Crossing Numbers of Beyond Planar Graphs Re-revisited: A Framework Approach

Markus Chimani Torben Donzelmann Nick Kloster
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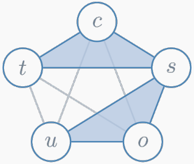
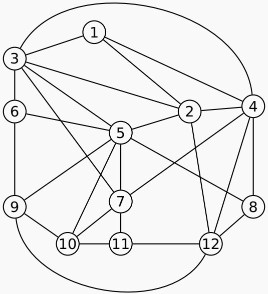


Theoretical Computer Science, Institute for Informatics, Osnabrück University

Crossing Number



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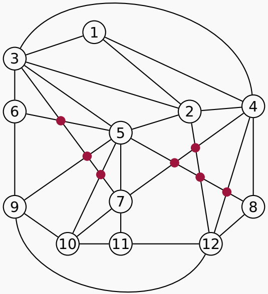
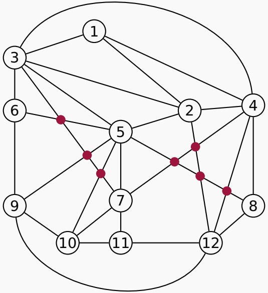


Crossing Number $\mathrm{cr}(G)$

minimum number of edge crossings over all drawings of G in the plane



$\mathrm{cr}(G) \leq 7$

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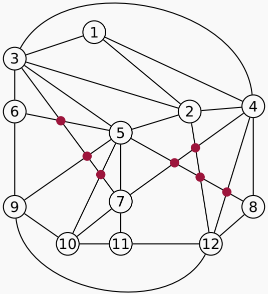
Crossing Number



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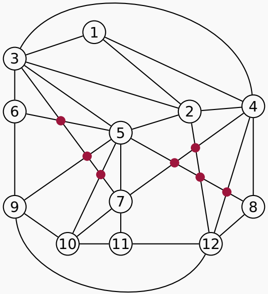
$\mathfrak{B}(k)$**-Crossing Number** $\mathrm{cr}_{\mathfrak{B}(k)}(G)$

restriction to drawings respecting the beyond-planarity concept $\mathfrak{B}(k)$



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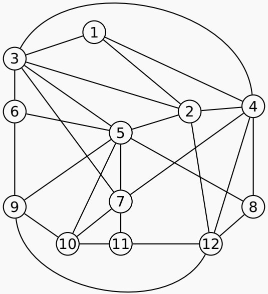
Examples of Beyond Planarity Concepts



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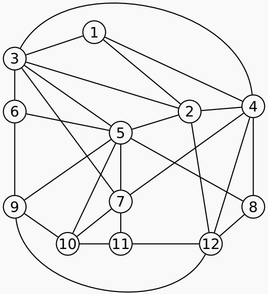

k-Planar

at most *k* crossings per edge

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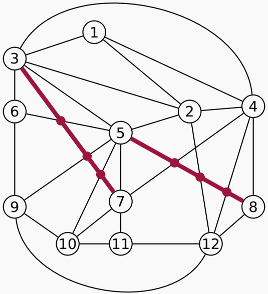
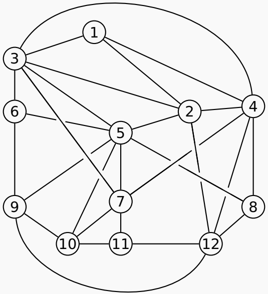


3-planar, but not 2-planar

k-Planar

at most *k* crossings per edge

Examples of Beyond Planarity Concepts



k-Planar

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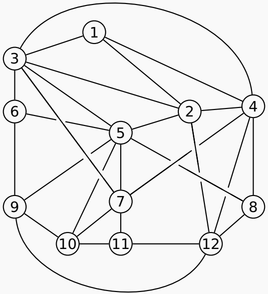
k-Gap-Planar

at most k gaps on each edge



1-gap-planar

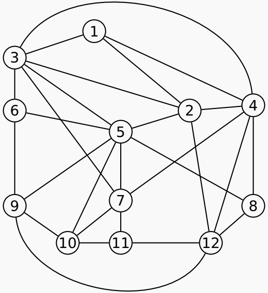
_k_-Planar

at most *k* crossings per edge

_k_-Gap-Planar

at most *k* gaps on each edge

Examples of Beyond Planarity Concepts



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k-Apex-Planar

there are k apex nodes whose removal makes the remaining subdrawing planar

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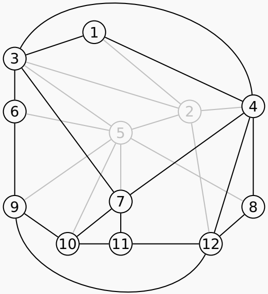

2-apex-planar

***k*-Planar**

at most *k* crossings per edge

***k*-Gap-Planar**

at most *k* gaps on each edge

***k*-Apex-Planar**

there are *k* apex nodes whose removal makes the remaining subdrawing planar

Crossing Ratio

Crossing Ratio of Beyond Planarity Concept $\mathfrak{B}(k)$

How much worse can the $\mathfrak{B}(k)$-crossing number be than the normal crossing number on graphs with n vertices?

$$\text{cr ratio}_{\mathfrak{B}(k)}(n) = \sup_{G \in \mathcal{G}_{\mathfrak{B}(k)}(n)} \frac{\text{cr}_{\mathfrak{B}(k)}(G)}{\text{cr}(G)}$$

