

Lower bound

Crossing Lemma

Upper bound

$\geq$

Lower bound

Crossing Lemma

Upper bound

2-planar graphs can be drawn with at most $(10n - 20)/3$ crossings.

Lower bound

Crossing Lemma

Upper bound

2-planar graphs can be drawn with at most $(10n - 20)/3$ crossings.

3-planar graphs can be drawn with at most $(33n - 66)/5$ crossings.

Lower bound

Crossing Lemma

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Lower bound

$\mathscr{X}$... graph family closed under taking induced subgraphs
$a, b \in R$ s.t. for every $H \in \mathscr{X}$ with ν vertices and μ edges: $cr(H) \geq a\mu - b\nu$.
Then:
For every $G \in \mathscr{X}$ with n vertices, m **edges** with $2am \geq 3bn$ we have:
$cr(G) \geq \frac{4a^3}{27b^2} \cdot \frac{m^3}{n^2}$

$\mathscr{X}$ ... graph family that is closed under taking subgraphs and disjoint unions
k positive integer
$\mu_i(n)$ upper bound on the number of edges for every i-planar graph from $\mathscr{X}$ on n vertices, for $0 \leq i \leq k-1$.
Then:
For every $G \in \mathscr{X}$ with $n \geq 4$ vertices and m **edges**:
$cr(G) \geq km - \sum_{i=0}^{k-1} \mu_i(n)$

Crossing Lemma

Upper bound

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3-planar graphs can be drawn with at most $(33n - 66)/5$ crossings.

Lower bound

Crossing number at least:

C_3-free	C_4-free	Girth 5
$0.049\frac{m^3}{n^2}$	$0.054\frac{m^3}{n^2}$	$0.071\frac{m^3}{n^2}$

On k-planar Graphs without Short Cycles

Question.

How many edges can a k-planar graph without short cycles have?

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0						
1		$3n$	$2.4n$	$2.5n$	$\frac{13n}{6}$	$2.4n$
2		$4n$	$2.5n$	$3.93n$	$\frac{16n}{7}$	$3.597n$
3		$5.12n$		$4.933n$	$2.5n$	$4.516n$
k		$3.19\sqrt{k}n$		$3.016\sqrt{k}n$		$2.642\sqrt{k}n$

Techniques upper bound

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Technique lower bound

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Techniques upper bound

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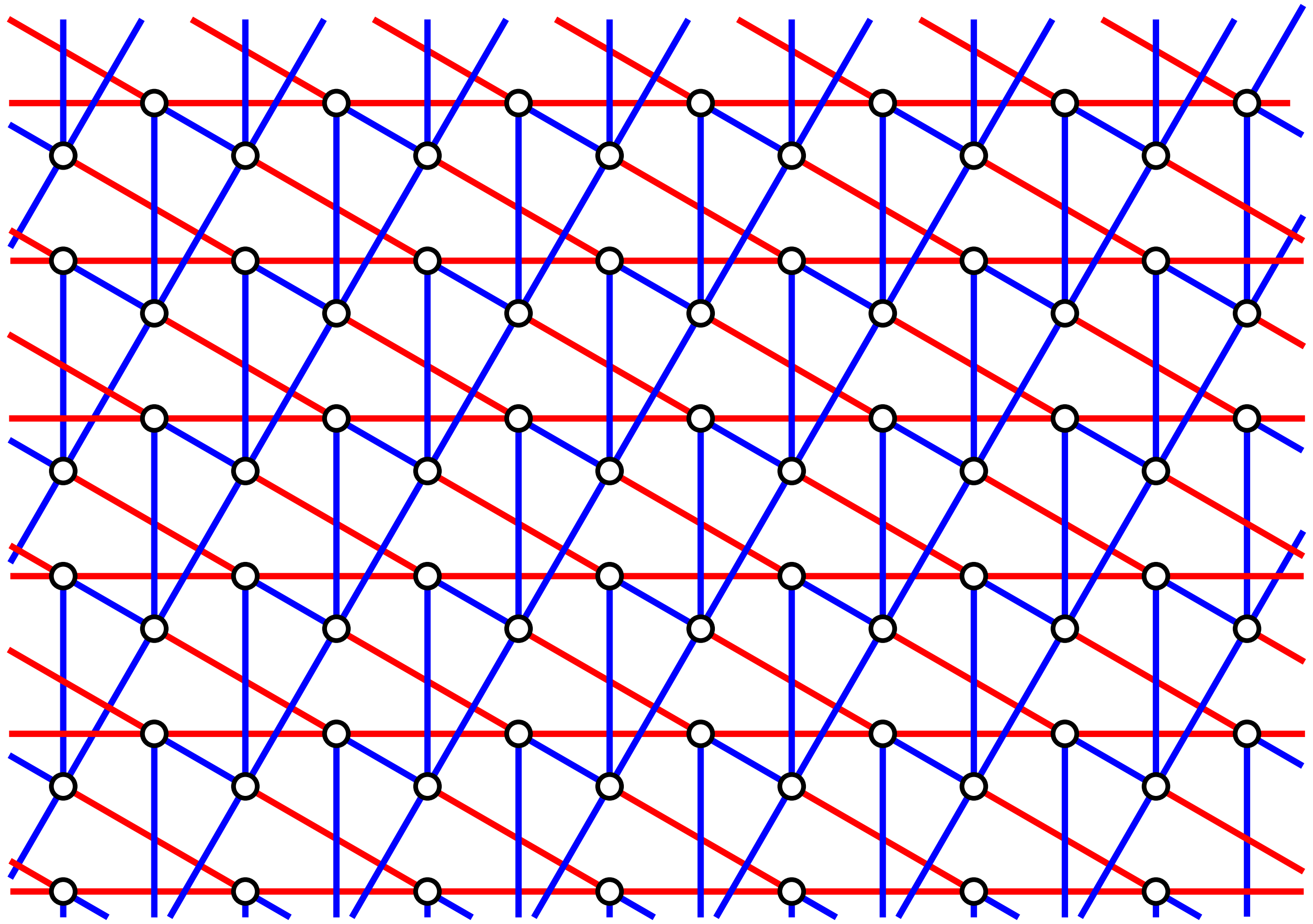
Technique lower bound

