The Density Formula

One Lemma to Bound them All

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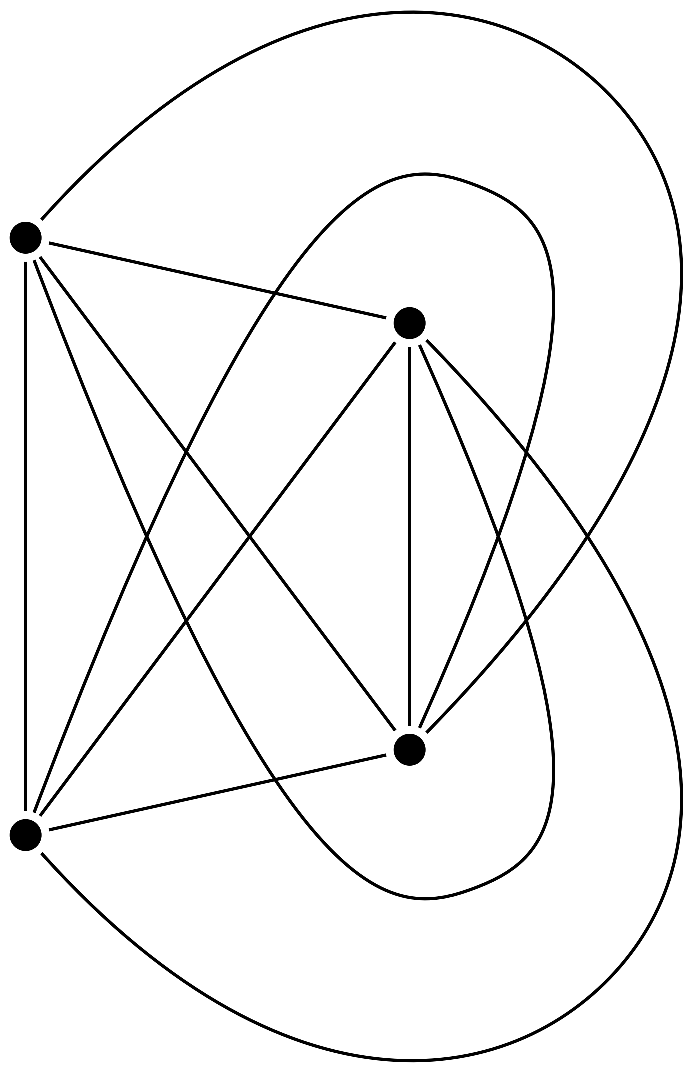
Drawings

- ▷ no loops
- ▷ parallel edges allowed
- ▷ connected, $n \geq 3$ vertices
- ▷ no self-crossings
- ▷ finite number of crossings
- ▷ no three edges through 1 point
- ▷ no touchings



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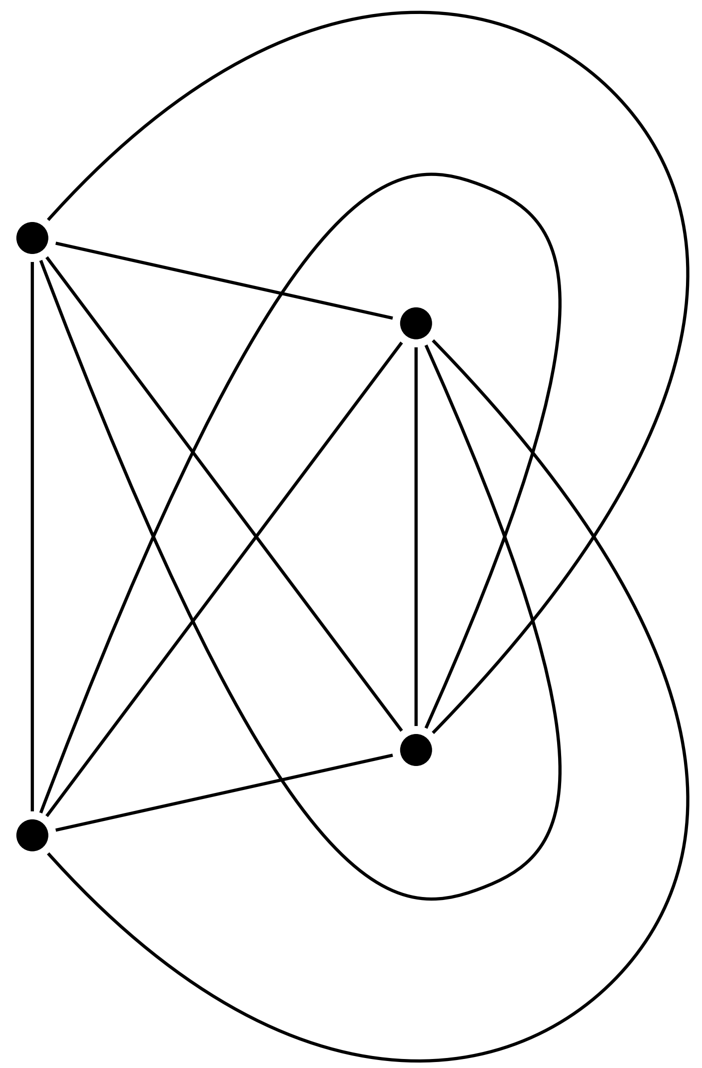

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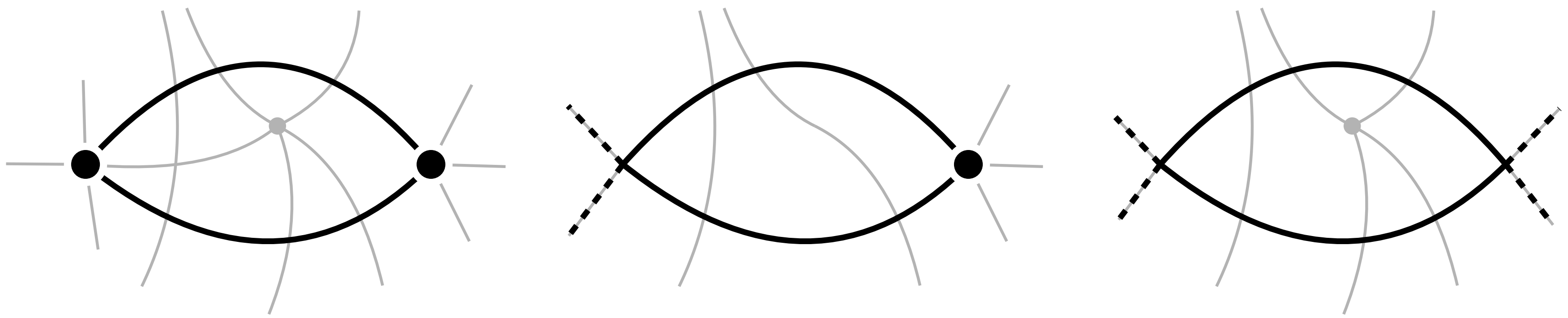
non-homotopic and simple drawings:

lens $=$ region bounded by two edge parts



non-homotopic drawing

Every lens contains
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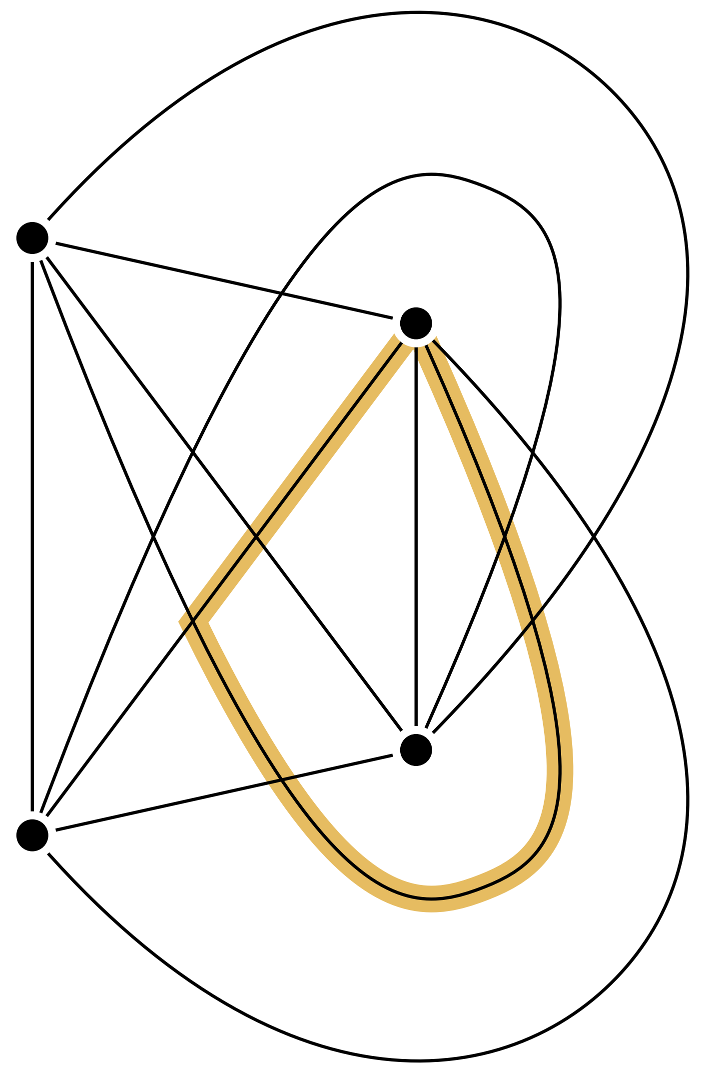

