Quantum Algorithms
$|f \otimes r\rangle$
One-Sided Crossing Minimization

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Roma Tre University

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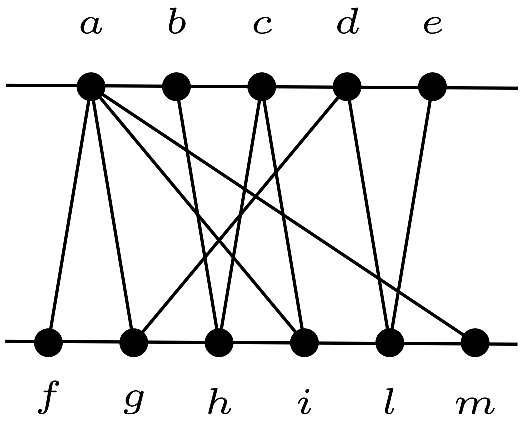

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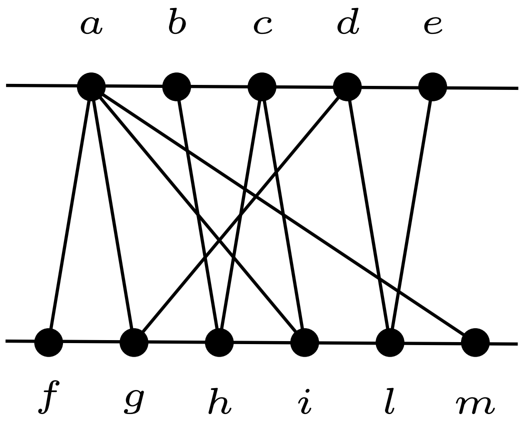

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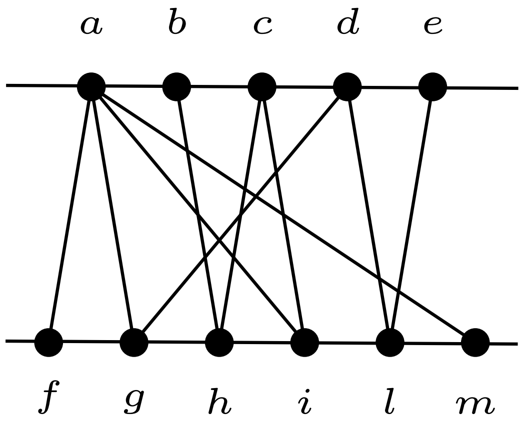

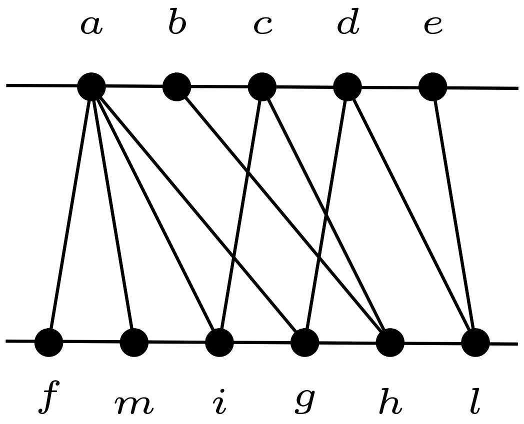

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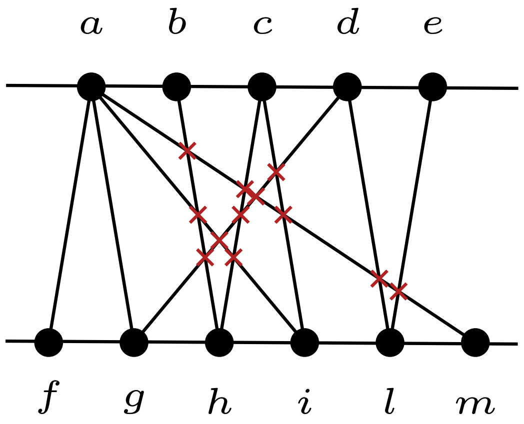
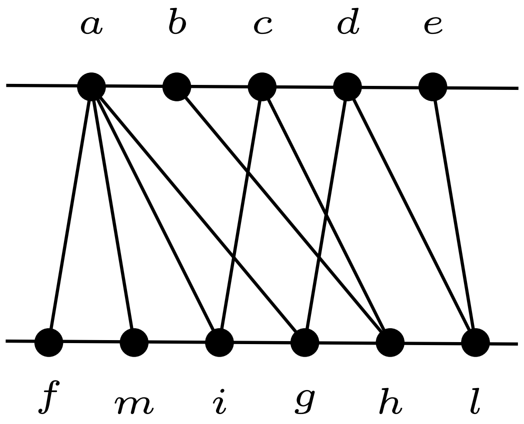


11 crossings



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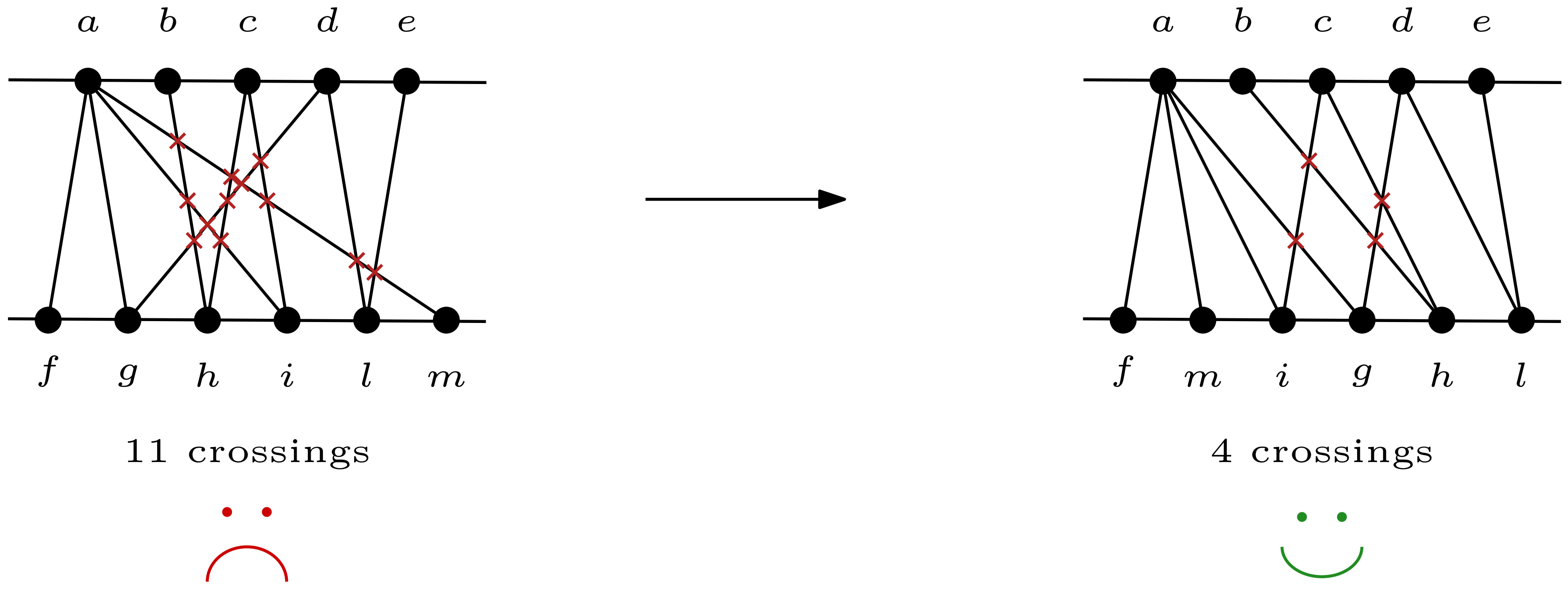

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  - Currently the best FPT results is $\mathcal{O}(k2^{\sqrt{2k}})$ [Kobayashi et al. 2015]