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Known Bounds

[Chimani, Kindermann, Montecchiani, Valtr ’19]

for some concepts, but

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specialized constructions for each concept

mostly non-tight bounds for simple drawings

Our contribution

framework that yields short proofs of tight crossing ratio bounds

	previous best	**our results**
k-planar	$\Omega(n/k) \cap \mathcal{O}(nk\sqrt{k})$	$\Theta(n)$
k-vertex-planar	—	$\Theta(n)$
IC-planar	—	$\Theta(n)$
NIC-planar	—	$\Theta(n)$
NNIC-planar	—	$\Theta(n^2)$
k-fan-crossing-free	$\Omega(n^2/k^3) \cap \mathcal{O}(n^2k^2)$	$\Theta(n^2/k)$
fan-planar (& variants)	$\Omega(n) \cap \mathcal{O}(n^2)$	$\Theta(n^2)$
k-edge-crossing	—	$\Theta(k)$
k-gap-planar	$\Omega(n/k^3) \cap \mathcal{O}(nk\sqrt{k})$	$\Theta(n/k)$
k-apex-planar	$\Omega(n/k) \cap \mathcal{O}(n^2k^2)$	$\Theta(n^2/k)$
skewness-k	$\Omega(n/k) \cap \mathcal{O}(nk)$	$\Theta(n)$

Bounding Crossing Ratios

$$\text{cr ratio}_{\mathfrak{B}(k)}(n) = \sup_{G \in \mathcal{G}_{\mathfrak{B}(k)}(n)} \frac{\text{cr}_{\mathfrak{B}(k)}(G)}{\text{cr}(G)}$$

Upper Bound

utilize lower bounds on crossing number

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$$\frac{\text{cr}_{k\text{-planar}}(G)}{\text{cr}(G)} \in \mathcal{O}$$

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$$\frac{\mathrm{cr}_{k\text{-planar}}(G)}{\mathrm{cr}(G)} \in \mathcal{O}\left(\begin{array}{cc} 1 & \text{if } \mathrm{cr}(G) \leq k \\ & \\ & \end{array} \right)$$

Bounding Crossing Ratios

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$$\frac{\text{cr}_{k\text{-planar}}(G)}{\text{cr}(G)} \in \mathcal{O}\begin{pmatrix} 1 & \text{if } \text{cr}(G) \leq k \\ mk/m & \text{if } m > 4n \\ & \end{pmatrix}$$

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Lower Bound

construct a family of $\mathfrak{B}(k)$-graphs $\{G_\ell\}$ that observes the bound

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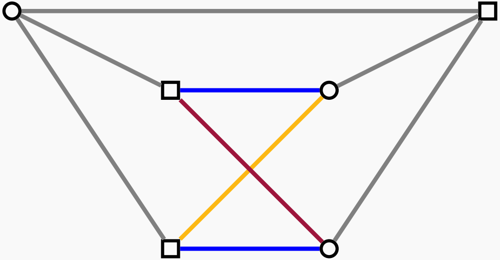
Lower Bound

construct a family of $\mathfrak{B}(k)$-graphs $\{G_\ell\}$ that observes the bound

$\text{cr ratio}_{k\text{-planar}}(n) \in \Omega(nk/k)$ with framework proof

Framework Definitions



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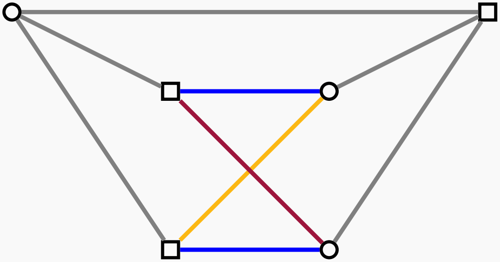

Frame

$K_{3,3}$ with colored **connections**

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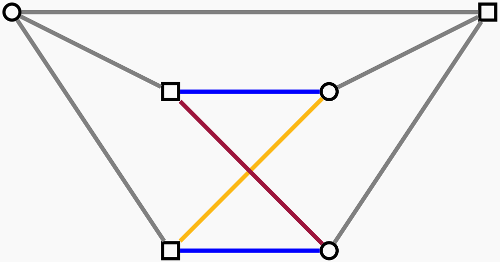

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Framework Graph G_ℓ

frame with connections replaced by **con-graphs**

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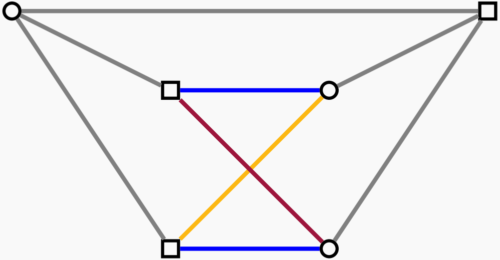

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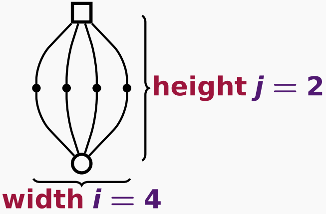

(i, j)-bundle

i parallel paths of length j

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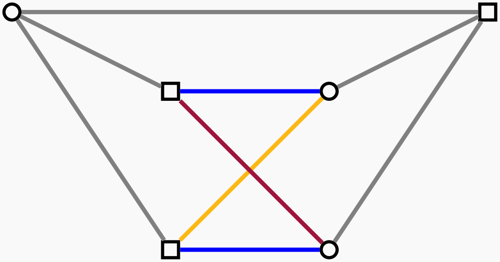

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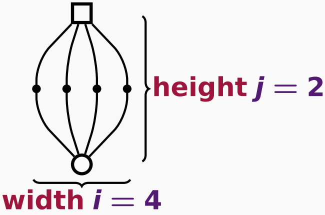