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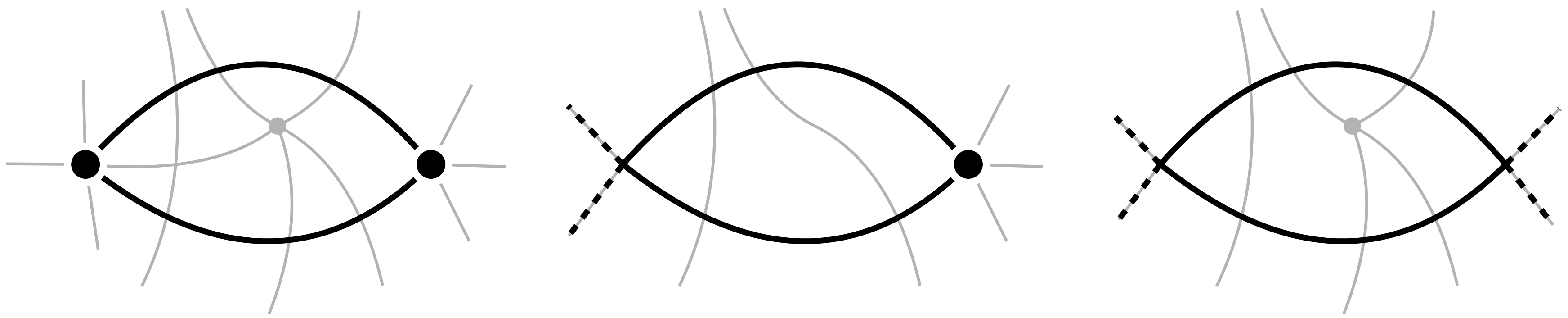

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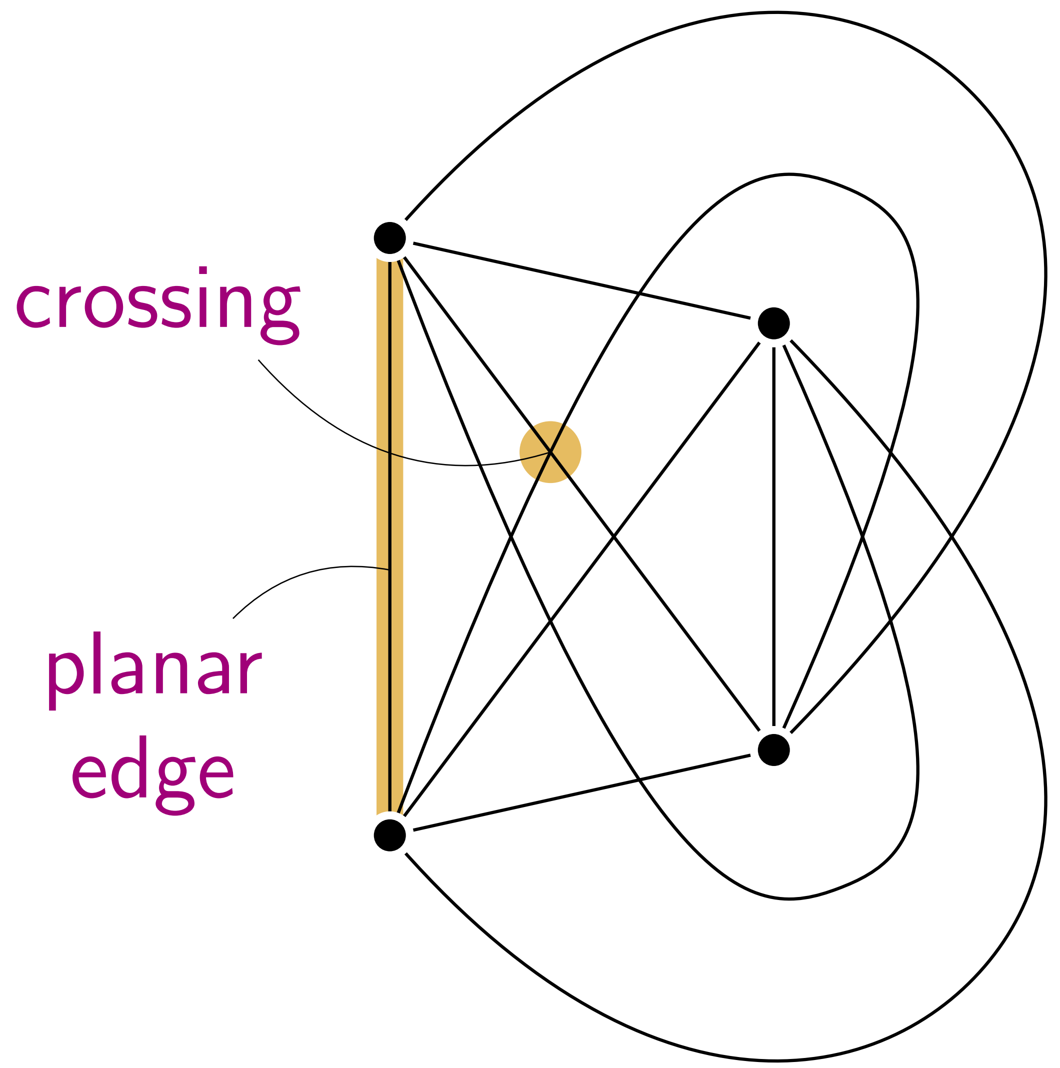
$E = \{\text{ edges }\} = E_{\text{x}} \cup E_{\text{p}}$

crossed

planar



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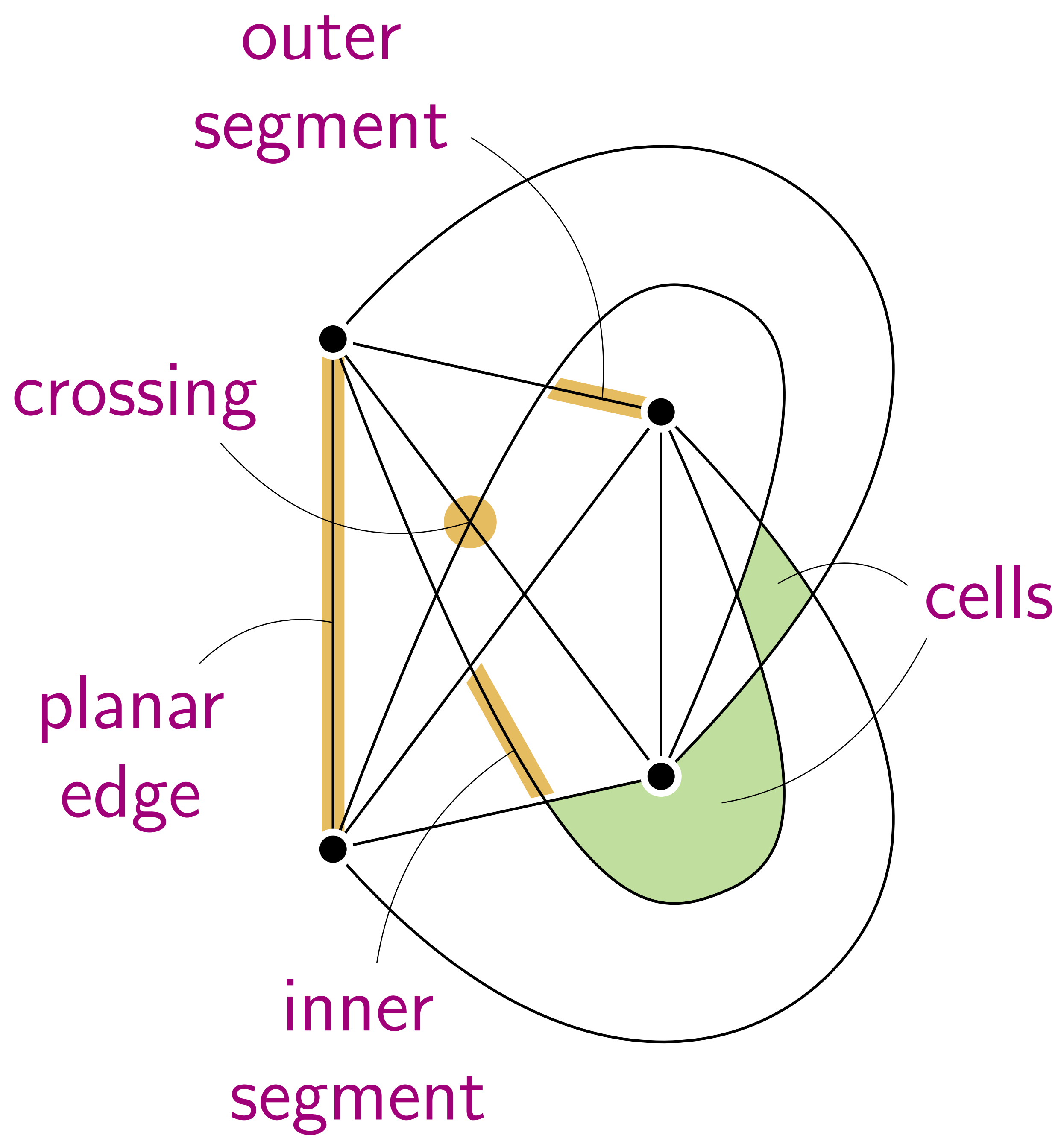

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$\mathcal{S} = \{\text{ segments }\} = \mathcal{S}_{\text{in}} \cup \mathcal{S}_{\text{out}}$ ($\mathcal{S}_{\text{in}}$: inner; $\mathcal{S}_{\text{out}}$: outer)