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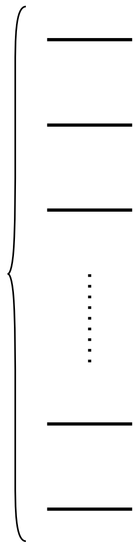

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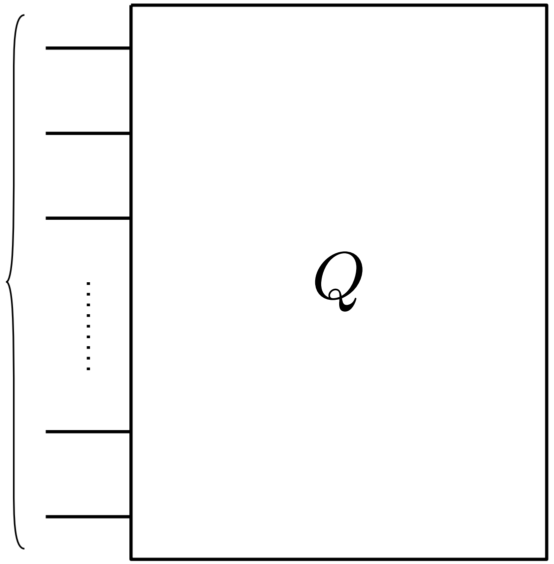

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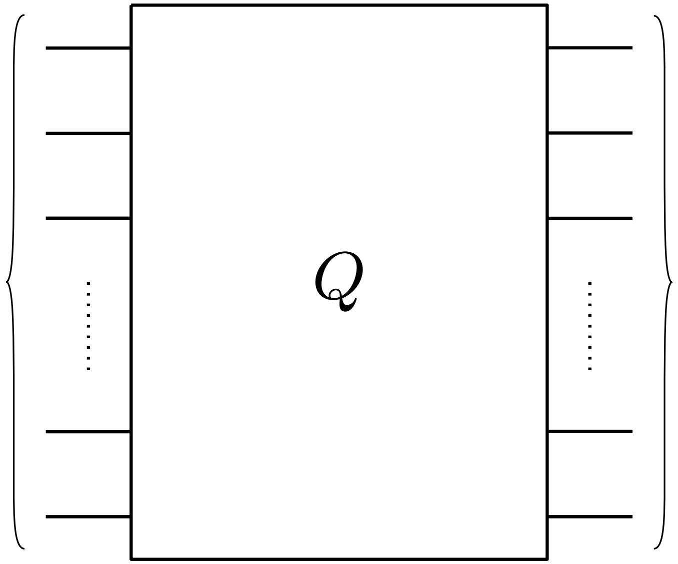

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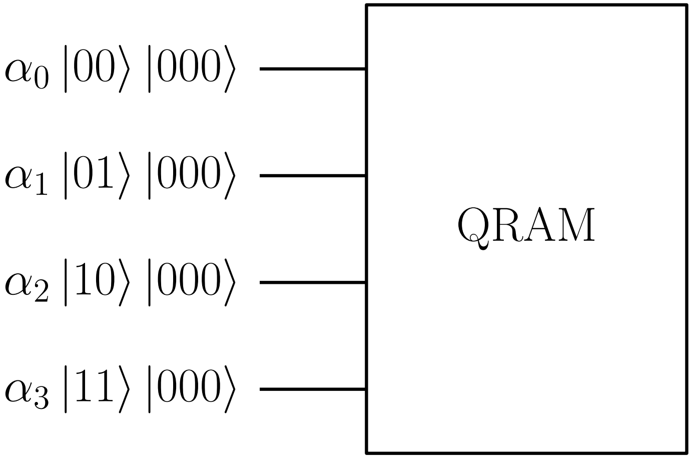

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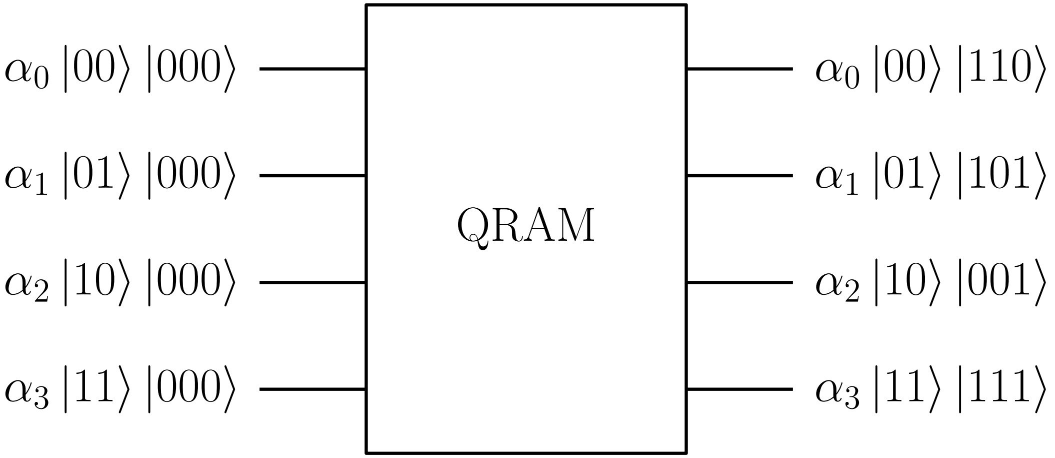

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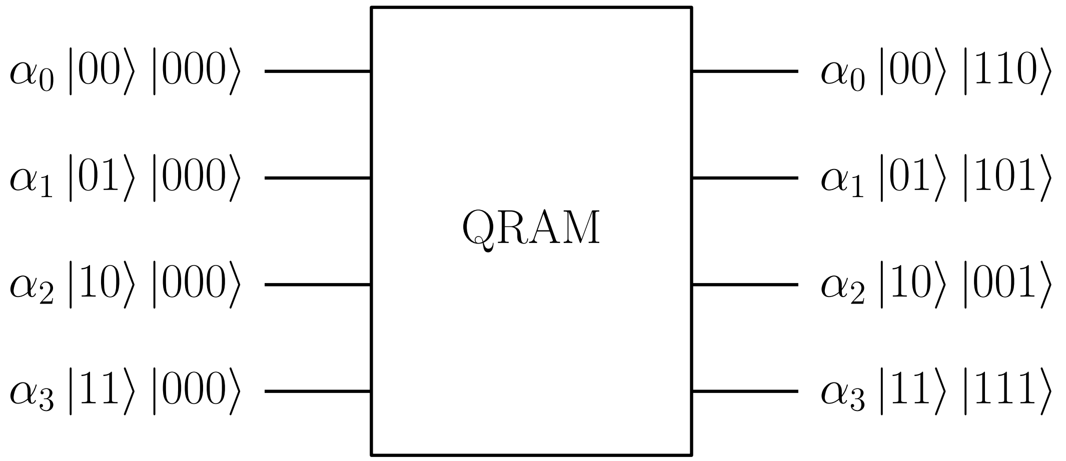

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  - Given a table T of size N the algorithms finds the index y such that $T[y]$ is minimized in time $\mathcal{O}(\sqrt{N})$

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Then, $OPT_{\mathcal{P}}(X)$ can be computed by a quantum algorithm that uses QRAM in $\mathcal{O}^*(1.728^n)$ time and space.