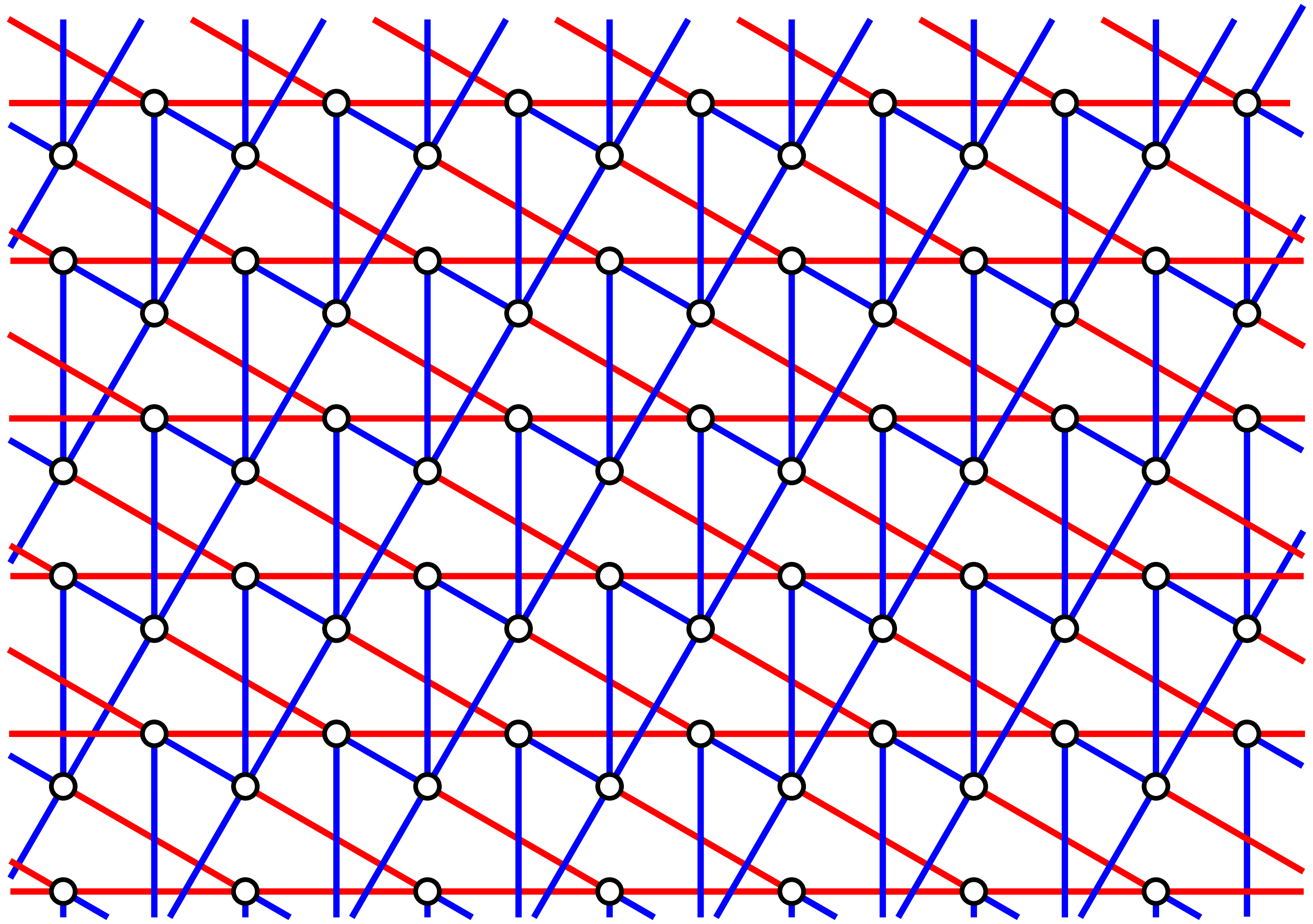
- Finding graphs

Lower bounds

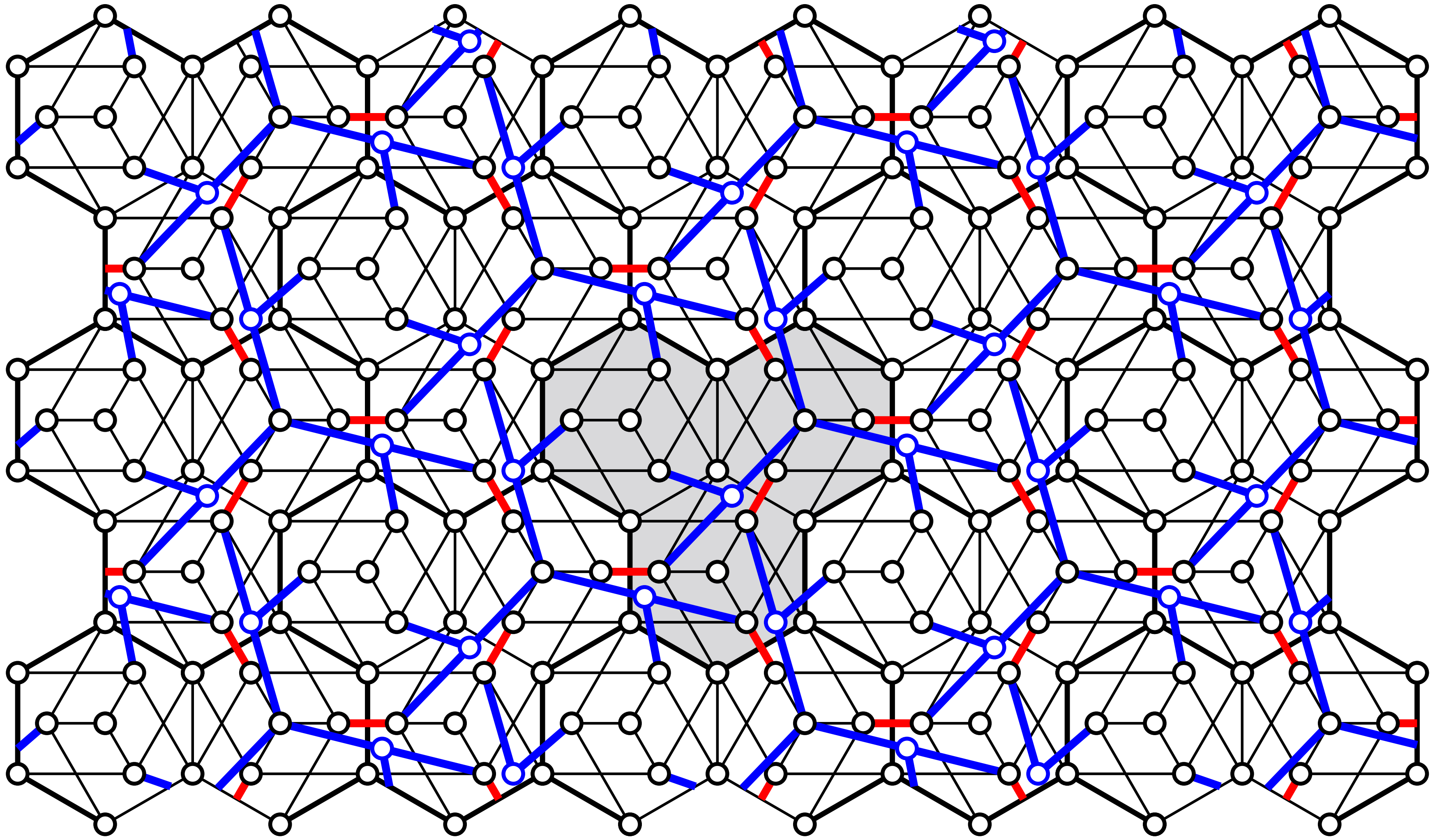


2-planar C_4-free

Lower bounds



2-planar C_4-free



2-planar, girth 5

On k-planar Graphs without Short Cycles

Question.

How many edges can a k-planar graph without short cycles have?

	C_3-free		C_4-free		Girth 5	
k	lower	upper	lower	upper	lower	upper
0						
1		$3n$	$2.4n$	$2.5n$	$\frac{13n}{6}$	$2.4n$
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Techniques upper bound

- Density Formula
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Technique lower bound

- Finding graphs

On k-planar Graphs without Short Cycles

Question.

How many edges can a k-planar graph without short cycles have?

k	C_3-free lower	C_3-free upper	C_4-free lower	C_4-free upper	Girth 5 lower	Girth 5 upper
0	$2n$	$2n$	$\frac{15n}{7}$ [3]	$\frac{15n}{7}$ [3]	$\frac{5n}{3}$	$\frac{5n}{3}$
1	$3n$ [2]	$3n$	$2.4n$	$2.5n$	$\frac{13n}{6}$	$2.4n$
2	$3.5n$ [1]	$4n$	$2.5n$	$3.93n$	$\frac{16n}{7}$	$3.597n$
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k		$3.19\sqrt{kn}$		$3.016\sqrt{kn}$ $O(\sqrt[3]{k})n$ [4]		$2.642\sqrt{kn}$ $O(\sqrt[3]{k})n$ [4]

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Technique lower bound

- Finding graphs

Summary and Open Problems

Number of edges

	C_3-free		C_4-free		Girth 5	
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0	$2n$	$2n$	$\frac{15n}{7}$ [3]	$\frac{15n}{7}$ [3]	$\frac{5n}{3}$	$\frac{5n}{3}$
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Number of crossings

	unrestricted		C_3-free	C_4-free	Girth 5
Graph class	lower	upper	lower	lower	lower
2-planar		$\frac{10n}{3}$			
3-planar		$\frac{33n}{5}$			
general	$0.034\frac{m^3}{n^2}$ [5]		$0.049\frac{m^3}{n^2}$	$0.054\frac{m^3}{n^2}$	$0.071\frac{m^3}{n^2}$

Summary and Open Problems

Number of edges

k	C_3-free lower	C_3-free upper	C_4-free lower	C_4-free upper	Girth 5 lower	Girth 5 upper
0	$2n$	$2n$	$\frac{15n}{7}$ [3]	$\frac{15n}{7}$ [3]	$\frac{5n}{3}$	$\frac{5n}{3}$
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Open Problems

- Close the gaps
- Fill the blanks
- $k \geq 4$
- Other graph classes
- Time for recognition

Summary and Open Problems

Number of edges

Close the gaps!

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Summary and Open Problems

Number of edges

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Summary and Open Problems

Number of edges

$k \geq 4$

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Summary and Open Problems

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Other graph classes

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Open Problems

- Close the gaps
- Fill the blanks
- $k \geq 4$
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Summary and Open Problems

Number of edges

Time for recognition?