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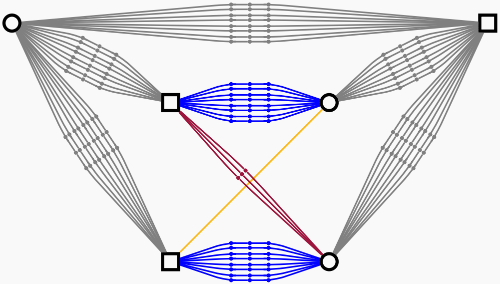

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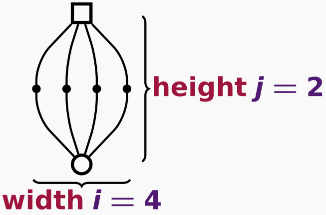

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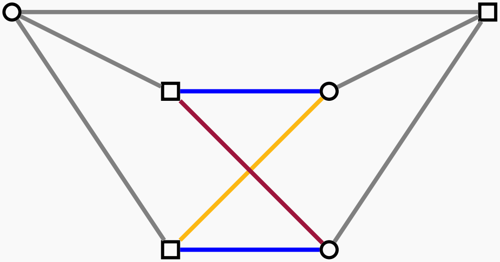
Framework Color Idea



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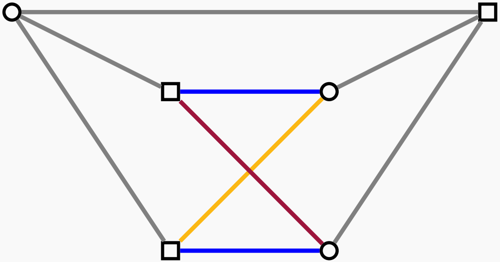

red and yellow con-graphs can cross each other cheaply, but not $\mathfrak{B}(k)$-legal

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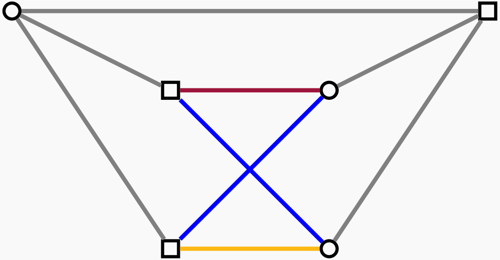

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blue con-graphs are $\mathfrak{B}(k)$-legally crossable, but costly

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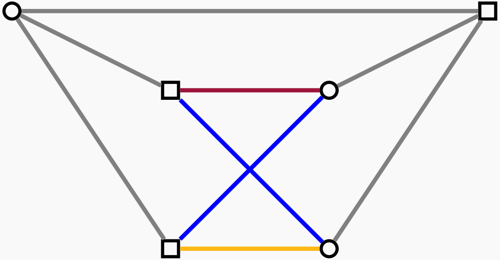

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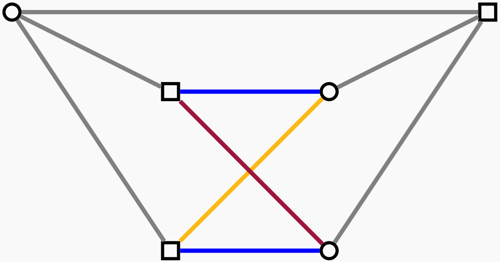
blue con-graphs are $\mathfrak{B}(k)$-legally crossable, but costly

gray con-graphs can cross neither cheaply, nor $\mathfrak{B}(k)$-legally

Lower Bound on $\mathrm{cr}_{k\text{-planar}}(G_\ell)/\mathrm{cr}(G_\ell)$



1. frame coloring

2. con-graphs

3. drawing for upper bound on $\mathrm{cr}(G_\ell)$

4. drawing for k-planarity

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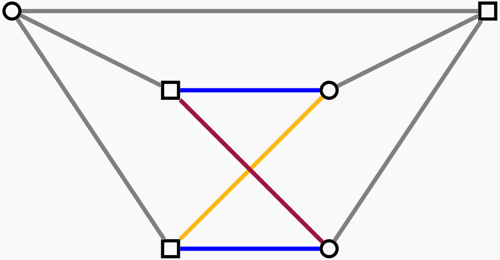

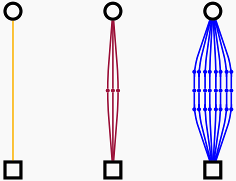

width	1	$k+1$	ℓk
height	1	2	ℓ

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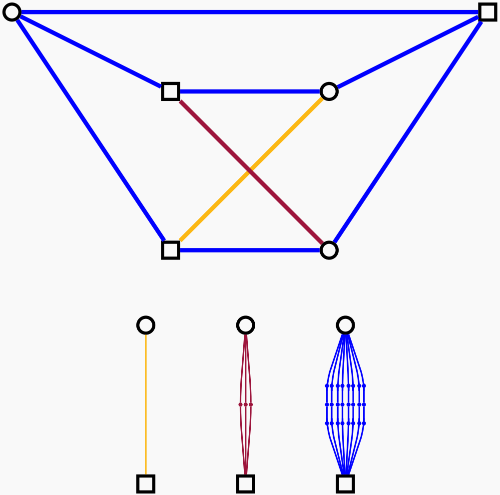