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- The main idea is to
  - Precompute solutions (pre-processing) for smaller subsets using classic dynamic programming and save the results in the QRAM
  - Recombine the results of the precomputation step to obtain the optimal solution for the whole set (recursively) applying Quantum Minimum Finding (QMF)

Quantum Dynamic Programming for Set Problems

Quantum Dynamic Programming for Set Problems

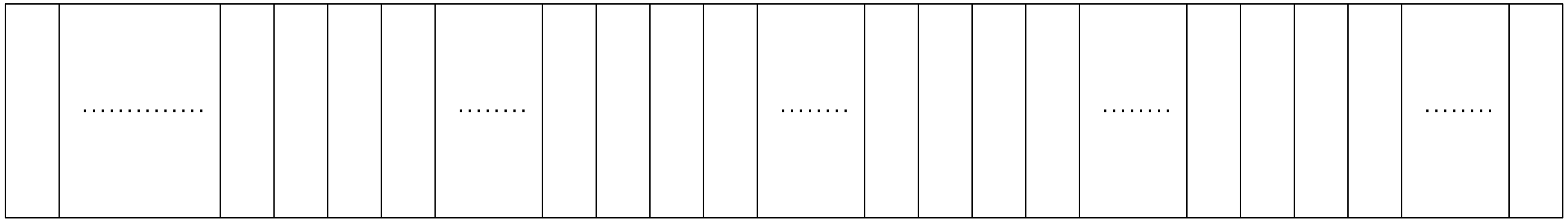
Quantum

Classic

Quantum Dynamic Programming for Set Problems

Quantum

Classic



QRAM

pre-processing
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Quantum Dynamic Programming for Set Problems

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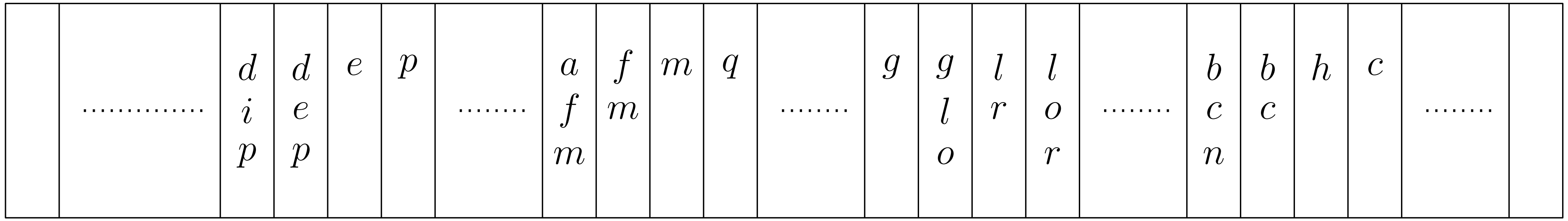

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QRAM

Quantum Dynamic Programming for Set Problems

$a \quad b \quad c \quad d \quad e \quad f \quad g \quad h \quad i \quad l \quad m \quad n \quad o \quad p \quad q \quad r$

Quantum

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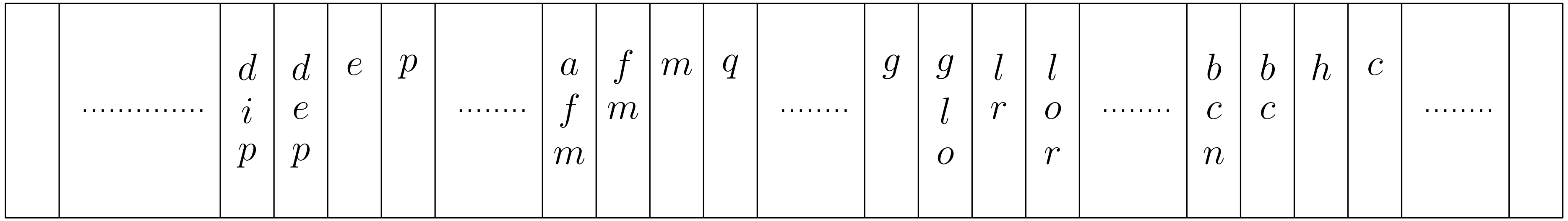

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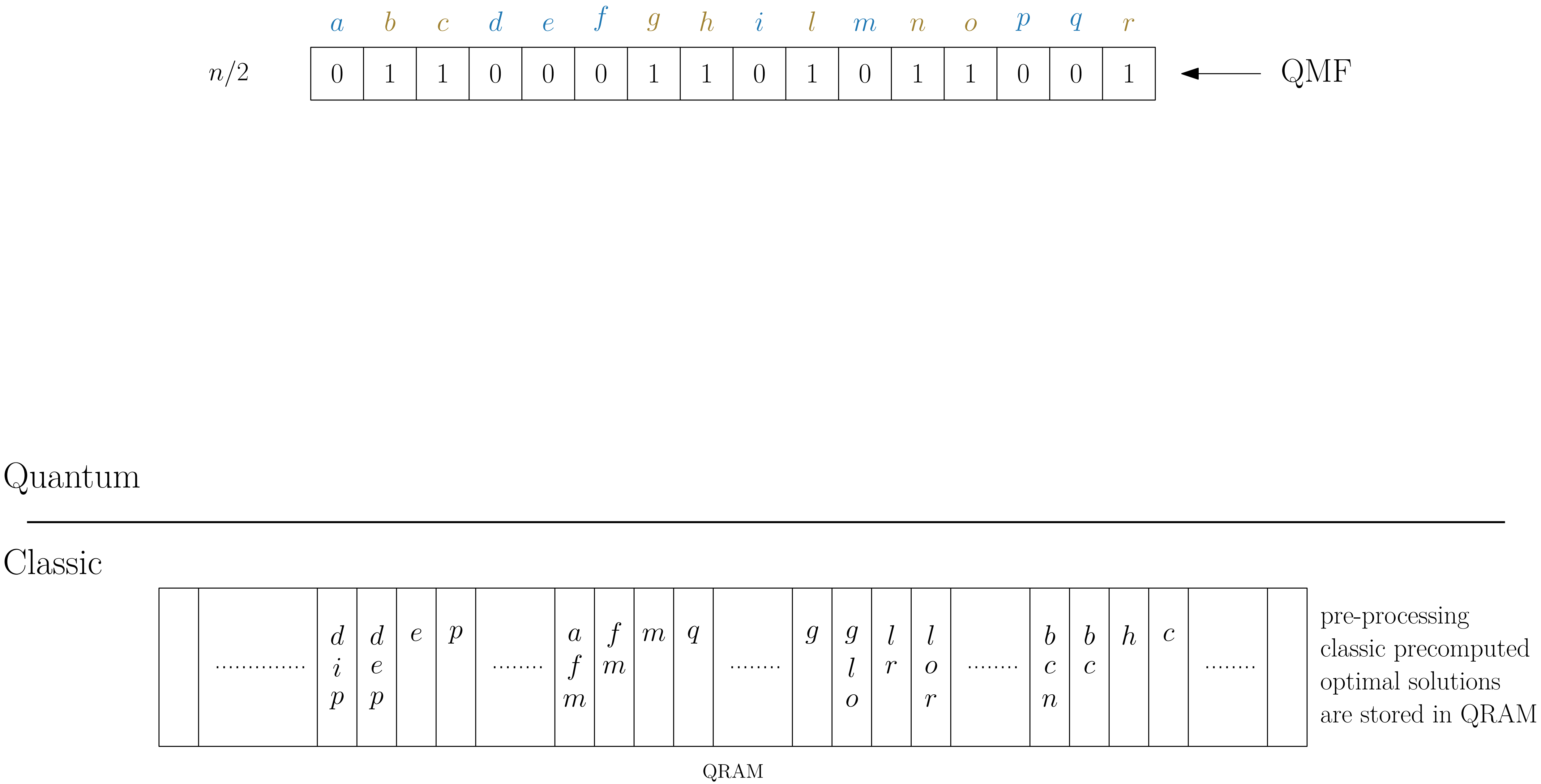