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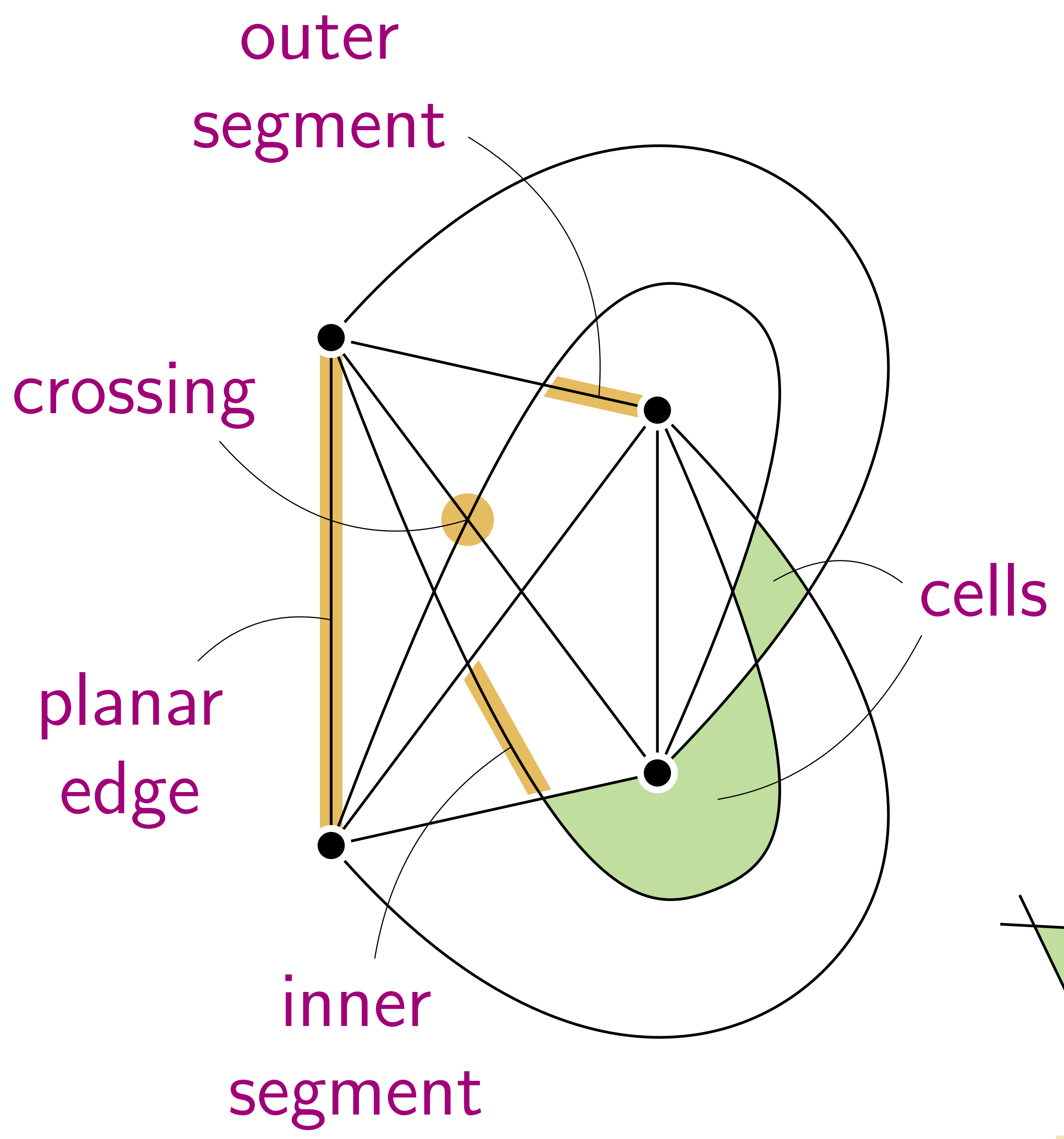

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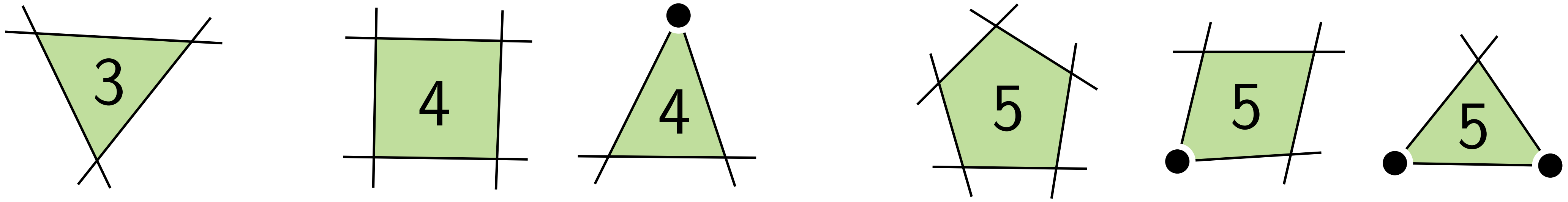
$||c|| =$ **size** of cell $c = \#$vertices $+ \#$ segments



$\mathcal{C}_3$ $\mathcal{C}_4$ $\mathcal{C}_5$

Density Formula

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special cases

$t = 4$ $$|E| = 4(|V| - 2) + \frac{7}{4}|\mathcal{C}_3| + |\mathcal{C}_4| + \frac{1}{4}|\mathcal{C}_5| - \frac{1}{2}|\mathcal{C}_6| - \cdots - |\mathcal{X}|$$

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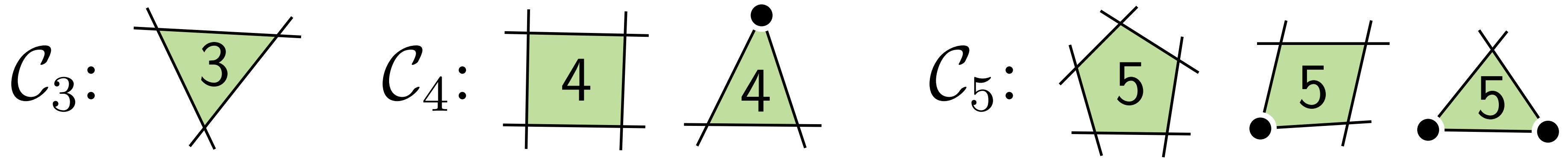

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General approach:

- apply the formula with a specific t
- upper bounds on $|\mathcal{C}_3|, |\mathcal{C}_4|, \ldots$ in terms of $|\mathcal{X}|$ and/or $|V|, |E|$

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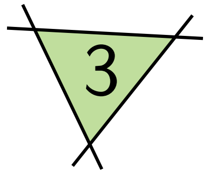

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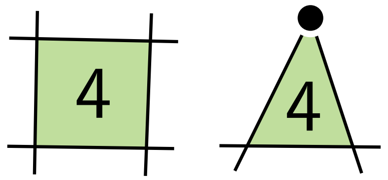

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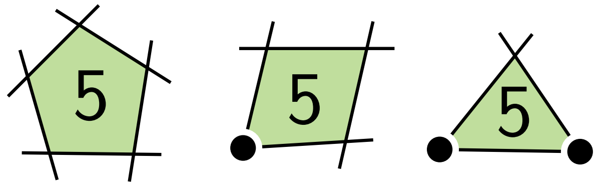

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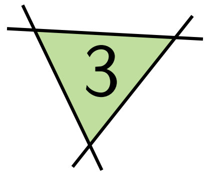

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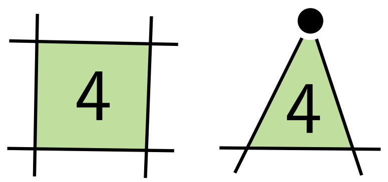

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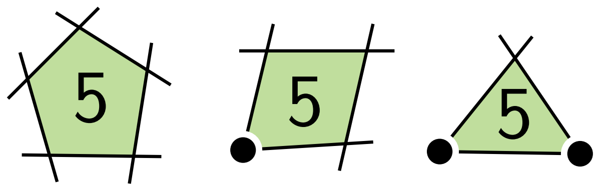

1-planar graphs

$$|E| = t(|V|-2) - \sum_{c \in \mathcal{C}} \left(\frac{t-1}{4} ||c|| - t \right) - |\mathcal{X}|$$

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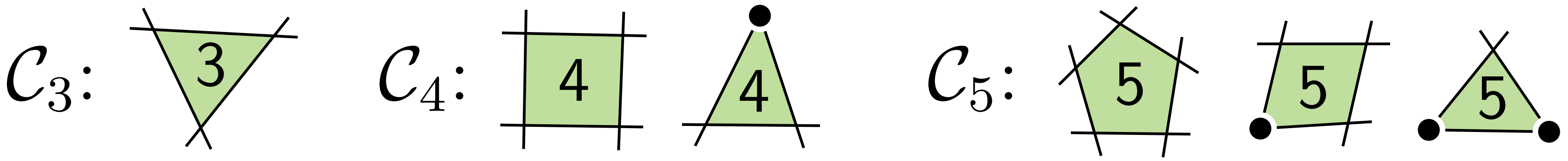

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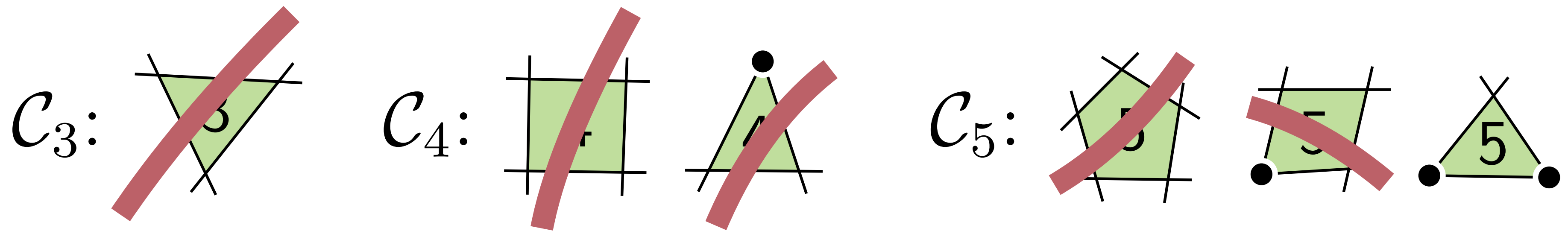