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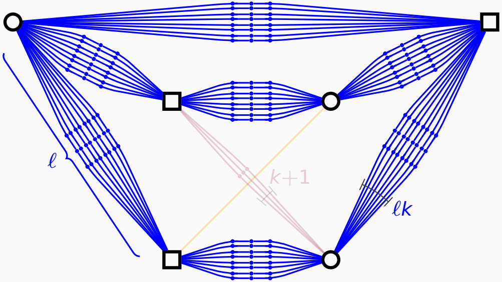

naïve drawing of frame and blue con-graphs

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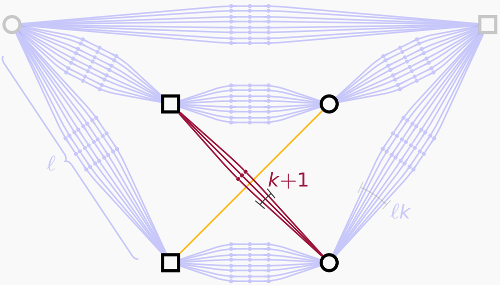

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width of red con-graph is $k+1$

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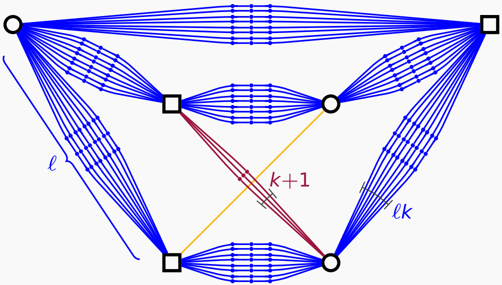

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width of red con-graph is $k+1$

$\mathrm{cr}(G_\ell) \leq k+1 \in \mathcal{O}(k)$

1. frame coloring
2. con-graphs
3. **drawing for upper bound on $\mathrm{cr}(G_\ell)$**
4. drawing for k-planarity
5. lower bound on $\mathrm{cr}_{k\text{-planar}}(G_\ell)$

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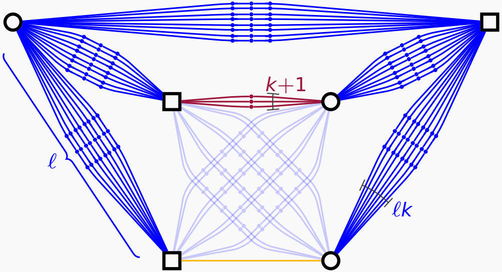

naïve drawing of frame and non-crossing con-graphs

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2. con-graphs
3. drawing for upper bound on $\mathrm{cr}(G_\ell)$
4. **drawing for k-planarity**
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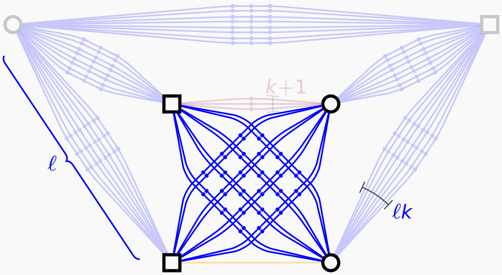
Lower Bound on $\mathrm{cr}_{k\text{-planar}}(G_\ell)/\mathrm{cr}(G_\ell)$



naïve drawing of frame and non-crossing con-graphs

$(\ell k)^2$ crossings on $2 \cdot \ell \cdot \ell k$ edges

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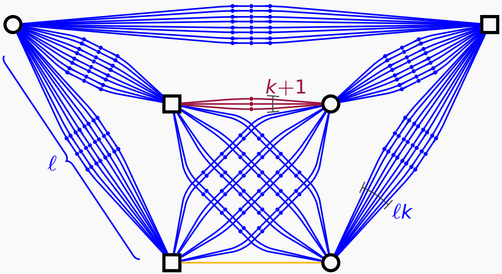


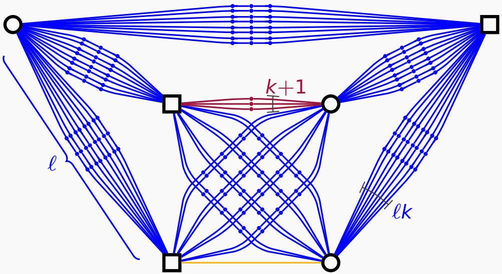
naïve drawing of frame and non-crossing con-graphs

$(\ell k)^2$ crossings on $2 \cdot \ell \cdot \ell k$ edges

at most k crossings per edge

1. frame coloring
2. con-graphs
3. drawing for upper bound on $\mathrm{cr}(G_\ell)$
4. **drawing for k-planarity**
5. lower bound on $\mathrm{cr}_{k\text{-planar}}(G_\ell)$

Lower Bound on $\mathrm{cr}_{k\text{-planar}}(G_\ell)/\mathrm{cr}(G_\ell)$



1. frame coloring

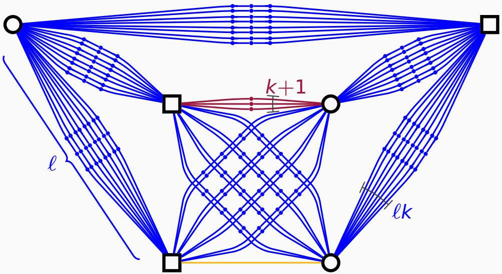
2. con-graphs

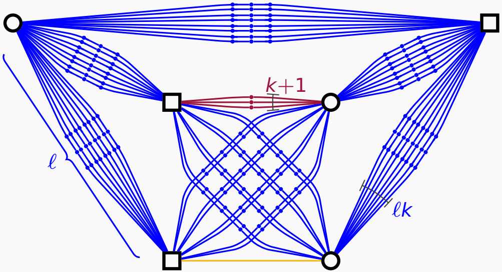
3. drawing for upper bound on $\mathrm{cr}(G_\ell)$