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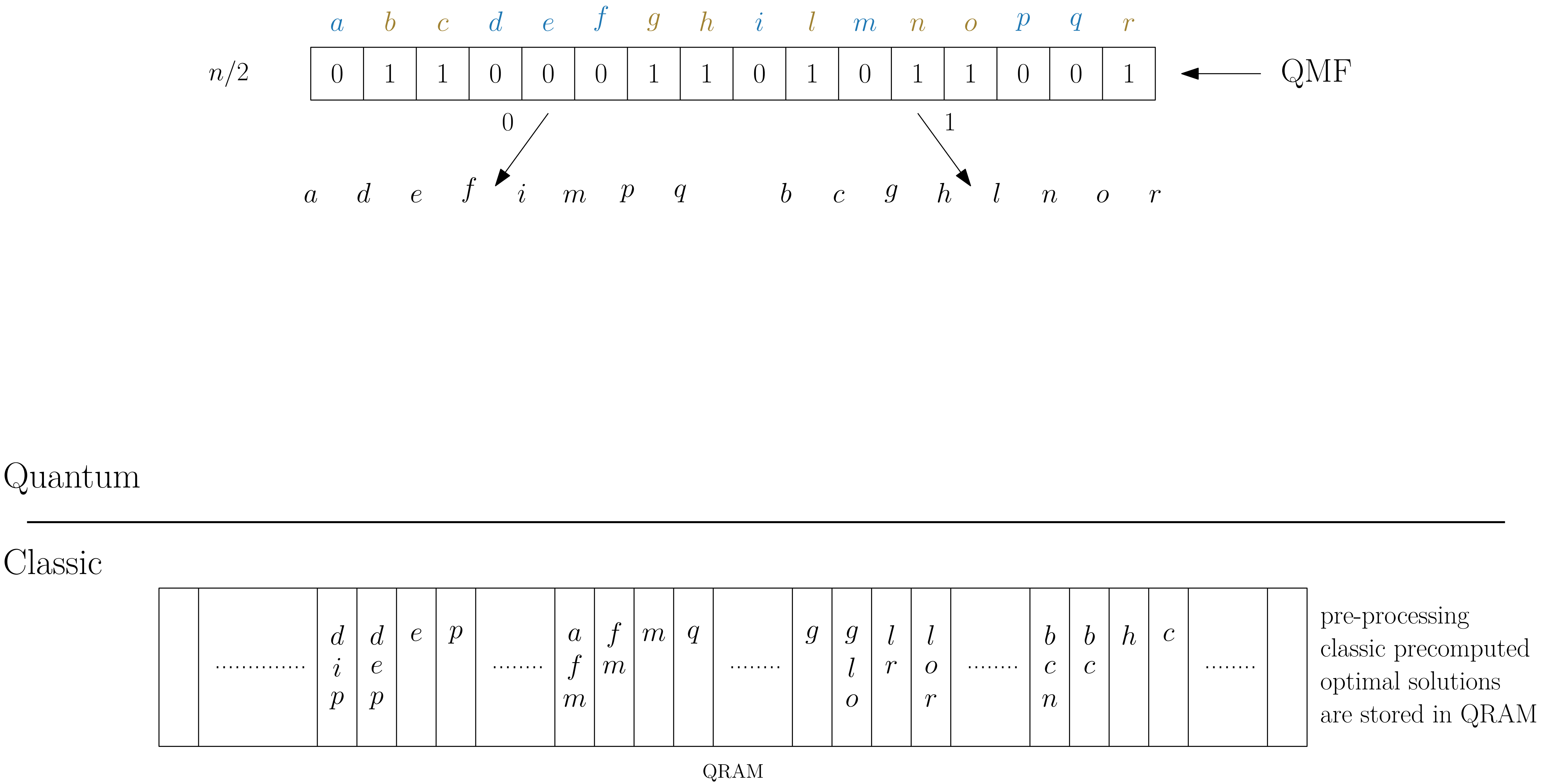

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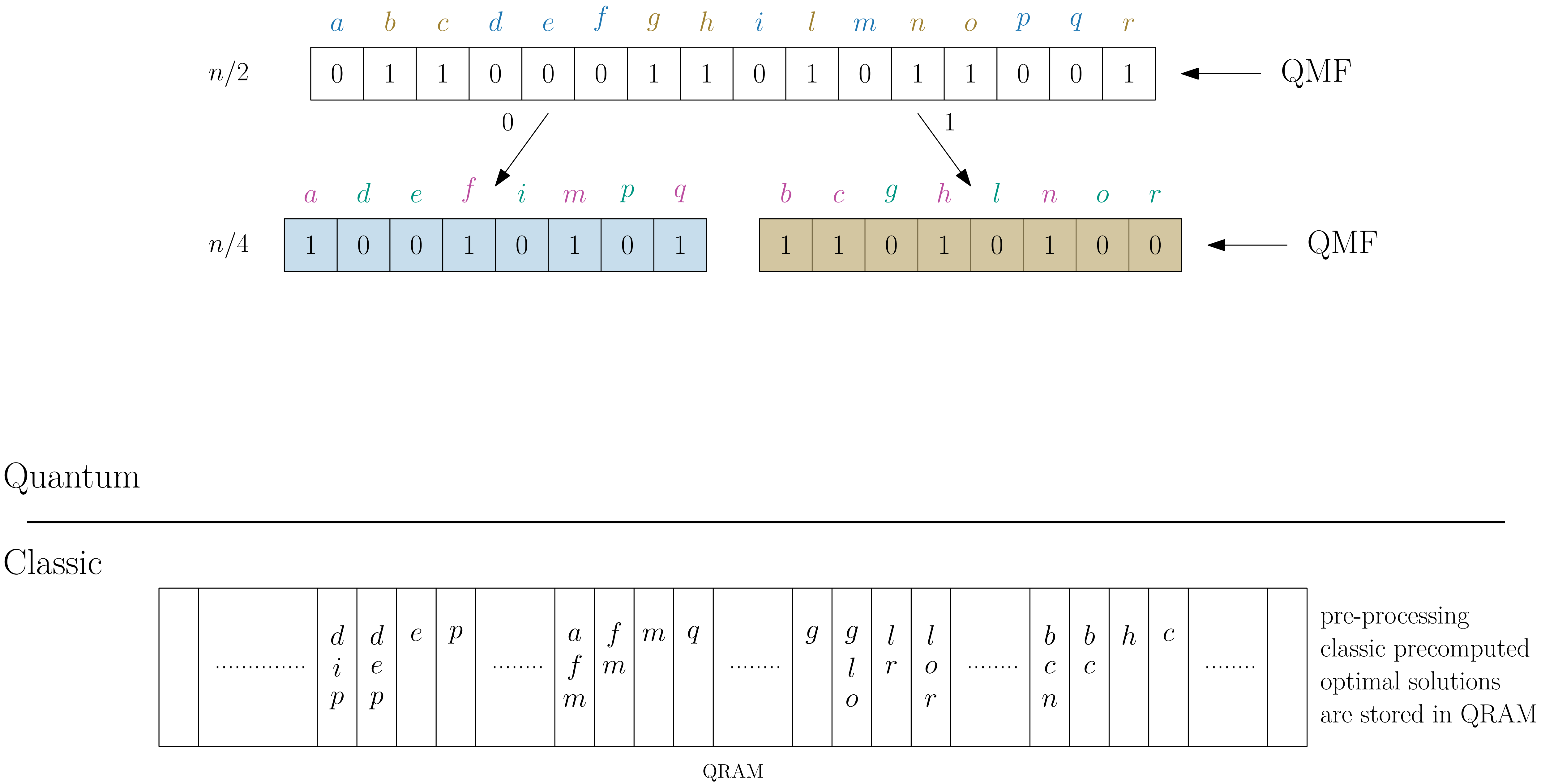