k	C_3-free lower	C_3-free upper	C_4-free lower	C_4-free upper	Girth 5 lower	Girth 5 upper
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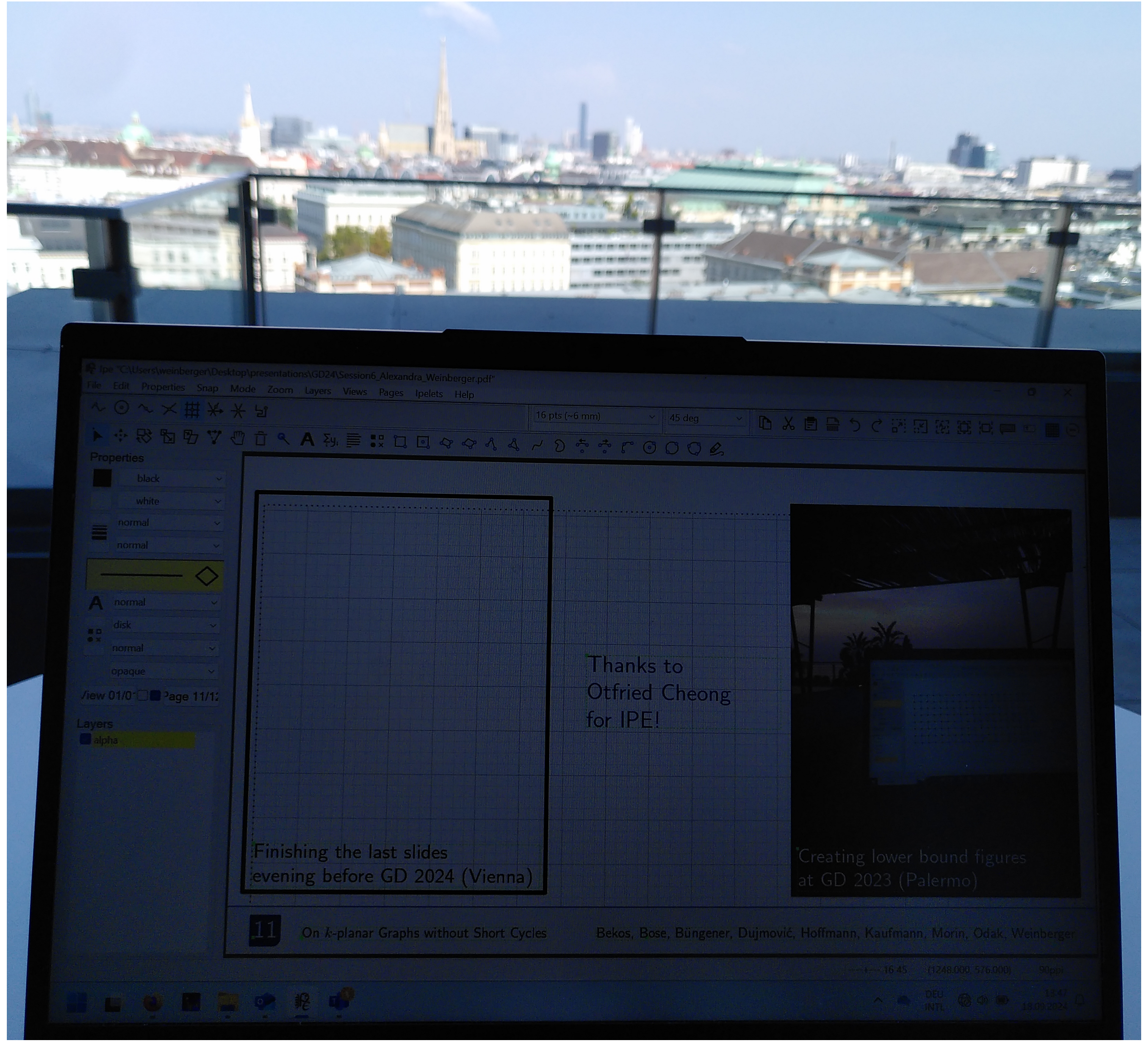
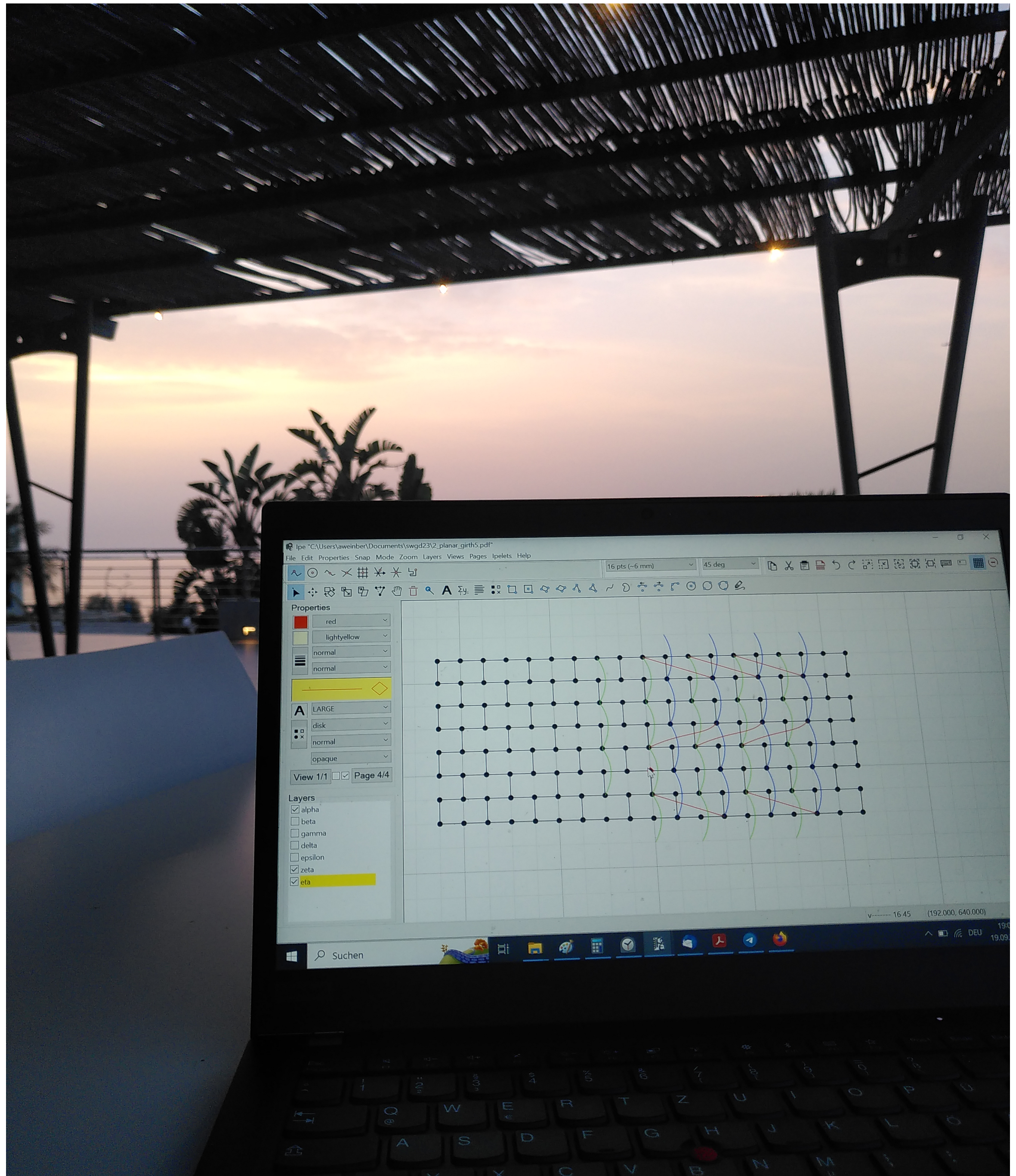


