Partitioning Complete Geometric Graphs on Dense Point Sets into Plane Subgraphs

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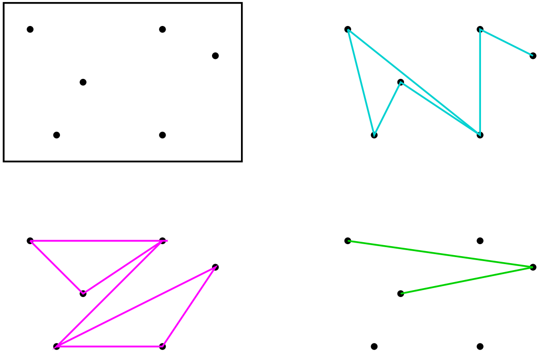
Abstract

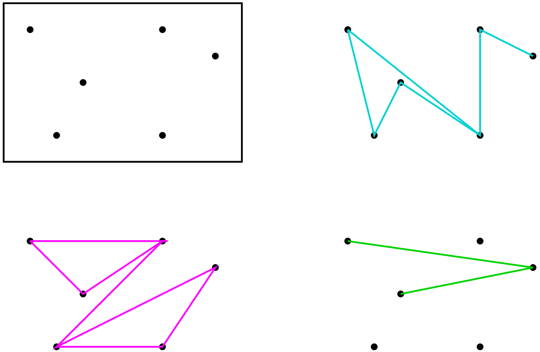
It is a well known question of Bose, Hurtado, Rivera-Campo, and Wood, whether there exists a positive constant $c < 1$, such that (the edges of) every complete geometric graph on n points can be partitioned into at most cn plane graphs (that is, noncrossing subgraphs).

Example with $n = 6$ *and* $c = 1/2$*:*



Abstract

We answer 'yes' in the special case where the underlying point set P is *dense*, which means that the ratio between the maximum and the minimum distances in P is of the order of $\Theta(\sqrt{n})$.



And leave the general question open!

Background

- Points in *general position*: no three points are collinear
- Obviously, every complete geometric graph of n vertices can be decomposed into $n-1$ plane stars.
- There is an intimate relationship between the above problem and another old (and still unsolved) question in combinatorial geometry, due to Aronov, Erdős, Goddard, Kleitman, Klugerman, Pach, and Schulman (1991).

  Two segments are said to *cross* each other if they do not share an endpoint and they have an interior point in common.

Problem

Does there exist a positive constant c such that every complete geometric graph on n vertices has cn pairwise crossing edges?

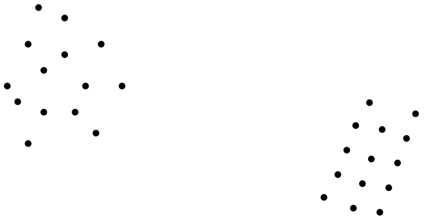
Dense point sets

For a set A of n points in the plane, consider the ratio

$$D(A) = \frac{\max\{|ab| : a, b \in A, a \neq b\}}{\min\{|ab| : a, b \in A, a \neq b\}},$$

where $|ab|$ is the Euclidean distance between points a and b.
An n-element point set A satisfying the condition $D(A) \leq \alpha\, n^{1/2}$, for some constant $\alpha \geq \alpha_0$, is said to be α-*dense*.

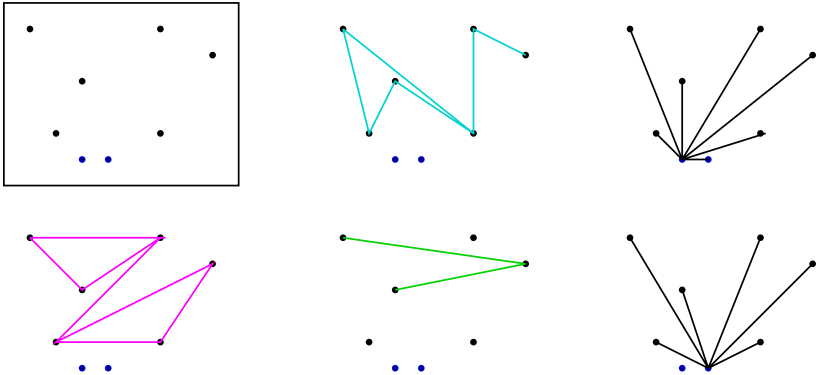
An example with $\alpha = 5$*:*



Preliminaries

Observation. If a suitable large subset $P' \subset P$ (i.e., $|P'| = \Omega(n)$) can be partitioned into at most $c'|P'|$ noncrossing subgraphs, where $c' < 1$, then the entire set P can be partitioned into at most cn noncrossing subgraphs, where $c = c(c') < 1$.