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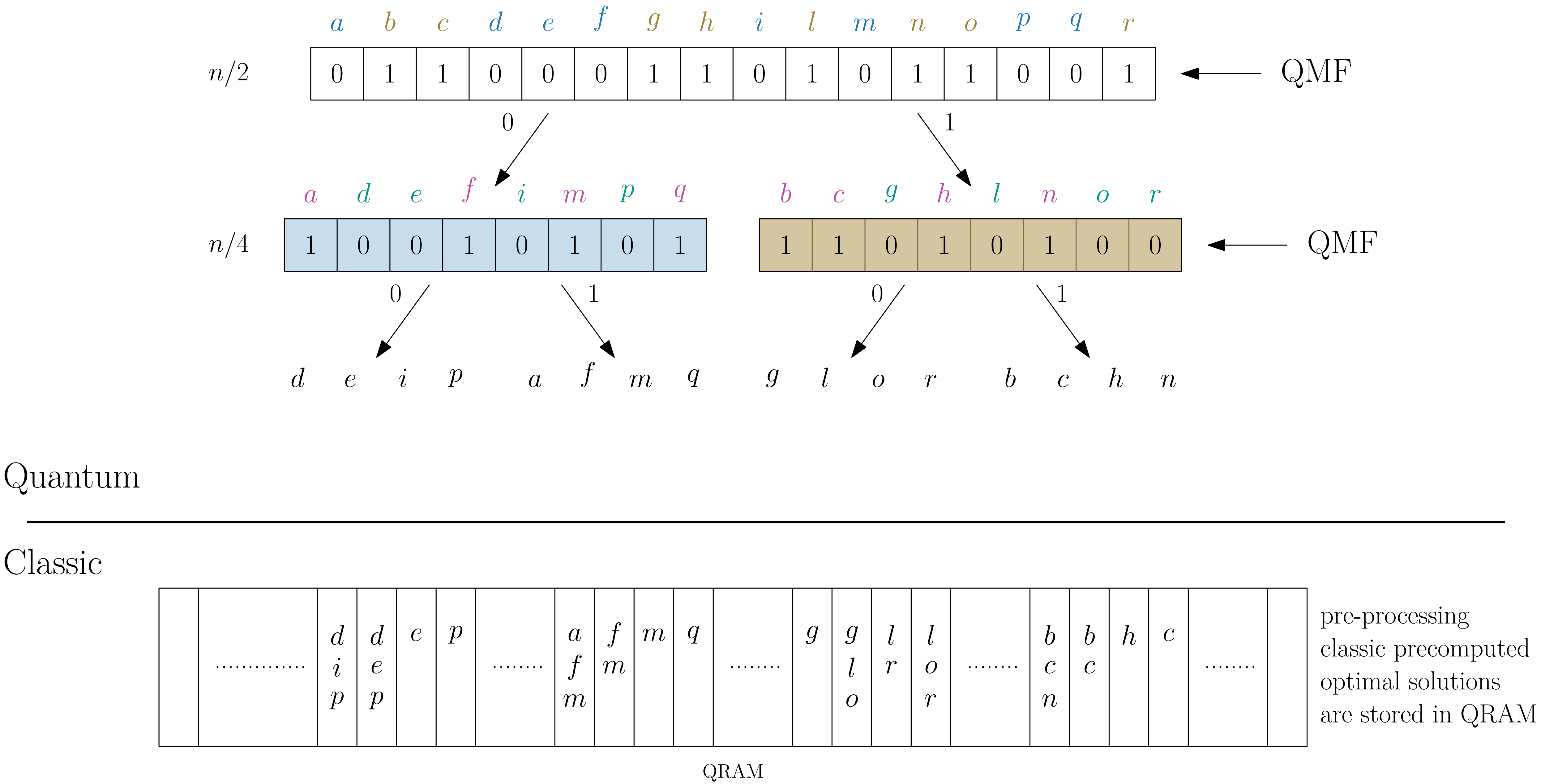

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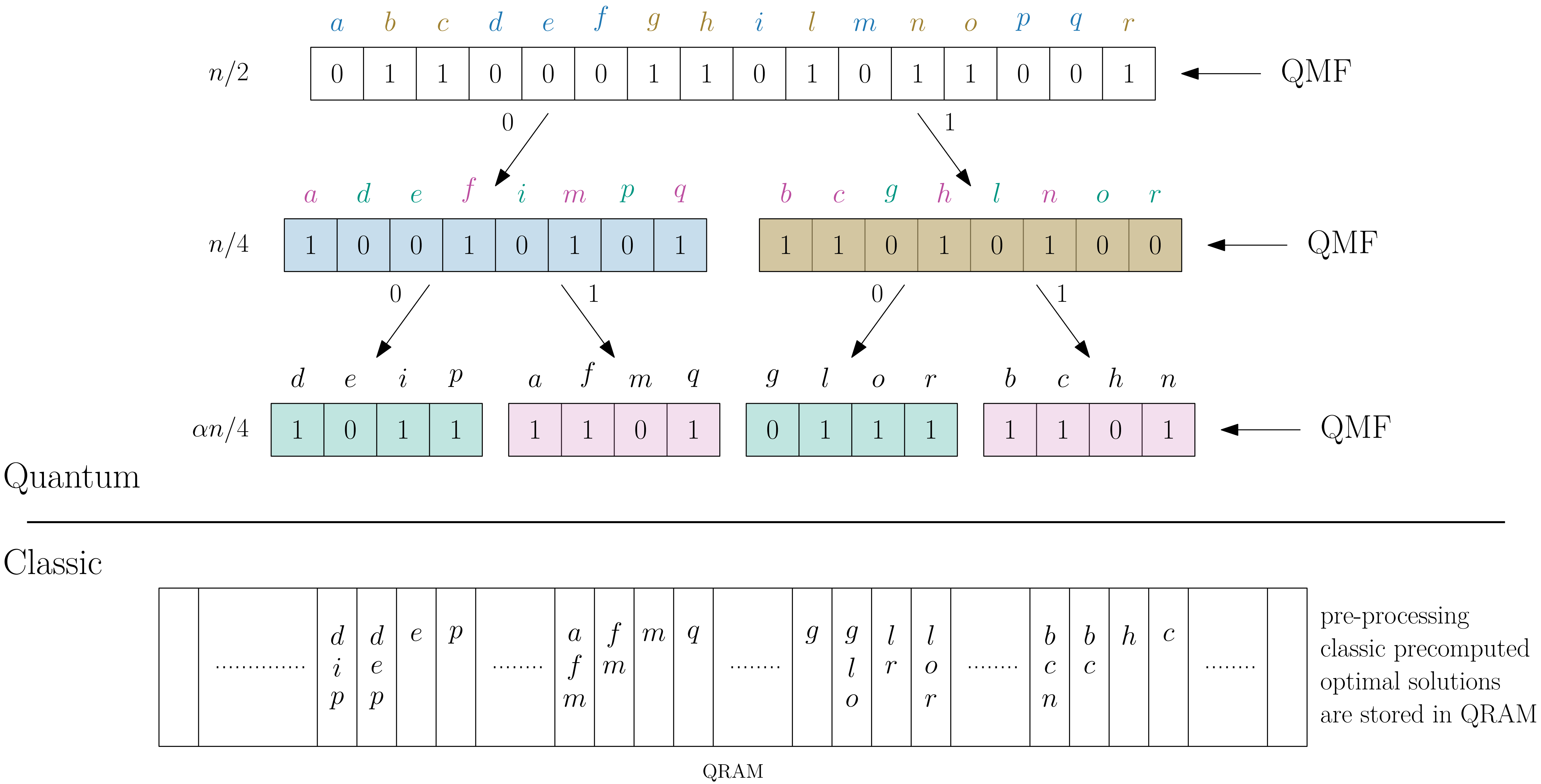