4. drawing for k-planarity

5. lower bound on $\mathrm{cr}_{k\text{-planar}}(G_\ell)$

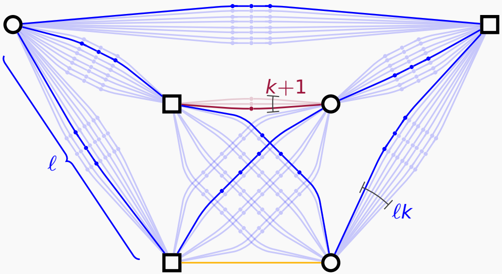
Kuratowski Subdivisions





Kuratowski Subdivision [Kuratowski ’30]

a subdivided $K_{3,3}$ (or K_5)



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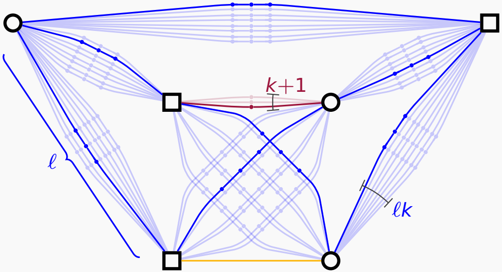
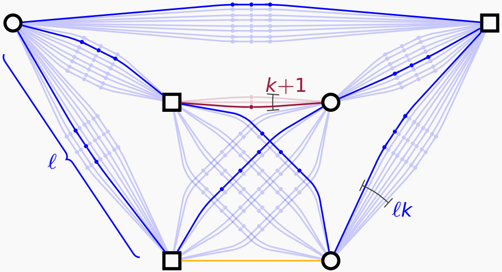
Kuratowski Subdivisions



Kuratowski Subdivision [Kuratowski '30]

a subdivided $K_{3,3}$ (or K_5)

has to be covered by a crossing



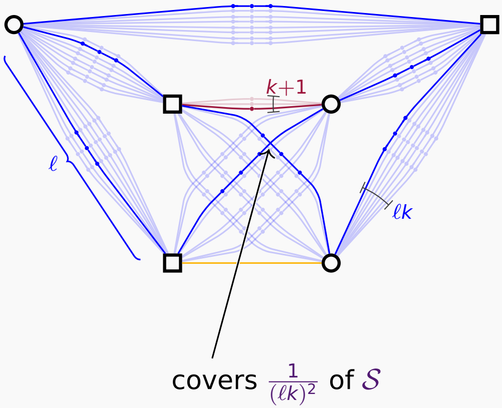
Kuratowski Subdivision [Kuratowski '30]

a subdivided $K_{3,3}$ (or K_5)

has to be covered by a crossing

$\mathcal{S}$ **:** Kuratowski Subdivisions with the frame nodes as Kuratowksi nodes

Kuratowski Subdivisions



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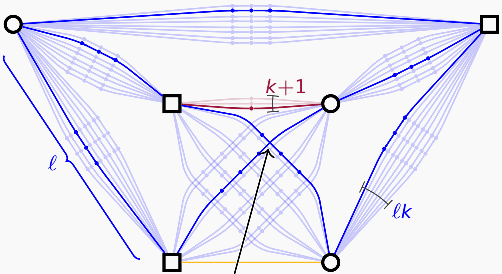

covers $\frac{1}{(\ell k)^2}$ of $\mathcal{S}$

Kuratowski Subdivision [Kuratowski '30]

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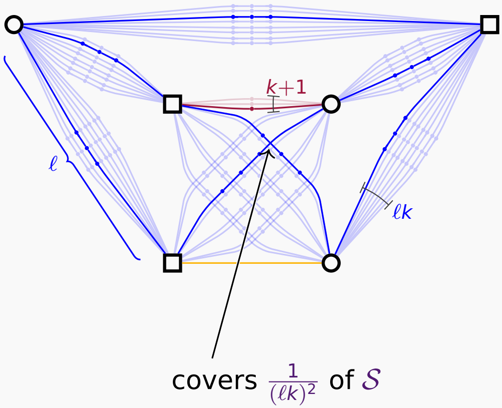

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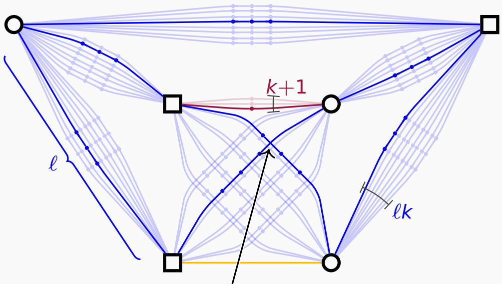

Kuratowski Subdivision

[Kuratowski '30]

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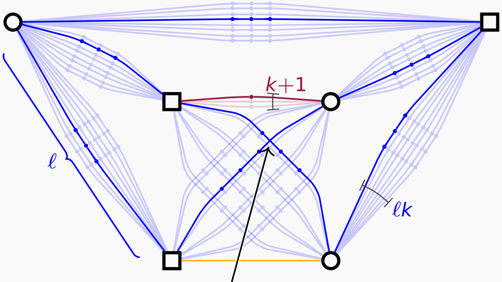
$\mathcal{S}$: Kuratowski Subdivisions with the frame nodes as Kuratowksi nodes

Kuratowski Subdivisions



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Kuratowski Subdivision [Kuratowski '30]