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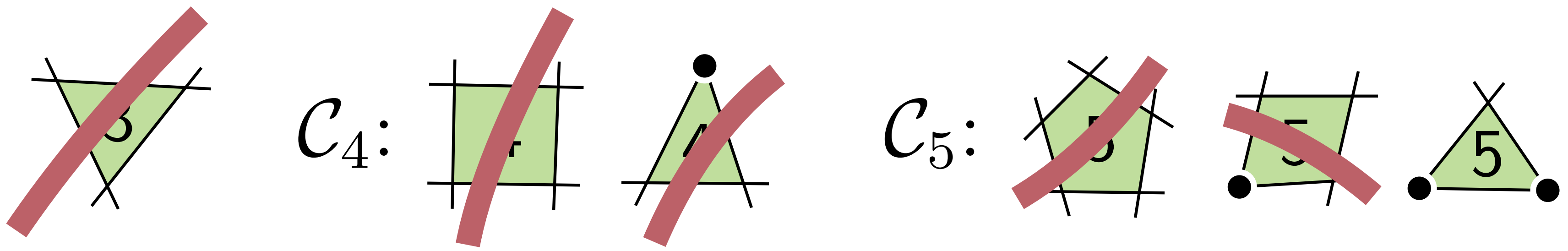

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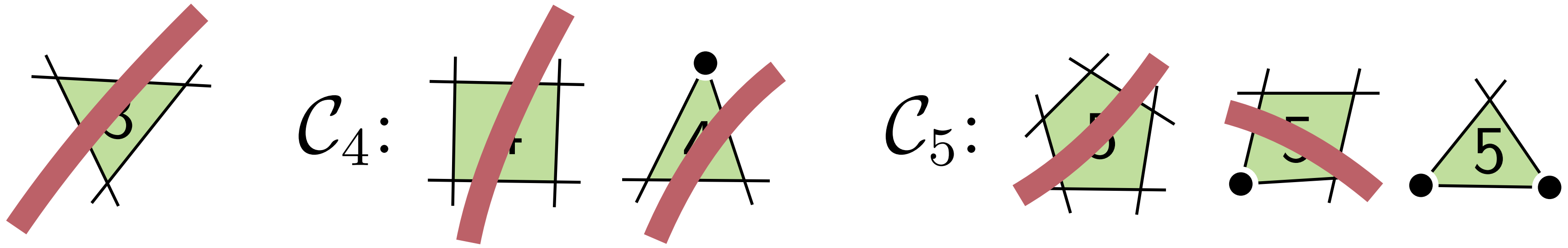
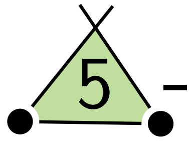


\# 

-cells $\leq 4|\mathcal{X}|$ $\implies$ $\frac{1}{4}|\mathcal{C}_5| \leq |\mathcal{X}|$ $\implies$ $|E| \leq 4n - 8$ $\square$

If $|E| = 4n - 8$, all cells must be 

Overview

density = max. # edges for n vertices

1-planar	$4n - 8$	Pach-Tóth '97
2-planar	$5n - 10$	Pach-Tóth '97
3-planar	$5.5n - \Theta(1)$	Pach-Radoičić-Tardos-Tóth '06
4-planar	$6n - \Theta(1)$	Ackerman '15
simple quasiplanar	$6.5n - 20$	Ackerman-Tardos '07
non-hom. quasiplanar	$8n - 20$	Ackerman-Tardos '07
0-bend RAC	$4n - 10$	Didimo-Eades-Liotta '11
1-bend RAC	$5n - 10$	Kaufmann-Klemz-Knorr-Reddy-S.-Ueckerdt '24
2-bend RAC	$10n - \Theta(1)$	Kaufmann-Klemz-Knorr-Reddy-S.-Ueckerdt '24
1^+-real face	$5n - 10$	Binucci-Di Battista-Didimo-Hong-Kaufmann-Liotta-Morin-Tappini '23
2^+-real face	$4n - 8$	Binucci-Di Battista-Didimo-Hong-Kaufmann-Liotta-Morin-Tappini '23
k^+-real face	$\frac{k}{k-2}(n-2)$	Binucci-Di Battista-Didimo-Hong-Kaufmann-Liotta-Morin-Tappini '23
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Proof.

- $|\mathcal{S}| = |E| + 2|\mathcal{X}|$
- $\sum_{c \in \mathcal{C}} ||c|| = 2|\mathcal{S}| + \sum_{v \in V} \deg(v) = 4|E| + 4|\mathcal{X}|$

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$$\implies 0 = |V| - 2 - (|E| + |\mathcal{X}| - |\mathcal{C}|) \implies 0 = |V| - 2 - \sum_{c \in \mathcal{C}} \left(\frac{||c||}{4} - 1 \right)$$

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$\implies 0 = |V| - 2 - (|E| + |\mathcal{X}| - |\mathcal{C}|) \implies$

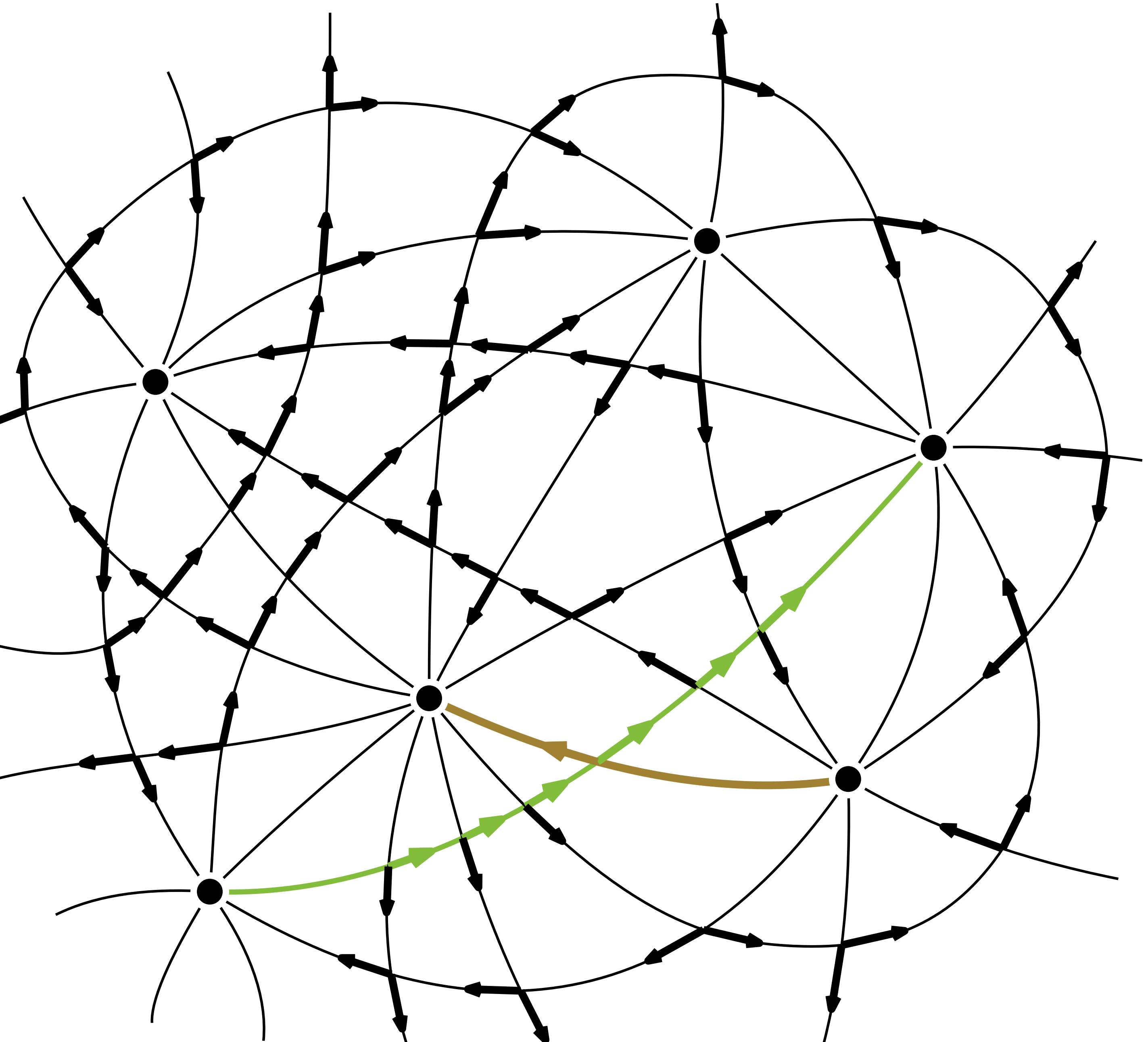
$$|E| = |V| - 2 + |\mathcal{C}| - |\mathcal{X}| \qquad \Big| \times 1$$

$$0 = |V| - 2 - \sum_{c \in \mathcal{C}} \left(\frac{||c||}{4} - 1 \right) \qquad \Big| \times (t-1)$$

$$|E| = t(|V| - 2) - \sum_{c \in \mathcal{C}} \left(\frac{t-1}{4} ||c|| - t \right) - |\mathcal{X}|$$