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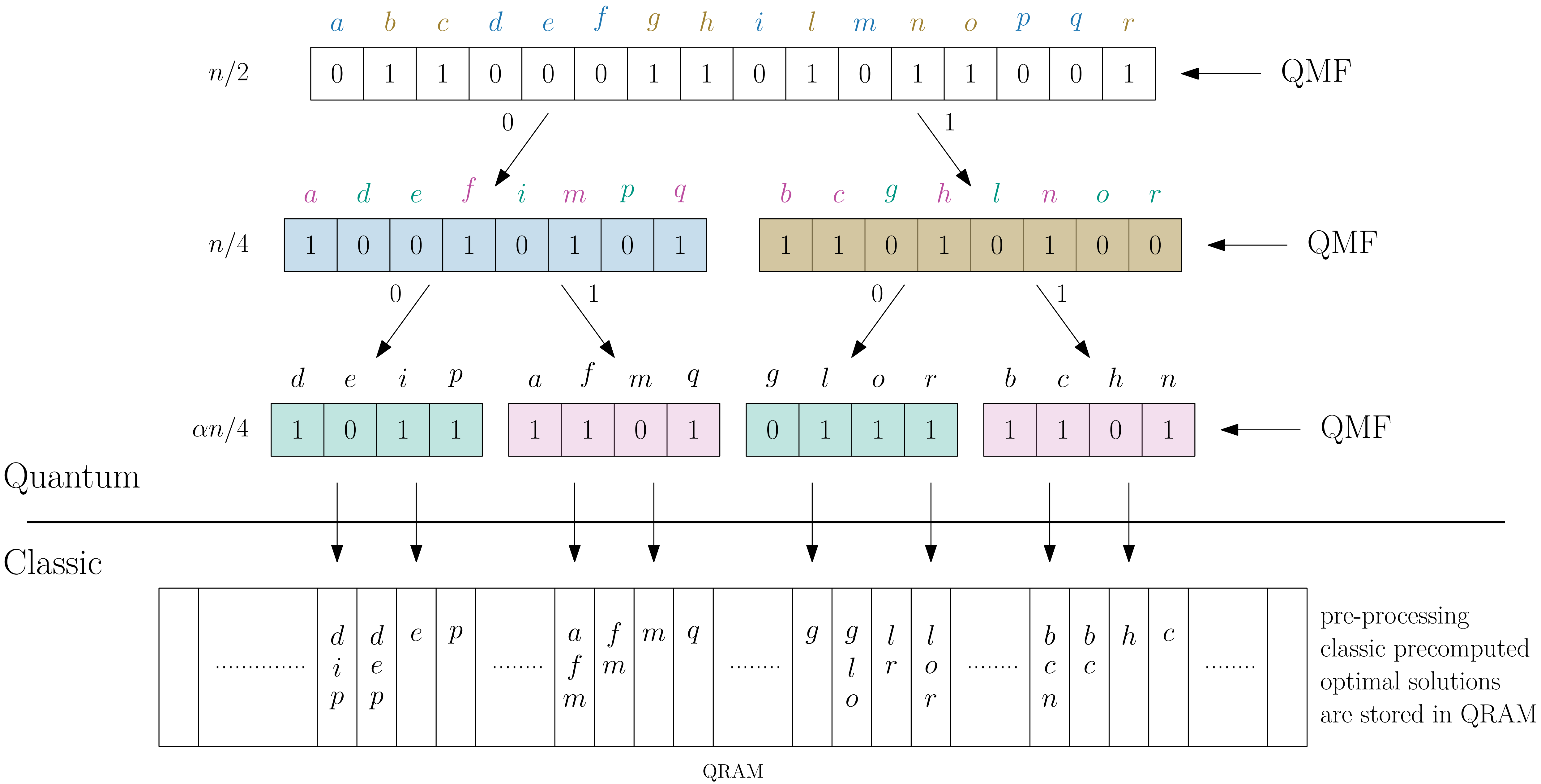

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  - The time and space complexity of the best classic algorithm is $\mathcal{O}^*(2^n)$

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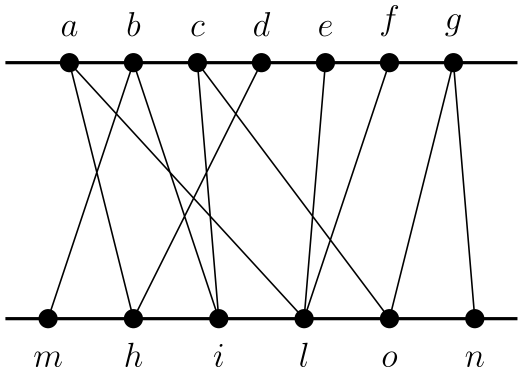

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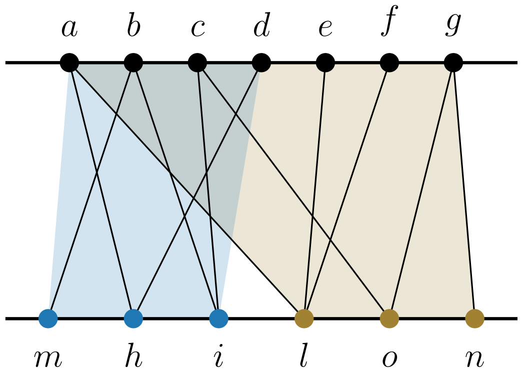

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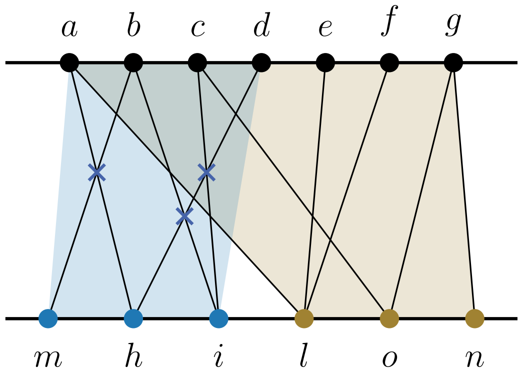

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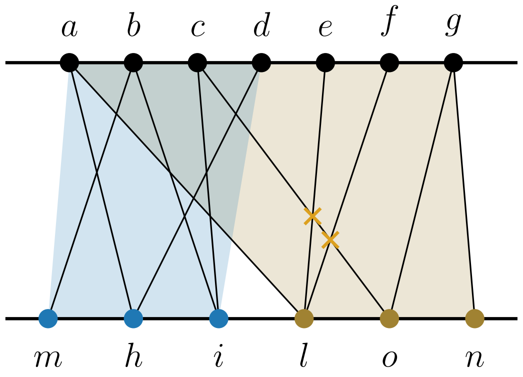

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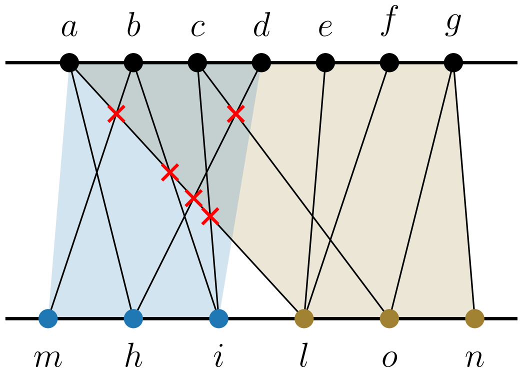

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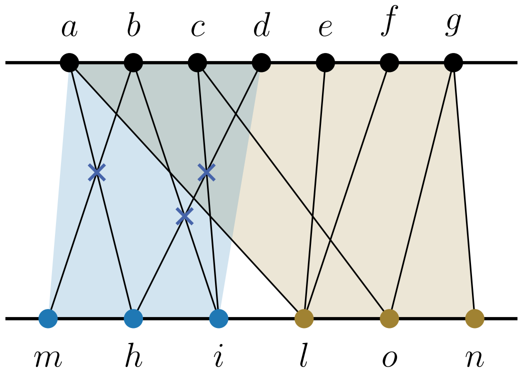
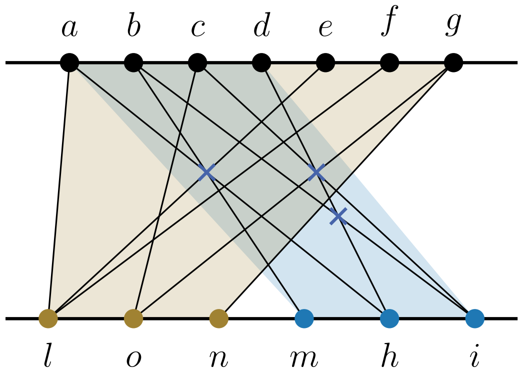