How?: add *stars* to the decomposition of P': *Example with* $n = 8$ *and* $c = 5/8$*:*



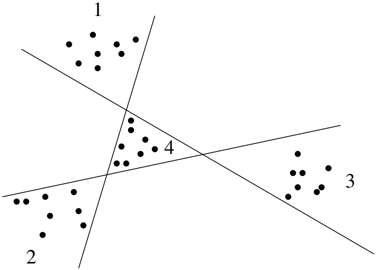
Starting the proof

Q: What should be P'? It can be B!

Lemma

(Pach, Saghafian, and Schnider 2023). *Let $B = \bigcup_{i=1}^{4} B_i$ be a set of $4m$ points, where $|B_1| = |B_2| = |B_3| = |B_4| = m$, such that for every choice $p_i \in B_i$, for $i = 1, 2, 3, 4$, p_4 lies inside the convex hull of $\{p_1, p_2, p_3\}$. Then the complete geometric graph $K_{4m}[B]$ can be decomposed into at most $3m$ plane subgraphs.*

Example with $m = 8$:



Decomposition of $K_{4m}[B]$:

1. all stars emanating from points in B_1 connecting to all points in B_1 and B_2 together with all stars emanating from points in B_3 connecting to all points in B_3 and B_4
2. all stars emanating from points in B_2 connecting to all points in B_2 and B_3 together with all stars emanating from points in B_4 connecting to all points in B_4 and B_1
3. all stars emanating from points in B_1 connecting to all points in B_1 and B_3 together with all stars emanating from points in B_2 connecting to all points in B_2 and B_4

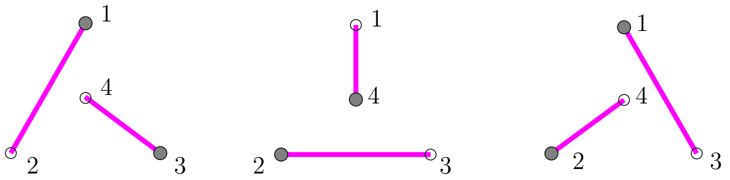


Figure: Sketch of the $3m$ plane subgraphs in the lemma.

Proof sketch

We show that every α-dense n-element point set A contains a subset B with $m = \Omega(n)$ and satisfying the conditions.

Let $k = k(\alpha) \geq 3\alpha^2$ and set $n_0 = \lceil 12k^2/\alpha^2 \rceil$. We distinguish between $n \leq n_0$, and $n \geq n_0$.

- $n \leq n_0$: Recall that there is a decomposition of $K_n[A]$ into $n-1$ stars. Note that $n - 1 \leq cn$ for $n \leq n_0$ provided that $c < 1$ is large enough.
- $n \geq n_0$: Let A be an n-element α-dense set.
- Since $D(A) \leq \alpha\sqrt{n}$, we may assume that A is contained in an axis-aligned square Q of side-length $\alpha\sqrt{n}$.
- Subdivide Q into k^2 axis-parallel squares, called *cells*, of side-length $\alpha\sqrt{n}/k$. Let Σ be the set of all k^2 cells in Q.

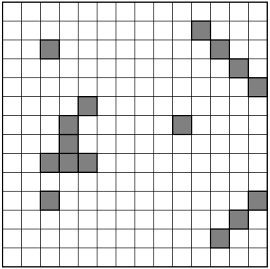
Proof sketch

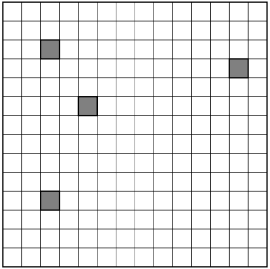
A cell $\sigma \in \Sigma$ is said to be *rich* if it contains at least $n/(3k^2)$ points of A, and *poor* otherwise. Let $\mathsf{R} \subset \Sigma$ denote the set of rich cells.

Lemma

There are at least $\frac{k^2}{3\alpha^2}$ rich cells; that is, $|\mathsf{R}| \geq \frac{k^2}{3\alpha^2}$.

