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Joint work with Géza Tóth

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# Matchstick graphs

Panna Gehér On 1-planar unit distance graphs

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## Motivation – Euler's formula

For any planar graph with  $n \ge 3$ , we have:  $e \le 3n - 6$ 

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For any planar graph with  $n \ge 3$ , we have:  $e \le 3n - 6$ and equality holds for triangulations



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#### Definition

A matchstick graph is a plane graph with unit length edges.

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## Matchstick graphs Construction:

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#### Matchstick graphs Construction:



### Theorem (Swanepoel, Lavollée, 2022)

The maximum number of edges a matchstick graph on n vertices can have, is

$$\lfloor 3n - \sqrt{12n - 3} \rfloor$$
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A drawn graph is 1-plane if each edge is crossed at most once.



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Theorem (Pach, Tóth, 1997)

The maximum number of edges of a 1-planar graph is

4*n* – 8.

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## Theorem (Pach, Tóth, 1997)

The maximum number of edges of a 1-planar graph is

4*n* – 8.

## Theorem (G., Tóth, 2023)

For the maximum number of edges a 1-planar unit distance graph can have,  $u_1(n)$ , we have

$$\lfloor 3n - \sqrt{12n - 3} \rfloor \leq u_1(n) \leq 3n - \sqrt[4]{n}/10.$$

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1. Take a maximal plane subgraph *G* 

2. Add the remaining edges in red

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2. Add the remaining edges in red

3. Split each red edge into two halfedges

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3. Split each red edge into two halfedges

4. Count the halfedges for each face

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## Triangles



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## Triangles



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## Triangles



# Halfedges: 0

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## Triangles



# Halfedges: 0

#### Quadrilaterals



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## Triangles



# Halfedges: 0

#### Quadrilaterals



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## Triangles



# Halfedges: 0

#### Quadrilaterals



# Halfedges:  $\leq$  2

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## Triangles



#### Quadrilaterals



# Halfedges: 0

# Halfedges:  $\leq$  2

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Key Lemma

A face  $\phi$  with  $m \ge 5$  edges has at most

 $2(t(\phi) - m/10)$ 

halfedges, where  $t(\phi)$  is the number of edges needed to triangulate  $\phi$ .

# Meaning of the Key Lemma



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# Meaning of the Key Lemma



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A forbidden subgraph for k = 4:



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Theorem (Pach, Tóth, 1997)

The maximum number of edges of a k-planar graph is:  $c \cdot n\sqrt{k}$ .

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Theorem (Pach, Tóth, 1997)

The maximum number of edges of a k-planar graph is:  $c \cdot n\sqrt{k}$ .

Theorem (Rote, 2023 and G., Tóth, 2023)

For the maximum number of edges of a k-planar unit distance graph,  $u_k(n)$ , we have

$$2^{\Omega(\log k/\log\log k)}n \leq u_k(n) \leq c\sqrt[4]{k}n.$$

#### Lower bound:

we use the construction of Erdős



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► Upper bound:

we use a strengthened version of the 'Crossing lemma'

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#### Lower bound:

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► Upper bound:

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if 
$$e \ge 4n$$
, then  $cr(G) \ge c \cdot \frac{e^5}{n^4}$ 

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if 
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but a k-planar graph can have at most  $\frac{ke}{2}$  crossings

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#### Lower bound:

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Upper bound:

we use a strengthened version of the 'Crossing lemma'

if 
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, then  $cr(G) \geq c \cdot \frac{e^{\epsilon}}{r^2}$ 

but a k-planar graph can have at most  $\frac{ke}{2}$  crossings  $\checkmark$ 

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1. Regular matchstick graphs:



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1. Regular matchstick graphs:



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1. Regular matchstick graphs:



#### Question:

Is there a 5-regular 1-planar unit distance graph?

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2. Constructions with more than  $\lfloor 3n - \sqrt{12n - 3} \rfloor$  edges

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2. Constructions with more than  $\lfloor 3n - \sqrt{12n - 3} \rfloor$  edges

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2. Constructions with more than  $\lfloor 3n - \sqrt{12n - 3} \rfloor$  edges



#### $\Rightarrow$ 3-planar unit distance graph with $\approx$ 3.5*n* edges

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2. Constructions with more than  $|3n - \sqrt{12n - 3}|$  edges

▶ k = 2: construction with  $\approx 3n - \sqrt{8.3n}$  edges (Simon)

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#### Thank you for your attention!



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