3 crossings



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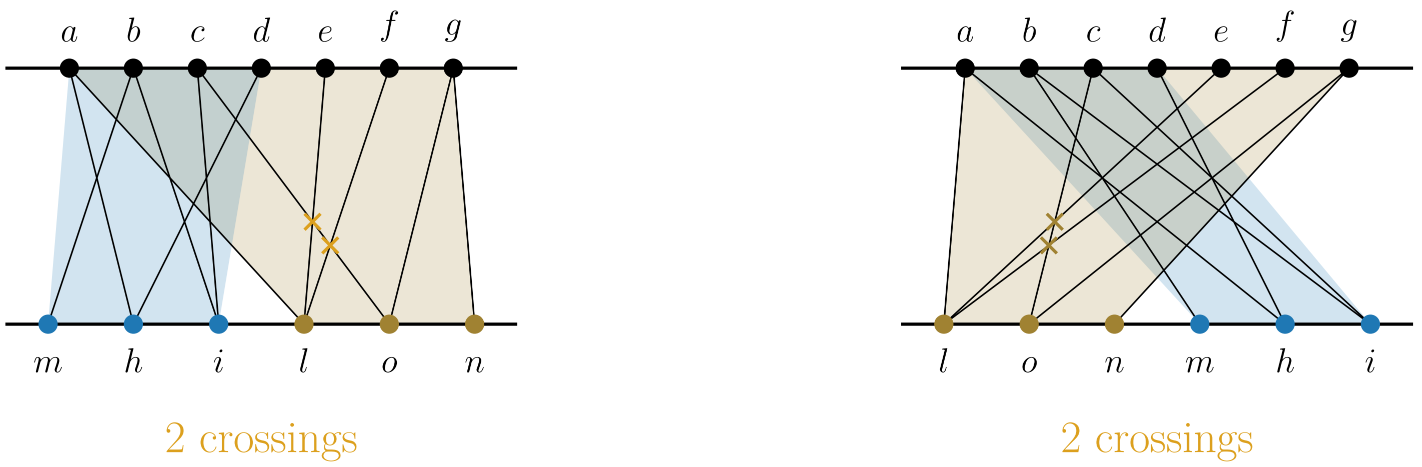

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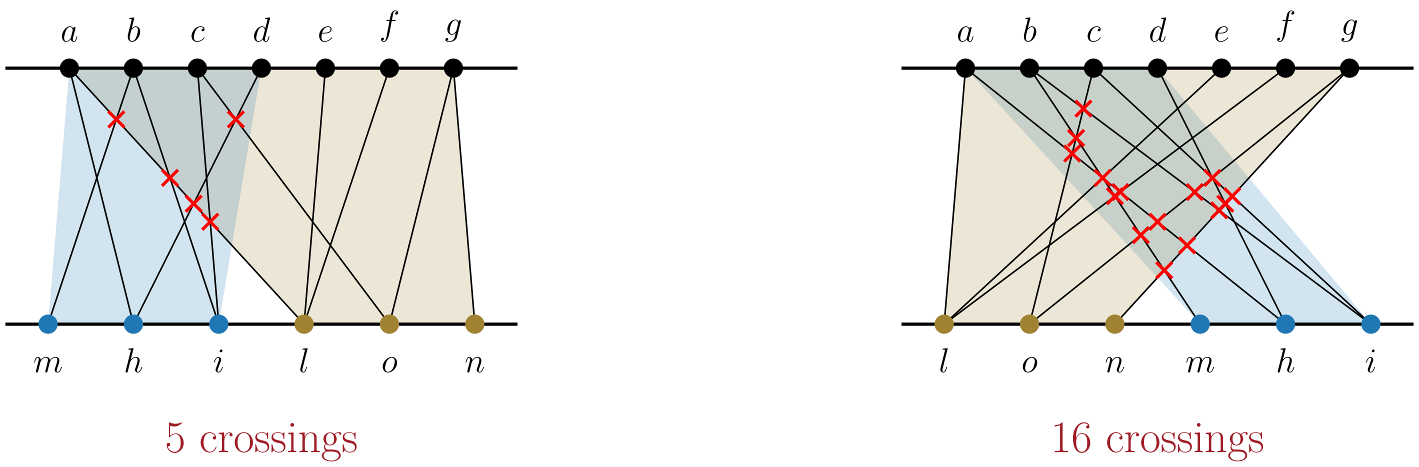

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- There exists a quantum algorithm that solves OSCM using $\mathcal{O}^*(1.728^n)$ time and space

Quantum Divide and Conquer for Set Problems

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Lemma *Let $\mathcal{P}$ be an optimization problem over a set X. Let $|X| = n$ and let $OPT_{\mathcal{P}}(X)$ be the optimal value for $\mathcal{P}$ over X. Suppose that there exists a polynomial-time computable function $f_{\mathcal{P}} : 2^X \times 2^X \to \mathbb{R}$ and a constant $c_{\mathcal{P}}$ such that, for any $S \subseteq X$, it holds that:*

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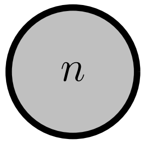

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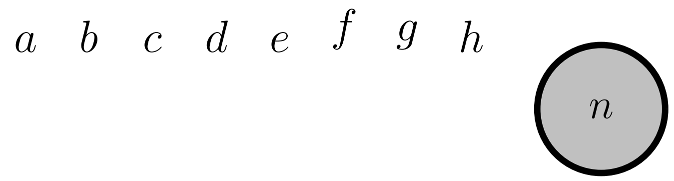

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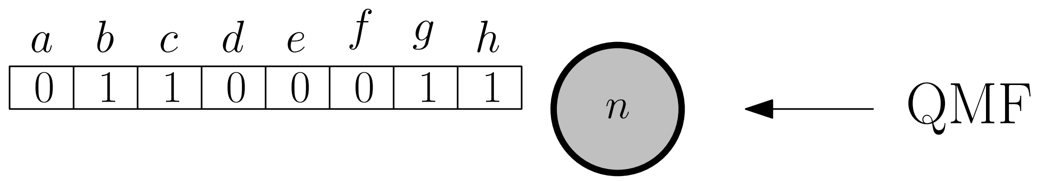

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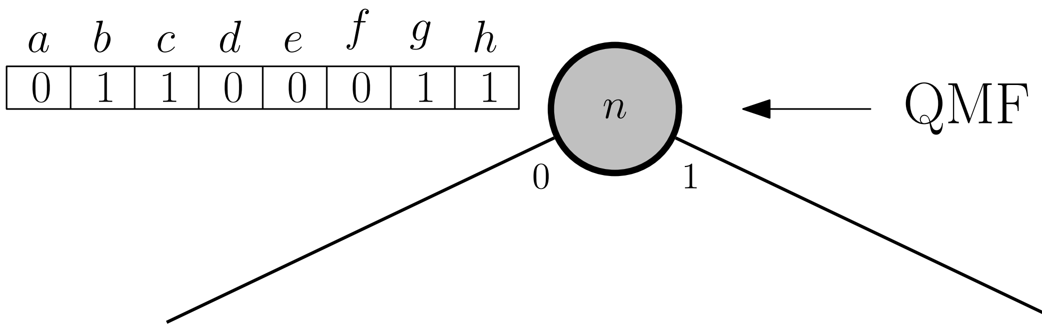

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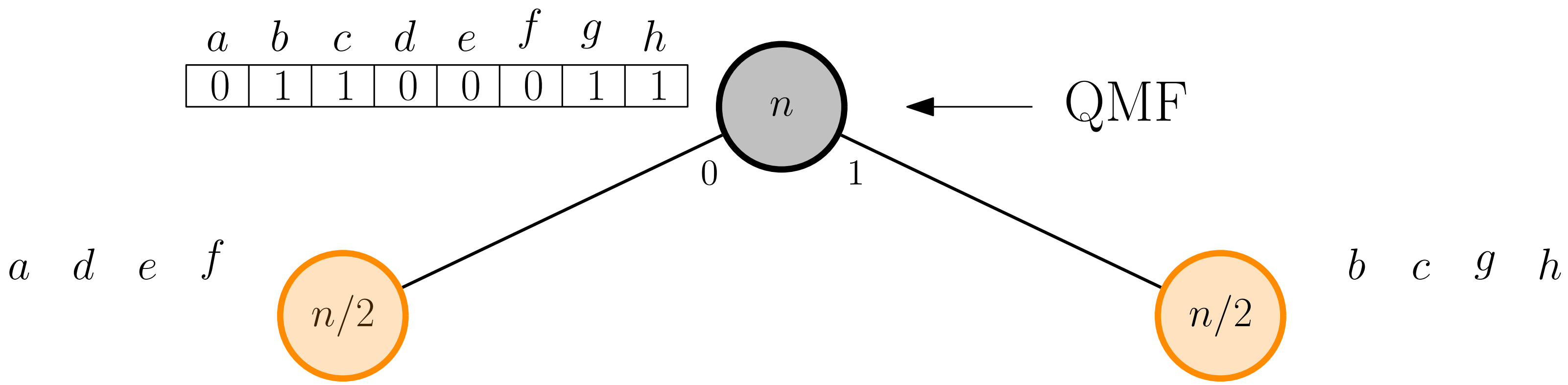