It remains to find four rich cells in a suitable configuration – with one in the middle:



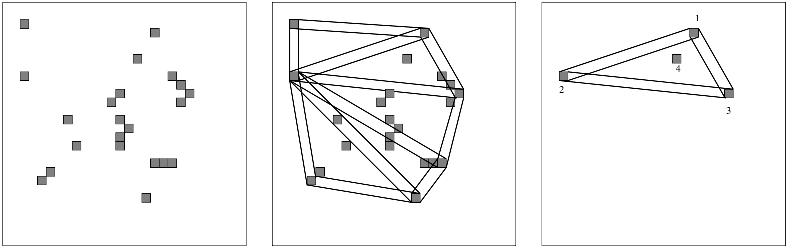


Proof sketch

- Let $P = \texttt{conv}(\mathsf{R})$. Note that P is a lattice polygon whose vertices are in the $(k+1)\times(k+1)$ grid $\mathcal{G}$ subdividing Q.
- As a lattice polygon P has $v(P) \le c' k^{2/3}$ vertices in $\mathcal{G}$.
- For $k > (24c' \cdot \alpha^2)^3$ we have

$$\frac{k^2}{3\alpha^2} - 8c'\, k^{5/3} > 0.$$

Setting $k(\alpha) = \lceil (24c' \cdot \alpha^2)^3 \rceil + 1$ will ensure the desired four-cell configuration:



Proof sketch

Lemma

There exist four rich cells $\sigma'_1, \sigma'_2, \sigma'_3, \sigma'_4$, such that for any four points $a_i \in \sigma'_i \cap A$, $i = 1, 2, 3, 4$, we have $a_4 \in \Delta a_1 a_2 a_3$.

Let C $\subset$ R denote the set of rich cells incident to vertices of P.

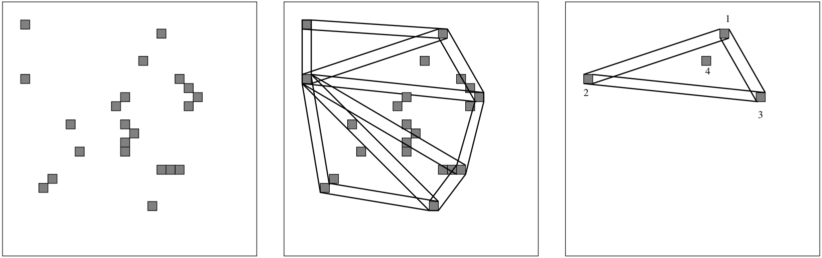


Figure: Left: The set of rich cells in Q. Center: the star triangulation K from a boundary cell in C. Here $|\mathsf{R}| = 22$ and $|\mathsf{C}| = 7$. Right: a set of four rich cells as in the lemma.

Finishing the proof

We use the point set structure guaranteed by the lemma.

- A cell $\sigma \in \Sigma$ is *rich* if it contains at least $n/(3k^2)$ points in A, where $k(\alpha) \sim \alpha^6$.
- Consider four rich cells $\sigma_1, \sigma_2, \sigma_3, \sigma_4$, such that for any four points $a_i \in \sigma_i \cap A$, $i = 1, 2, 3, 4$, we have $a_4 \in \Delta a_1 a_2 a_3$.
- Let $B_i = A \cap \sigma_i$, for $i = 1, 2, 3, 4$. We may assume that $|B_1| = |B_2| = |B_3| = |B_4| = m = \lceil n/(3k^2) \rceil$.

We conclude that the edge-set of $K_n[A]$ can be decomposed into at most

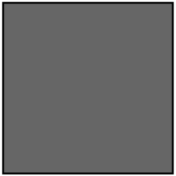
$$n - 4m + 3m = n - m \leq \left(1 - \frac{1}{3k^2}\right) n$$

plane subgraphs. We can set $c(\alpha) = 1 - \frac{1}{3k^2(\alpha)}$, thus $c(\alpha) \leq 1 - \Omega\left(\alpha^{-12}\right)$. $\square$

Another scenario: random point sets

Corollary

Let A be a set of n random points uniformly distributed in $[0,1]^2$, and let $n \to \infty$. There exists an absolute constant $c < 1$ such that, with probability tending to 1, the complete geometric graph induced by A can be decomposed into at most cn plane subgraphs.



The result can be deduced from that for dense sets; however, here is a direct proof:

Random point sets

Proof:

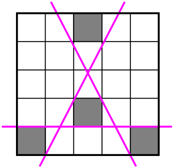


Figure: The four distinguished subsquares are shaded.

- The expected number of points in each subsquare is $n/25$. With probability tending to 1 as $n \to \infty$, each of the four subsquares contains at least $n/50$ points in A.
- As such, a structure with four rich cells is obtained, as before, and the corollary follows.

$\square$

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END