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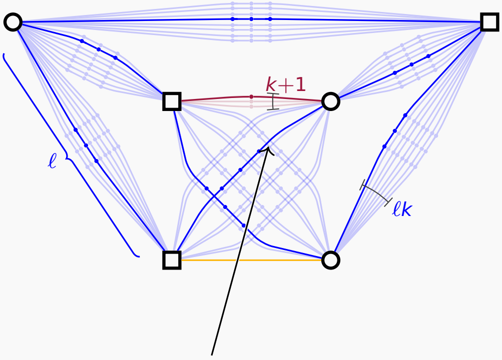
$\mathcal{S}$ **:** Kuratowski Subdivisions with the frame nodes as Kuratowksi nodes



covers $\frac{1}{(\ell k)^2}$ of $\mathcal{S}$

Kuratowski Subdivision

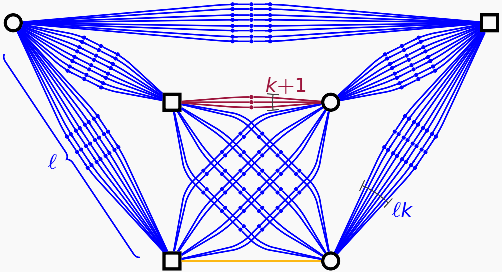
[Kuratowski '30]

a subdivided $K_{3,3}$ (or K_5)

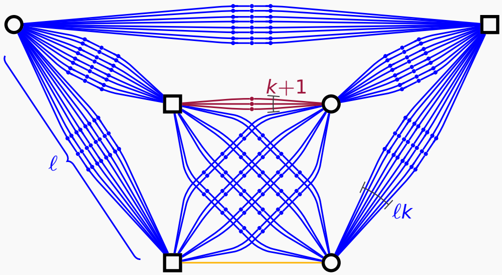
has to be covered by a crossing

$\mathcal{S}$ **:** Kuratowski Subdivisions with the frame nodes as Kuratowksi nodes

Lower Bound on $\mathrm{cr}_{k\text{-planar}}(G_\ell)$



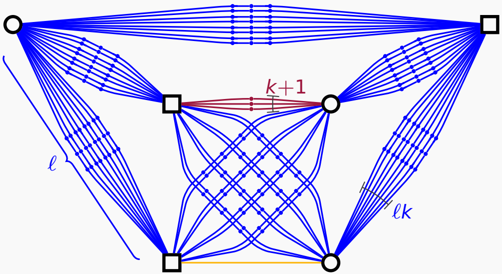
Lower Bound on $\mathrm{cr}_{k\text{-planar}}(G_\ell)$



***k*-planar drawing**

at most k crossings on yellow

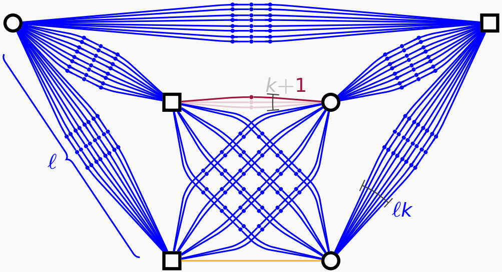
Lower Bound on $\mathrm{cr}_{k\text{-planar}}(G_\ell)$



***k*-planar drawing**

at most k crossings on yellow

at least one red path R not crossed by yellow edge



k-planar drawing

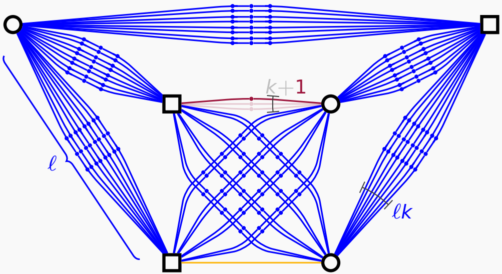
at most k crossings on yellow

at least one red path R not crossed by yellow edge

subdrawing with R as red con-graph

has no red-yellow crossings

Lower Bound on $\mathrm{cr}_{k\text{-planar}}(G_\ell)$



k-planar drawing

at most k crossings on yellow

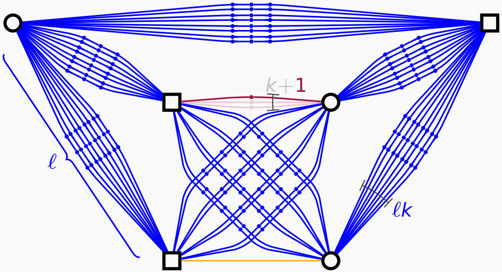
at least one red path R not crossed by yellow edge

subdrawing with R as red con-graph

has no red-yellow crossings

$\mathcal{S}' \subset \mathcal{S}$ **:** Kuratowski Subdivisions with R as the red Kuratowski path

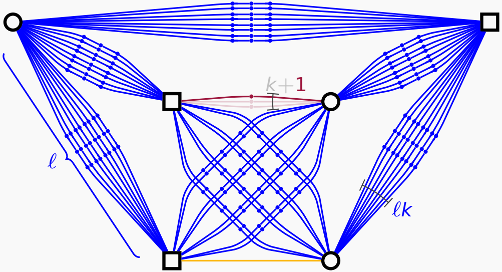
Lower Bound on $\mathrm{cr}_{k\text{-planar}}(G_\ell)$



crossings on red and yellow

cover at most $3 \cdot k \cdot \frac{1}{1 \cdot \ell k}$ of $\mathcal{S}'$

Lower Bound on $\mathrm{cr}_{k\text{-planar}}(G_\ell)$



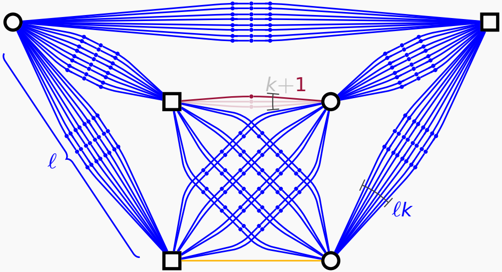
crossings on red and yellow

cover at most $3 \cdot k \cdot \frac{1}{1 \cdot \ell k}$ of $\mathcal{S}'$

for $\ell \geq 4$:

at least $\frac{1}{4}$ of $\mathcal{S}'$ is covered by:

Lower Bound on $\mathrm{cr}_{k\text{-planar}}(G_\ell)$



crossings on red and yellow

cover at most $3 \cdot k \cdot \frac{1}{1 \cdot \ell k}$ of $\mathcal{S}'$

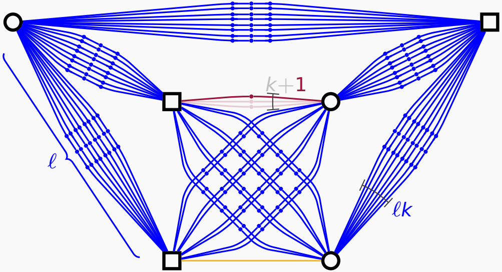
for $\ell \geq 4$:

at least $\frac{1}{4}$ of $\mathcal{S}'$ is covered by:

blue-blue crossings

each covers at most $\frac{1}{(\ell k)^2}$ of $\mathcal{S}'$

Lower Bound on $\mathrm{cr}_{k\text{-planar}}(G_\ell)$



crossings on red and yellow

cover at most $3 \cdot k \cdot \frac{1}{1 \cdot \ell k}$ of $\mathcal{S}'$

for $\ell \geq 4$:

at least $\frac{1}{4}$ of $\mathcal{S}'$ is covered by:

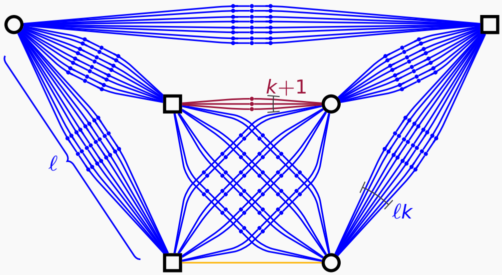
blue-blue crossings

each covers at most $\frac{1}{(\ell k)^2}$ of $\mathcal{S}'$

total number of crossings

$\Omega((\ell k)^2) = \Omega(nk)$ crossings

Lower Bound on $\text{cr}_{k\text{-planar}}(G_\ell)/\text{cr}(G_\ell)$



1. frame coloring

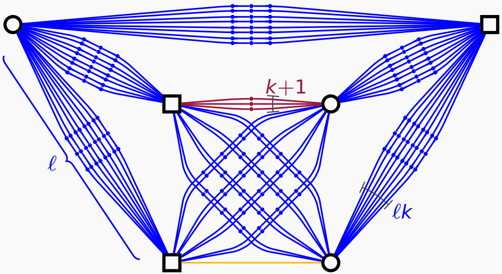
2. con-graphs

3. drawing for upper bound on $\text{cr}(G_\ell)$
$\rightarrow \mathcal{O}(k)$

4. drawing for k-planarity

5. lower bound on $\text{cr}_{k\text{-planar}}(G_\ell)$
$\rightarrow \Omega(nk)$

Lower Bound on $\mathrm{cr}_{k\text{-planar}}(G_\ell)/\mathrm{cr}(G_\ell)$



1. frame coloring

2. con-graphs

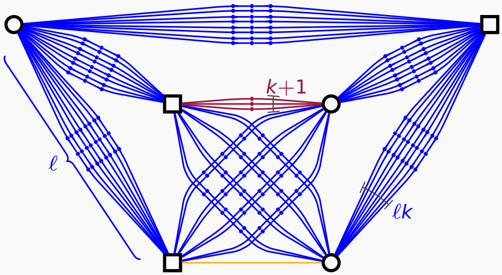
3. drawing for upper bound on $\mathrm{cr}(G_\ell)$
$\rightarrow \mathcal{O}(k)$

4. drawing for k-planarity

5. lower bound on $\mathrm{cr}_{k\text{-planar}}(G_\ell)$
$\rightarrow \Omega(nk)$

Lower Bound on $\mathrm{cr}_{k\text{-planar}}(G_\ell)/\mathrm{cr}(G_\ell)$
$\rightarrow \Omega(nk/k)$

Lower Bound on $\mathrm{cr}_{k\text{-planar}}(G_\ell)/\mathrm{cr}(G_\ell)$



1. frame coloring

2. con-graphs

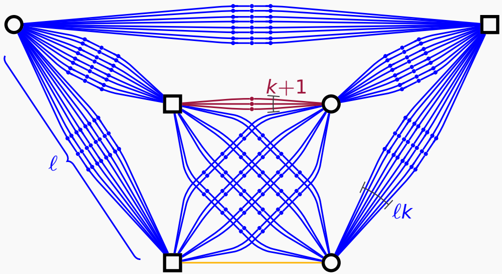
3. drawing for upper bound on $\mathrm{cr}(G_\ell)$
$\to \mathcal{O}(k)$

4. drawing for k-planarity

5. lower bound on $\mathrm{cr}_{k\text{-planar}}(G_\ell)$
$\to \Omega(nk)$

Lower Bound on $\mathrm{cr\ ratio}_{k\text{-planar}}(n)$
$\to \Omega(nk/k)$

Lower Bound on $\mathrm{cr\ ratio}_{k\text{-planar}}(n)$



1. frame coloring

2. con-graphs

3. drawing for upper bound on $\mathrm{cr}(G_\ell)$
$\rightarrow \mathcal{O}(k)$

4. drawing for k-planarity

5. lower bound on $\mathbf{cr}_{k\text{-planar}}(G_\ell)$
$\rightarrow \Omega(nk)$