Useful Lemmas

$$|\mathcal{S}_{\text{in}}| = 2|\mathcal{X}| - |E_{\text{x}}|$$



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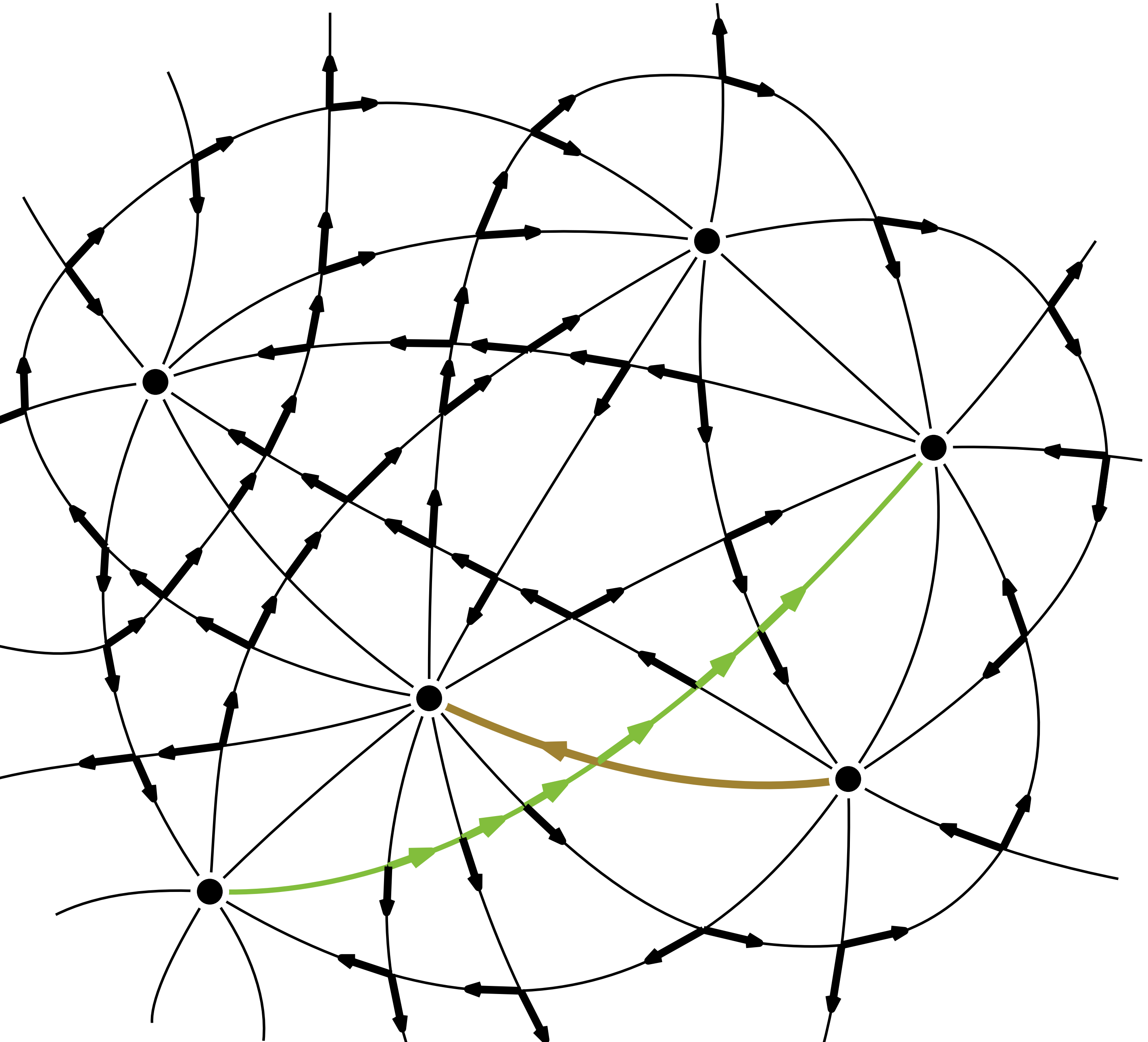

Lemma

In any non-homotopic drawing we have

$$|\mathcal{S}_{\text{in}}| \geq 3\# \text{(3)-cells} + 2\# \text{(4)-cells} + \# \text{(4)-cells}$$

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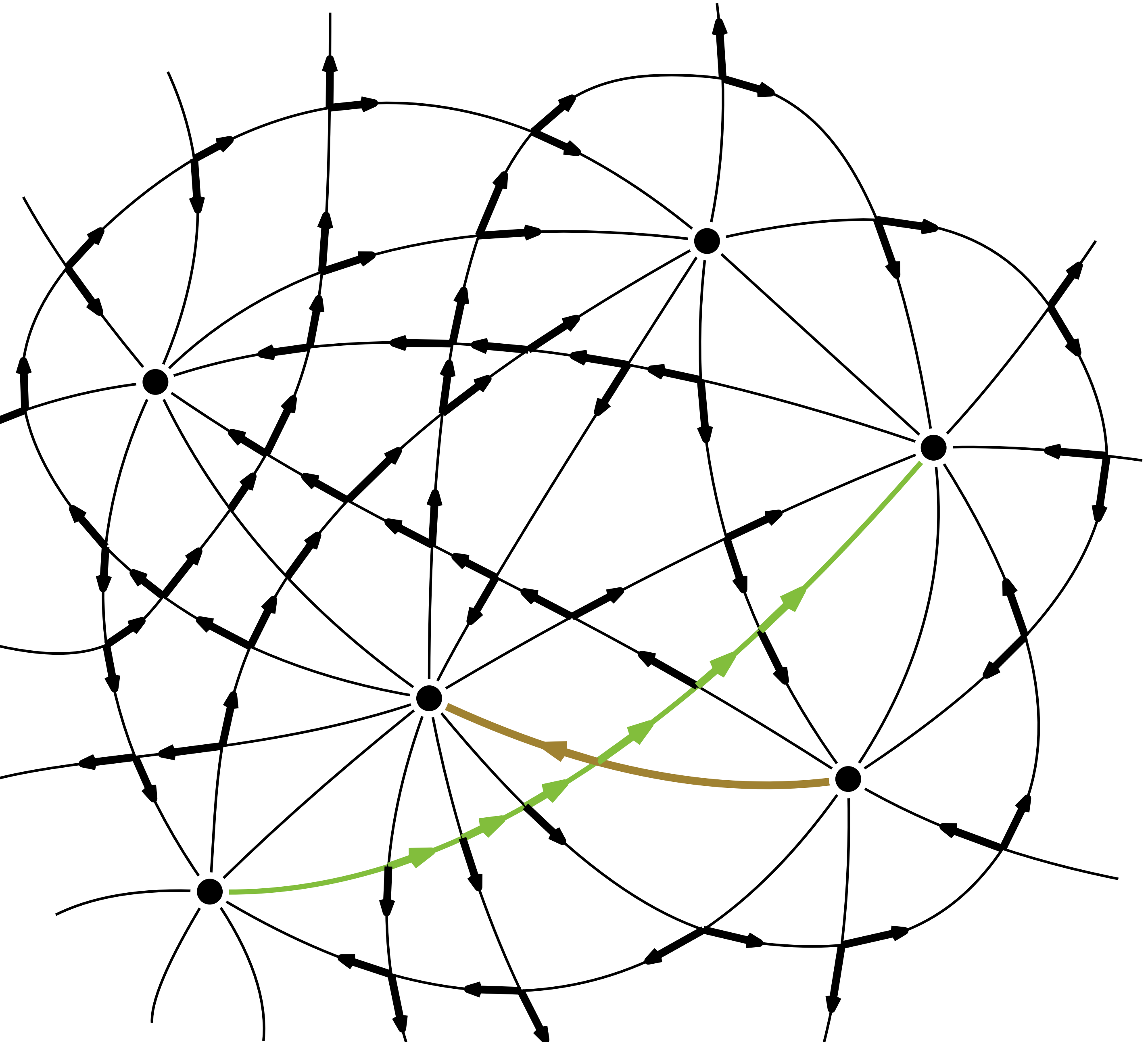

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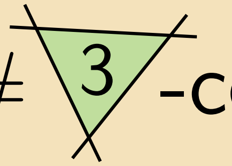
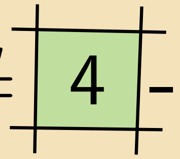
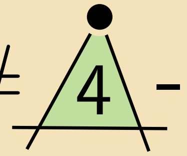
$|\mathcal{S}_{\text{in}}| \geq 3\#$ 