Making this very slide
at GD 2024 (Vienna)

Thanks to Otfried Cheong for IPE!



Creating lower bound figures
at GD 2023 (Palermo)

Summary and Open Problems

Number of edges

k	C_3-free lower	C_3-free upper	C_4-free lower	C_4-free upper	Girth 5 lower	Girth 5 upper
0	$2n$	$2n$	$\frac{15n}{7}$ [3]	$\frac{15n}{7}$ [3]	$\frac{5n}{3}$	$\frac{5n}{3}$
1	$3n$ [2]	$3n$	$2.4n$	$2.5n$	$\frac{13n}{6}$	$2.4n$
2	$3.5n$ [1]	$4n$	$2.5n$	$3.93n$	$\frac{16n}{7}$	$3.597n$
3	$4n$ [1]	$5.12n$	-	$4.933n$	$2.5n$	$4.516n$
k		$3.19\sqrt{kn}$		$3.016\sqrt{kn}$ $O(\sqrt[3]{k})n$ [4]		$2.642\sqrt{kn}$ $O(\sqrt[3]{k})n$ [4]

Number of crossings

Graph class	unrestricted lower	unrestricted upper	C_3-free lower	C_4-free lower	Girth 5 lower
2-planar		$\frac{10n}{3}$			
3-planar		$\frac{33n}{5}$			
general	$0.034\frac{m^3}{n^2}$ [5]		$0.049\frac{m^3}{n^2}$	$0.054\frac{m^3}{n^2}$	$0.071\frac{m^3}{n^2}$

Open Problems

- Close the gaps
- Fill the blanks
- $k \geq 4$
- Other graph classes
- Time for recognition