Lower Bound on $\mathrm{cr\ ratio}_{k\text{-planar}}(n)$
$\rightarrow \Omega(n)$

Framework: Short Proofs, Tight Bounds

	previous best	our results
***k*-planar**	$\Omega(n/k) \cap \mathcal{O}(nk\sqrt{k})$	$\boldsymbol{\Theta(n)}$
k-vertex-planar	—	$\Theta(n)$
IC-planar	—	$\Theta(n)$
NIC-planar	—	$\Theta(n)$
NNIC-planar	—	$\Theta(n^2)$
k-fan-crossing-free	$\Omega(n^2/k^3) \cap \mathcal{O}(n^2k^2)$	$\Theta(n^2/k)$
adjacency- & fan-crossing	$\Omega(n) \cap \mathcal{O}(n^2)$	$\Theta(n^2)$
weakly & strongly fan-planar	$\Omega(n) \cap \mathcal{O}(n^2)$	$\Theta(n^2)$
k-edge-crossing	—	$\Theta(k)$
k-gap-planar	$\Omega(n/k^3) \cap \mathcal{O}(nk\sqrt{k})$	$\Theta(n/k)$
k-apex-planar	$\Omega(n/k) \cap \mathcal{O}(n^2k^2)$	$\Theta(n^2/k)$
skewness-k	$\Omega(n/k) \cap \mathcal{O}(nk)$	$\Theta(n)$

Framework: Short Proofs, Tight Bounds

	previous best	our results
k-planar	$\Omega(n/k) \cap \mathcal{O}(nk\sqrt{k})$	$\Theta(n)$
k-vertex-planar	—	$\Theta(n)$
IC-planar	—	$\Theta(n)$
NIC-planar	—	$\Theta(n)$
NNIC-planar	—	$\Theta(n^2)$
k-fan-crossing-free	$\Omega(n^2/k^3) \cap \mathcal{O}(n^2k^2)$	$\Theta(n^2/k)$
adjacency- & fan-crossing	$\Omega(n) \cap \mathcal{O}(n^2)$	$\Theta(n^2)$
weakly & strongly fan-planar	$\Omega(n) \cap \mathcal{O}(n^2)$	$\Theta(n^2)$
k-edge-crossing	—	$\Theta(k)$
k-gap-planar	$\Omega(n/k^3) \cap \mathcal{O}(nk\sqrt{k})$	$\Theta(n/k)$
k-apex-planar	$\Omega(n/k) \cap \mathcal{O}(n^2k^2)$	$\Theta(n^2/k)$
skewness-k	$\Omega(n/k) \cap \mathcal{O}(nk)$	$\Theta(n)$

Framework: Short Proofs, Tight Bounds

	previous best	our results	
k-planar	$\Omega(n/k) \cap \mathcal{O}(nk\sqrt{k})$	$\Theta(n)$	+1
k-vertex-planar	—	$\Theta(n)$	+1
IC-planar	—	$\Theta(n)$	
NIC-planar	—	$\Theta(n)$	
NNIC-planar	—	$\Theta(n^2)$	
k-fan-crossing-free	$\Omega(n^2/k^3) \cap \mathcal{O}(n^2k^2)$	$\Theta(n^2/k)$	+1
adjacency- & fan-crossing	$\Omega(n) \cap \mathcal{O}(n^2)$	$\Theta(n^2)$	
weakly & strongly fan-planar	$\Omega(n) \cap \mathcal{O}(n^2)$	$\Theta(n^2)$	
k-edge-crossing	—	$\Theta(k)$	+1
k-gap-planar	$\Omega(n/k^3) \cap \mathcal{O}(nk\sqrt{k})$	$\Theta(n/k)$	~~+1~~
k-apex-planar	$\Omega(n/k) \cap \mathcal{O}(n^2k^2)$	$\Theta(n^2/k)$	+1
skewness-k	$\Omega(n/k) \cap \mathcal{O}(nk)$	$\Theta(n)$	+1

+1 : already holds for $\sup_{G\in\mathcal{G}_{\mathfrak{B}(k)}(n)} \frac{\mathrm{cr}_{\mathfrak{B}(k)}(G)}{\mathrm{cr}_{\mathfrak{B}(k+1)}(G)}$

Framework: Short Proofs, Tight Bounds

	previous best [simple]	our results	
k-planar	$\Omega(n/k) \cap \mathcal{O}(nk\sqrt{k})$	$\Theta(n)$	[+1]
k-vertex-planar	—	$\Theta(n)$	[+1]
IC-planar	—	$\Theta(n)$	
NIC-planar	—	$\Theta(n)$	
NNIC-planar	—	$\Theta(n^2)$	
k-fan-crossing-free	$\Omega(n^2/k^3) \cap \mathcal{O}(n^2k^2)$	$\Theta(n^2/k)$	[+1]
adjacency- & fan-crossing	$\Omega(n) \cap \mathcal{O}(n^2)$	$\Theta(n^2)$	[simple]
weakly & strongly fan-planar	$\Omega(n) \cap \mathcal{O}(n^2)$	$\Theta(n^2)$	
k-edge-crossing	—	$\Theta(k)$	[+1]
k-gap-planar	$\Omega(n/k^3) \cap \mathcal{O}(nk\sqrt{k})$	$\Theta(n/k)$	~~[+1]~~
k-apex-planar	$\Omega(n/k) \cap \mathcal{O}(n^2k^2)$	$\Theta(n^2/k)$	[+1]
skewness-k	$\Omega(n/k) \cap \mathcal{O}(nk)$	$\Theta(n)$	[+1]

[+1] : already holds for $\sup_{G\in\mathcal{G}_{\mathfrak{B}(k)}(n)} \frac{\mathrm{cr}_{\mathfrak{B}(k)}(G)}{\mathrm{cr}_{\mathfrak{B}(k+1)}(G)}$

[simple] : upper bound only for simple drawings