-cells $+2\#$ 

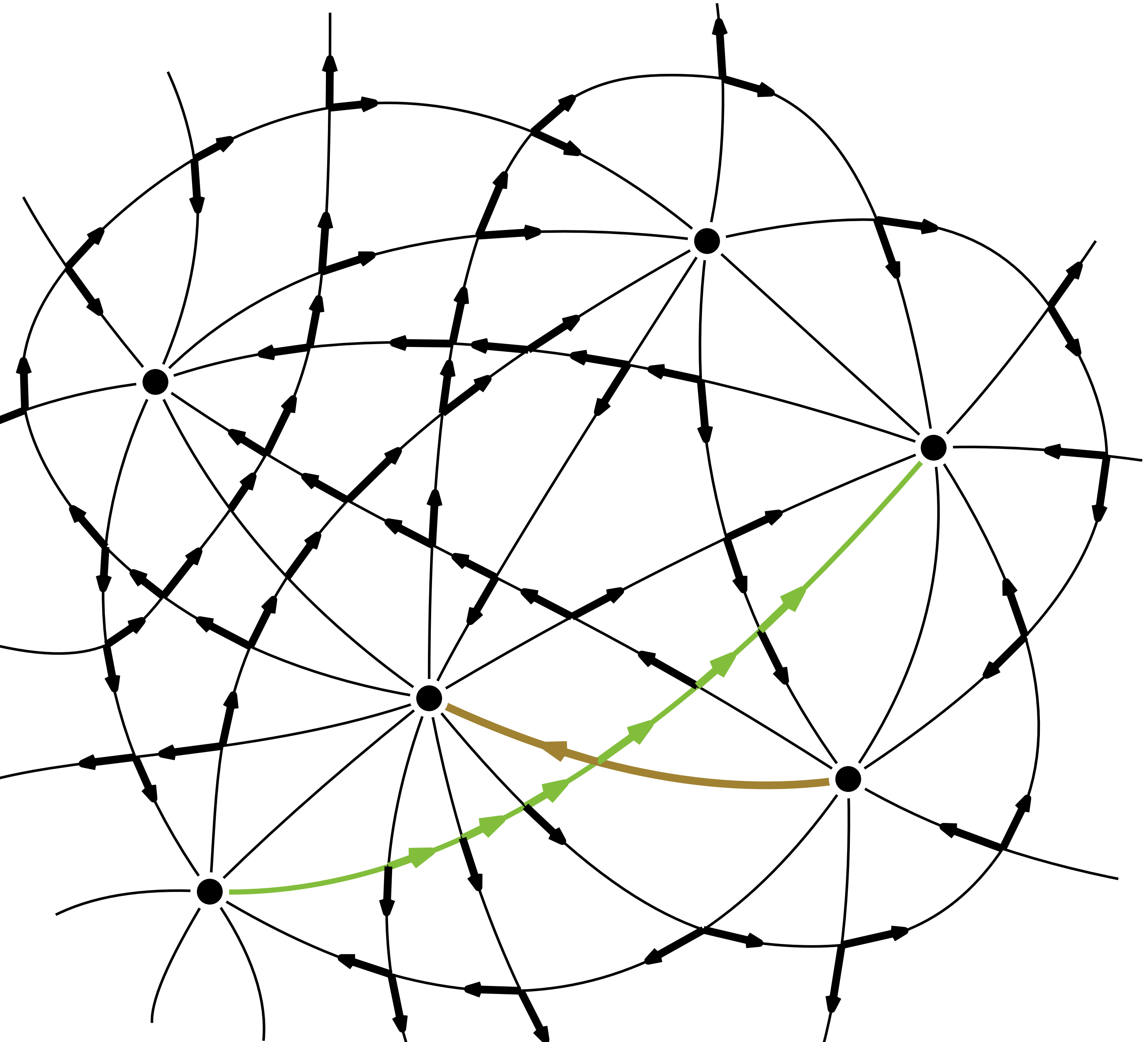
-cells $+\#$ 

-cells

Proof idea

Useful Lemmas

$$|\mathcal{S}_{\text{in}}| = 2|\mathcal{X}| - |E_{\text{x}}|$$

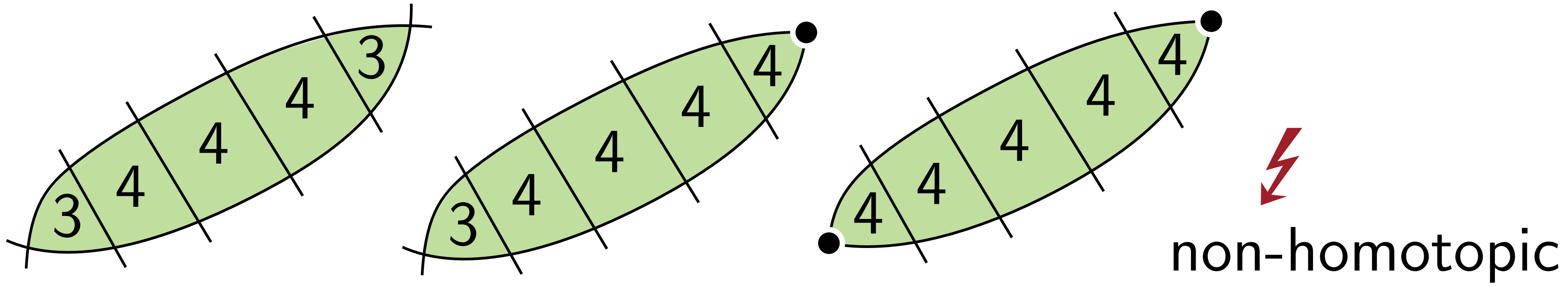


Lemma

In any non-homotopic drawing we have

$$|\mathcal{S}_{\text{in}}| \geq 3\# \text{ 3 -cells } + 2\# \text{ 4 -cells } + \# \text{ 4 -cells}$$

Proof idea



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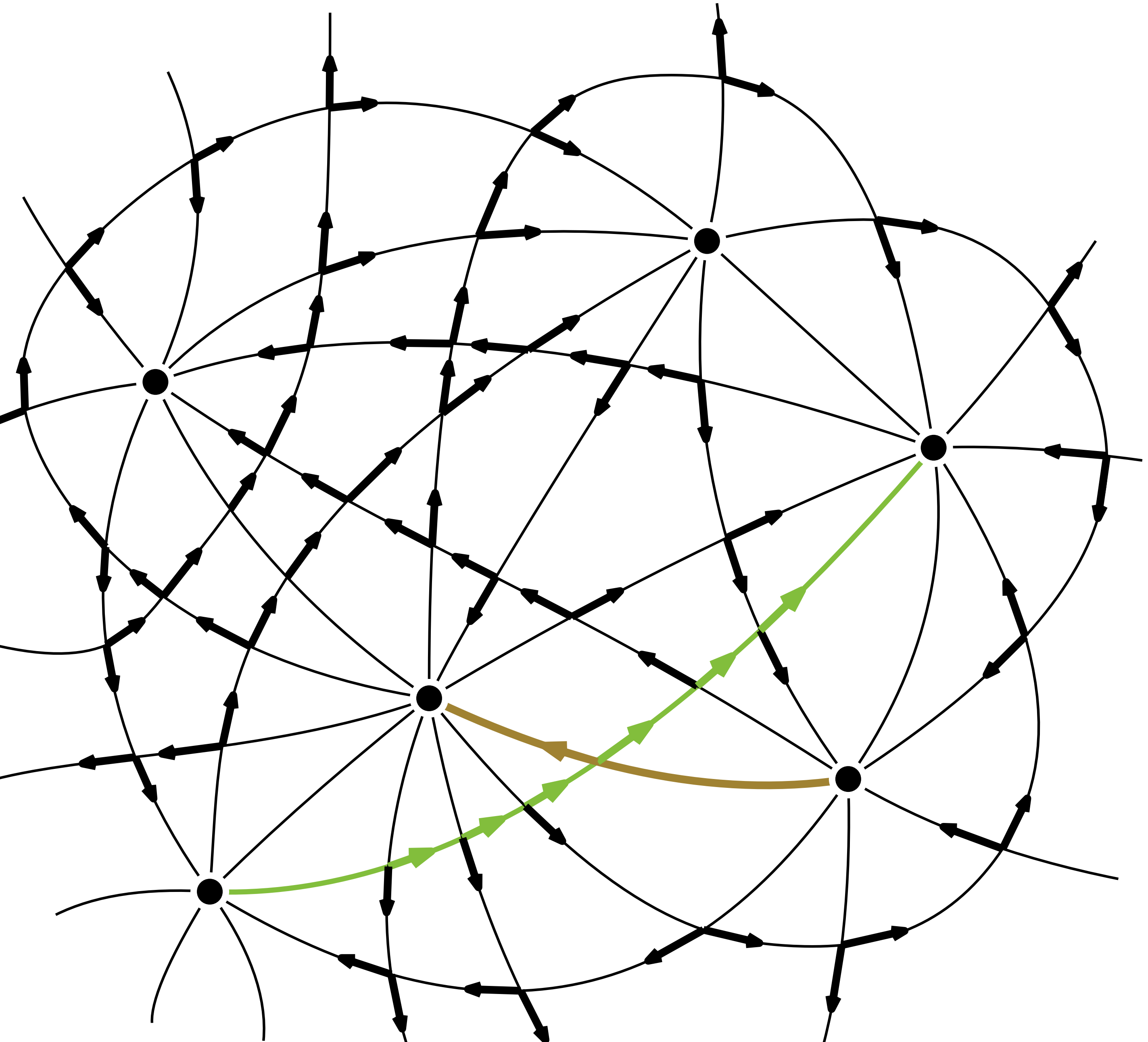

Lemma

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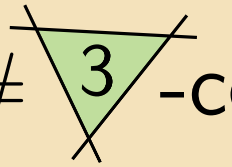
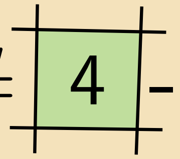
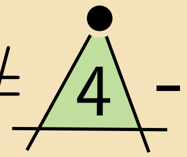
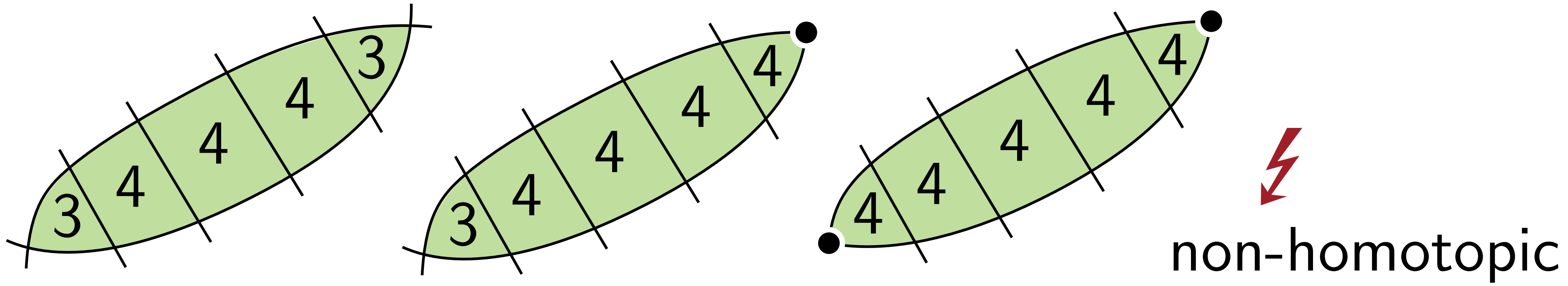
$|\mathcal{S}_{\text{in}}| \geq 3\#$ 

-cells $+2\#$ 

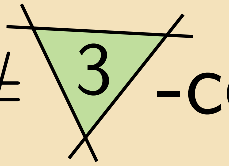
-cells $+\#$ 

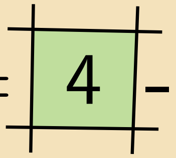
-cells

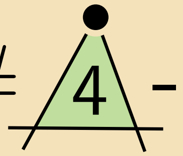
Proof idea



2-planar graphs

$|\mathcal{S}_{\text{in}}| \geq 3\#$ 

-cells $+2\#$ 

-cells $+\#$ 

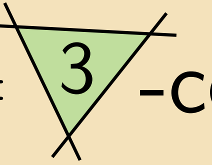
-cells

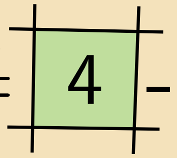
$$|E| = t(|V|-2) - \sum_{c \in \mathcal{C}} \left(\frac{t-1}{4} ||c|| - t \right) - |\mathcal{X}|$$