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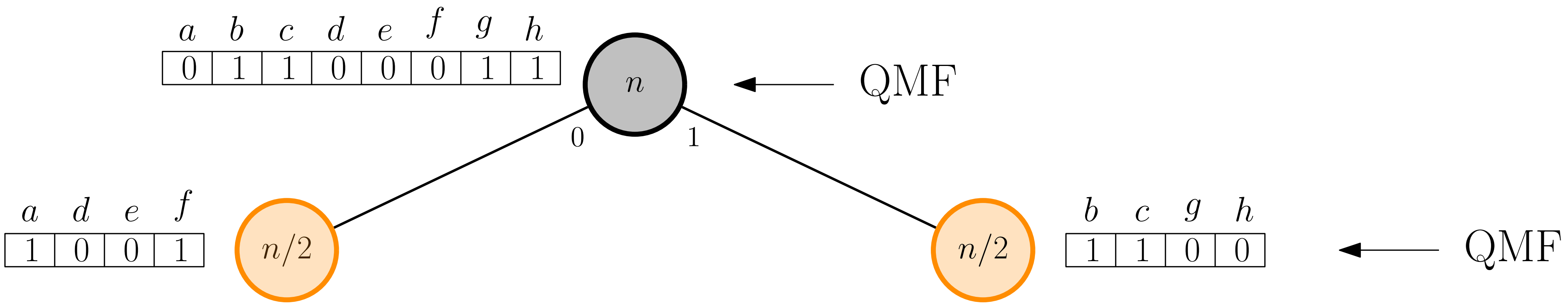

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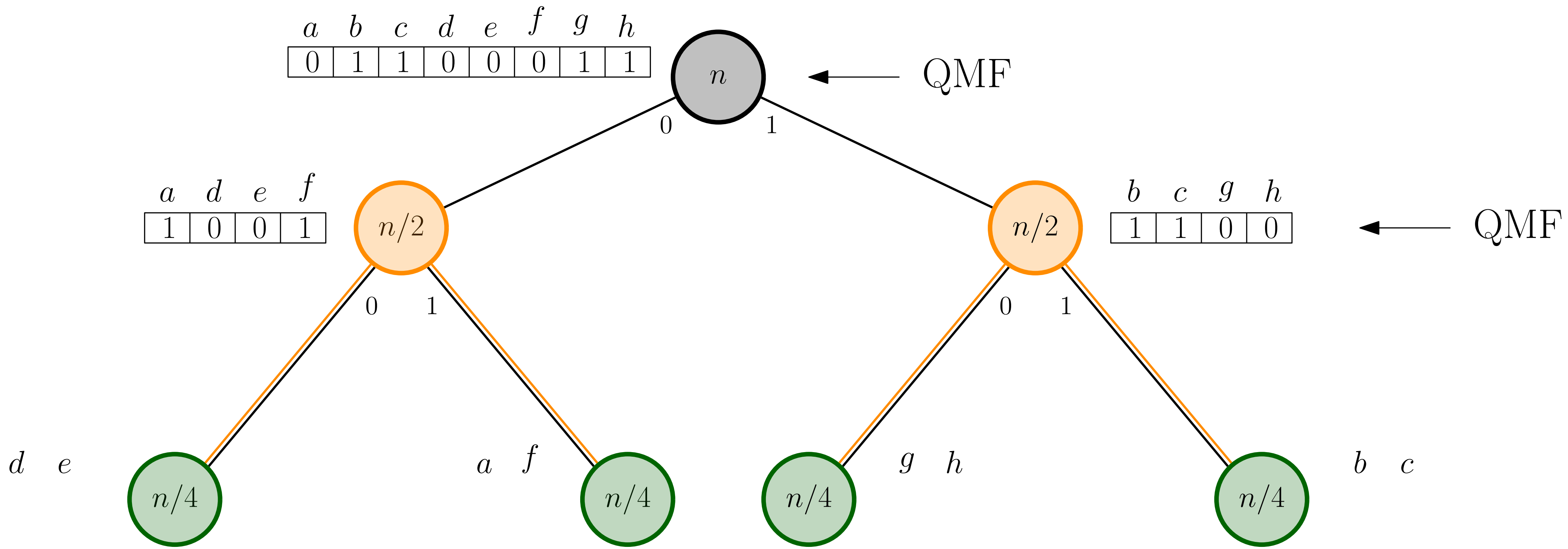

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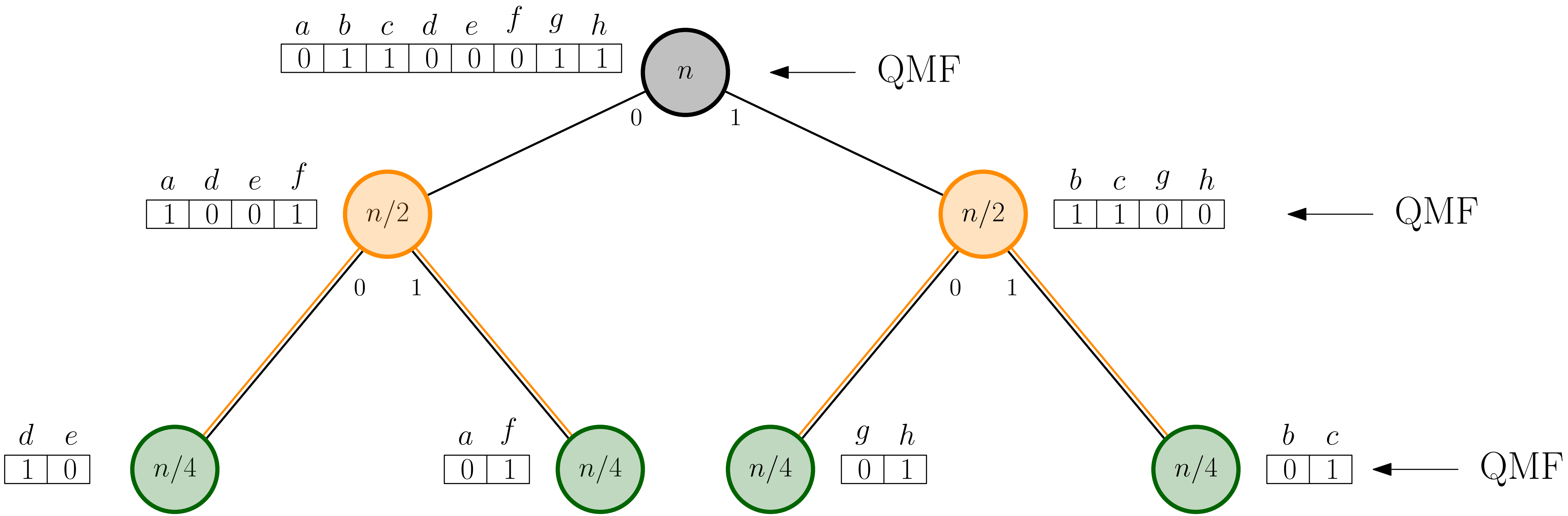

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  - Recall that the time complexity of the best classic algorithm using polynomial space is $\mathcal{O}^*(4^n)$

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Thank you for your attention!