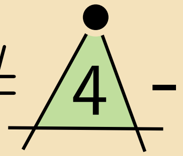
Theorem (Pach + Tóth, 1997)

n-vertex 2-planar graphs have $\leq 5n - 10$ edges

2-planar graphs

$|\mathcal{S}_{\text{in}}| \geq 3\#$ 

-cells $+2\#$ 

-cells $+\#$ 

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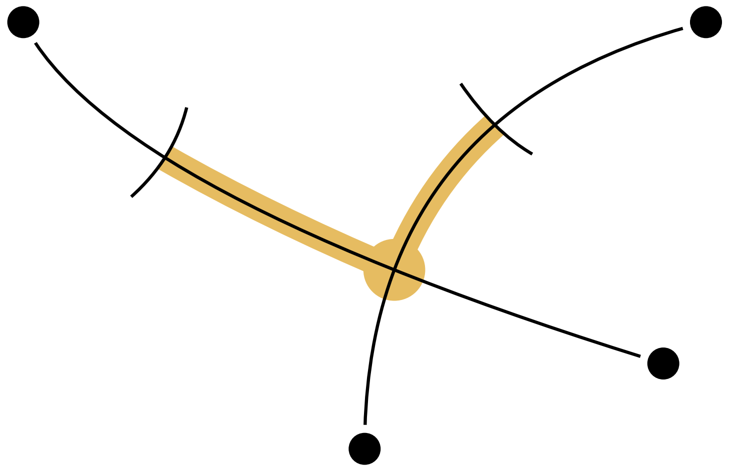
Theorem (Pach + Tóth, 1997)

n-vertex 2-planar graphs have $\leq 5n - 10$ edges

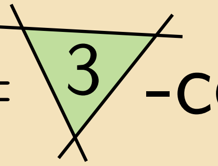
Proof using the Density Formula:

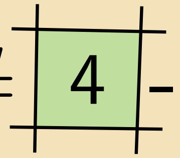
2 crossings at each inner segment

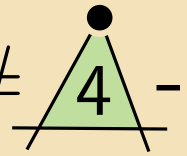
$\leq$ 2 inner segments at each crossing $\implies |\mathcal{X}| \geq |\mathcal{S}_{\text{in}}|$



2-planar graphs

$|\mathcal{S}_{\text{in}}| \geq 3\#$ 

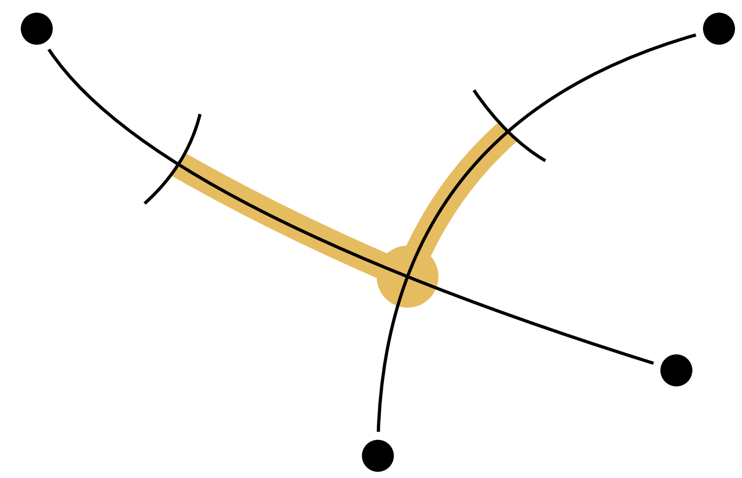
-cells $+2\#$ 

-cells $+ \#$ 

-cells

$$|E| = t(|V| - 2) - \sum_{c \in \mathcal{C}} \left(\frac{t-1}{4} ||c|| - t \right) - |\mathcal{X}|$$

Theorem (Pach + Tóth, 1997)
n-vertex 2-planar graphs have $\leq 5n - 10$ edges



Proof using the Density Formula:

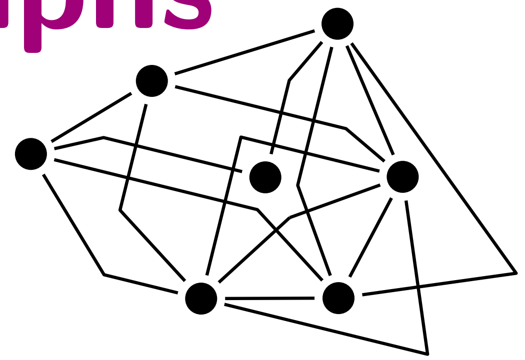
2 crossings at each inner segment
≤ 2 inner segments at each crossing $\implies$ $|\mathcal{X}| \geq |\mathcal{S}_{\text{in}}|$

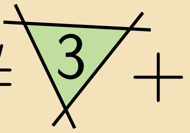
$t = 5$ $\quad |E| = 5(n-2) + 2|\mathcal{C}_3| + |\mathcal{C}_4| + 0|\mathcal{C}_5| - 1|\mathcal{C}_6| - \cdots - |\mathcal{X}|$

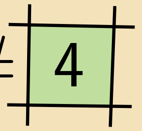
$\implies \quad |E| \leq 5(n-2) + |\mathcal{S}_{\text{in}}| - |\mathcal{X}| \leq 5(n-2)$ □

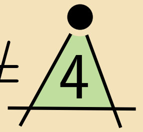
k-bend RAC graphs

edges with $\leq k$ bends
Right-**A**ngle **C**rossings



$2|\mathcal{X}| - |E_{\mathrm{x}}| = |\mathcal{S}_{\mathrm{in}}| \geq 3\#$ 

$+ 2\#$ 

$+ \#$ 

$t = 5 : \qquad |E| \leq 5(n-2) + 2|\mathcal{C}_3| + |\mathcal{C}_4| - |\mathcal{X}|$

Theorem.

n-vertex non-hom. 1-bend RAC graphs have $\leq 5n - 10$ edges.
n-vertex non-hom. 2-bend RAC graphs have $\leq 10n - 19$ edges.

k-bend RAC graphs

edges with $\leq k$ bends
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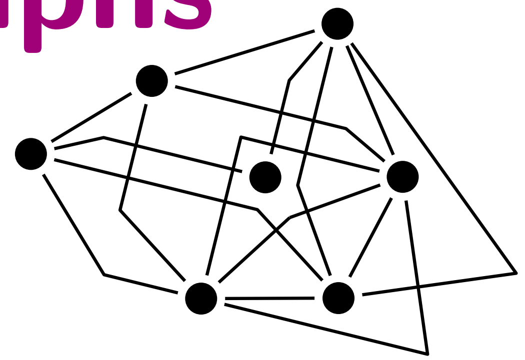
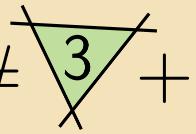
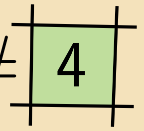
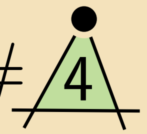


$2|\mathcal{X}| - |E_{\mathrm{x}}| = |\mathcal{S}_{\mathrm{in}}| \geq 3\#$  $+ 2\#$  $+ \#$ 

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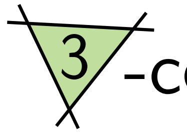
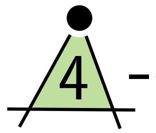
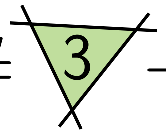
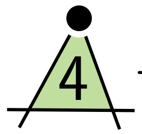
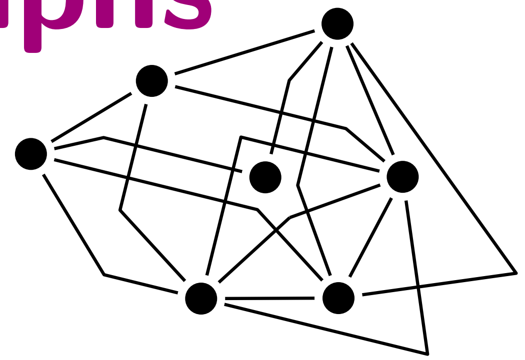
Proof.

≥ 1 convex bend at each*  -cell and  -cell $\implies$ $k|E_{\mathrm{x}}| \geq \#$  $+ \#$  $- 1$

k-bend RAC graphs

edges with $\leq k$ bends
Right-**A**ngle **C**rossings



$$2|\mathcal{X}| - |E_\mathrm{x}| = |\mathcal{S}_\mathrm{in}| \geq 3\#\triangledown_3 + 2\#\square_4 + \#\triangle^{\bullet}_4$$

$$t = 5: \qquad |E| \leq 5(n-2) + 2|\mathcal{C}_3| + |\mathcal{C}_4| - |\mathcal{X}|$$

Theorem.

n-vertex non-hom. 1-bend RAC graphs have $\leq 5n - 10$ edges.
n-vertex non-hom. 2-bend RAC graphs have $\leq 10n - 19$ edges.

Proof.

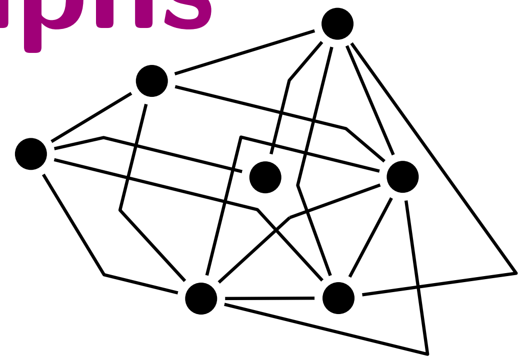
≥ 1 convex bend at each* $\triangledown_3$-cell and $\triangle^{\bullet}_4$-cell $\implies k|E_\mathrm{x}| \geq \#\triangledown_3 + \#\triangle^{\bullet}_4 - 1$

$$\implies 2|\mathcal{X}| + (k-1)|E_\mathrm{x}| \geq 4\#\triangledown_3 + 2\#\square_4 + 2\#\triangle^{\bullet}_4 - 1 = 4|\mathcal{C}_3| + 2|\mathcal{C}_4| - 1$$

$$\implies 2|\mathcal{C}_3| + |\mathcal{C}_4| - |\mathcal{X}| \leq \frac{k-1}{2}|E_\mathrm{x}| + \frac{1}{2}$$

k-bend RAC graphs

edges with $\leq k$ bends
Right-**A**ngle **C**rossings



$$2|\mathcal{X}| - |E_\text{x}| = |\mathcal{S}_\text{in}| \geq 3\#\nabla_{3} + 2\#\boxed{4} + \#\triangle_{4}$$

$$t = 5: \quad |E| \leq 5(n-2) + 2|\mathcal{C}_3| + |\mathcal{C}_4| - |\mathcal{X}|$$

Theorem.

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Proof.

≥ 1 convex bend at each* ∇_{3}-cell and $\triangle_{4}$-cell $\implies$ $k|E_\text{x}| \geq \#\nabla_{3} + \#\triangle_{4} - 1$

$$\implies 2|\mathcal{X}| + (k-1)|E_\text{x}| \geq 4\#\nabla_{3} + 2\#\boxed{4} + 2\#\triangle_{4} - 1 = 4|\mathcal{C}_3| + 2|\mathcal{C}_4| - 1$$

$$\implies 2|\mathcal{C}_3| + |\mathcal{C}_4| - |\mathcal{X}| \leq \tfrac{k-1}{2}|E_\text{x}| + \tfrac{1}{2}$$

$$|E| \leq 5(n-2) + 2|\mathcal{C}_3| + |\mathcal{C}_4| - |\mathcal{X}| \leq 5(n-2) + \tfrac{k-1}{2}|E| + \tfrac{1}{2}$$

$$k = 1: \quad |E| \leq 5(n-2) + \tfrac{1}{2}$$

$$k = 2: \quad \tfrac{|E|}{2} \leq 5(n-2) + \tfrac{1}{2} \implies |E| \leq 10(n-2) + 1$$

$\square$

Overview

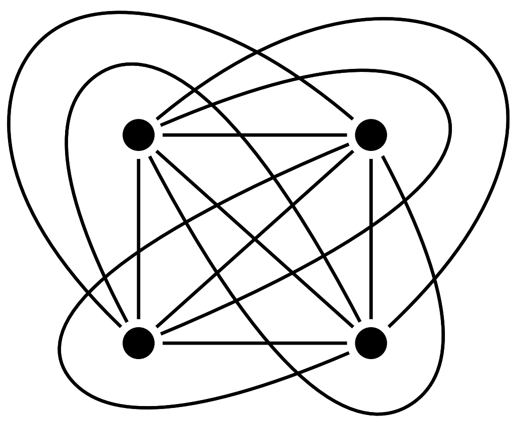
density = max. # edges for n vertices

Thank You!

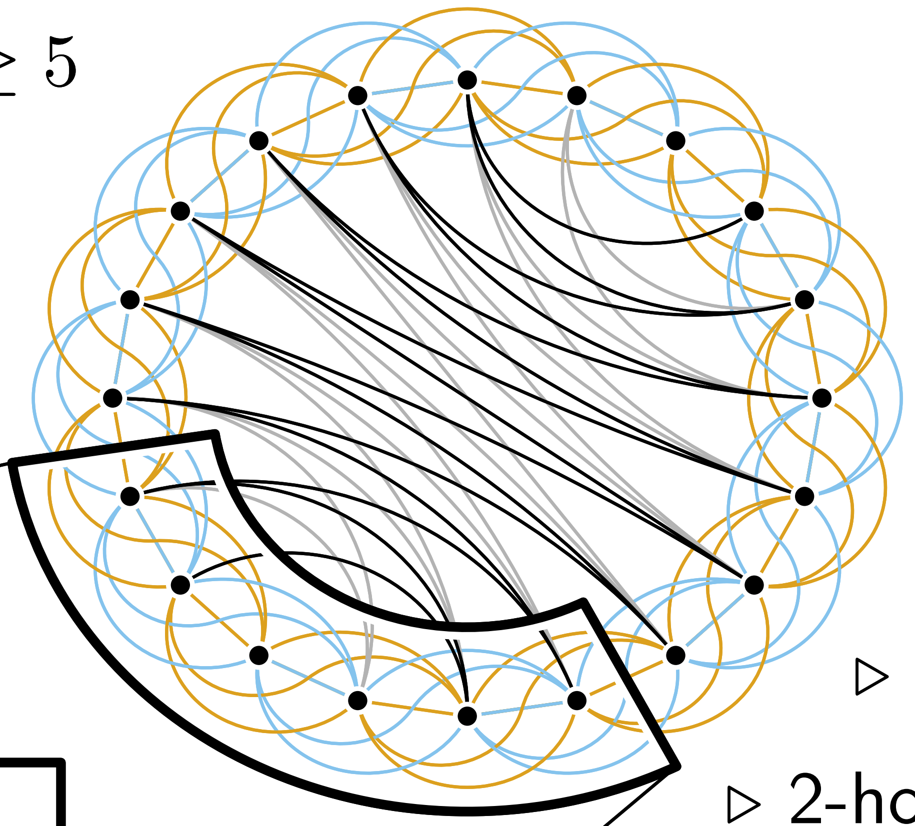
	density	
1-planar	$4n-8$	Pach-Tóth '97
2-planar	$5n-10$	Pach-Tóth '97
3-planar	$5.5n-\Theta(1)$	Pach-Radoičić-Tardos-Tóth '06
4-planar	$6n-\Theta(1)$	Ackerman '15
simple quasiplanar	$6.5n-20$	Ackerman-Tardos '07
non-hom. quasiplanar	$8n-20$	Ackerman-Tardos '07
0-bend RAC	$4n-10$	Didimo-Eades-Liotta '11
1-bend RAC	$5n-10$	Kaufmann-Klemz-Knorr-Reddy-S.-Ueckerdt '24
2-bend RAC	$10n-\Theta(1)$	Kaufmann-Klemz-Knorr-Reddy-S.-Ueckerdt '24
1^+-real face	$5n-10$	Binucci-Di Battista-Didimo-Hong-Kaufmann-Liotta-Morin-Tappini '23
2^+-real face	$4n-8$	Binucci-Di Battista-Didimo-Hong-Kaufmann-Liotta-Morin-Tappini '23
k^+-real face	$\frac{k}{k-2}(n-2)$	Binucci-Di Battista-Didimo-Hong-Kaufmann-Liotta-Morin-Tappini '23
fan-planar	$5n-10$	(Kaufmann-Ueckerdt '22) Ackerman-Keszegh '23

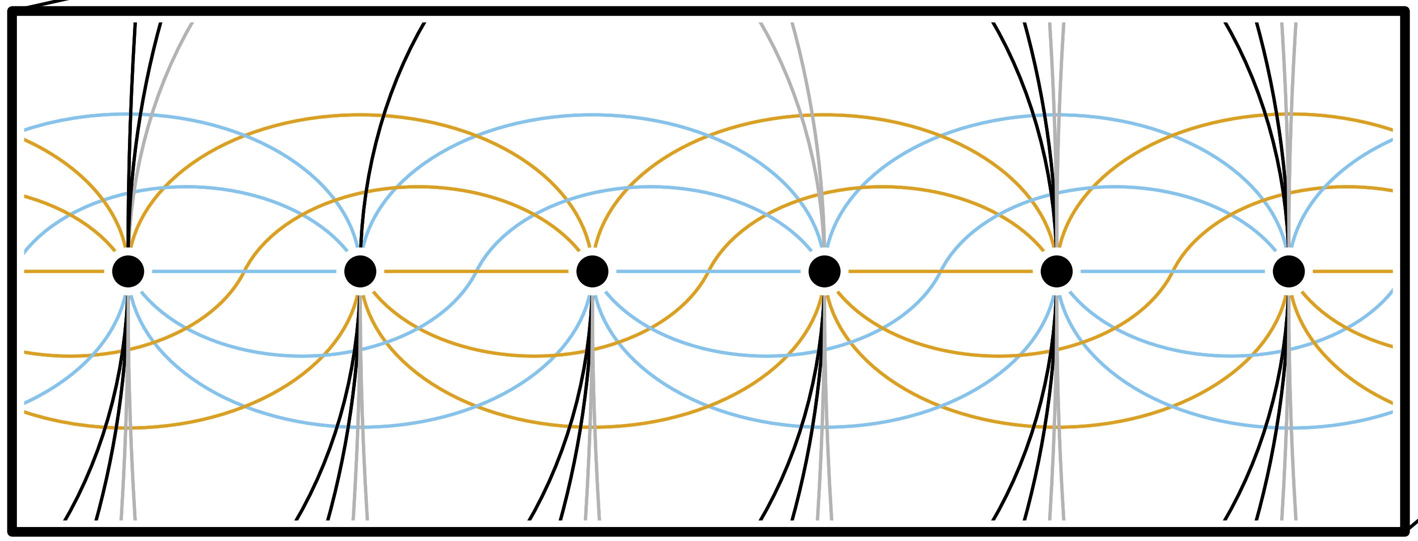
Lower Bound for Quasiplanar Graphs

$n = 4$



$n \geq 5$





- ▷ cycle C (n edges)
- ▷ 2-hops inside C (n edges)
- ▷ 2-hops outside C (n edges)
- ▷ 3-hops along C (n edges)
- ▷ 2 zig-zag paths inside C ($2(n-5)$ edges)
- ▷ 2 zig-zag paths outside C ($2(n-5)$ edges)