On 1-planar unit distance graphs

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Joint work with Géza Tóth

Matchstick graphs

Motivation – Euler's formula

For any planar graph with $n \geq 3$, we have: $e \leq 3n - 6$

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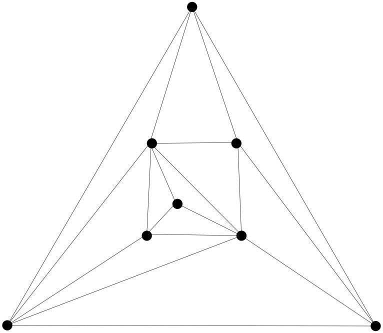

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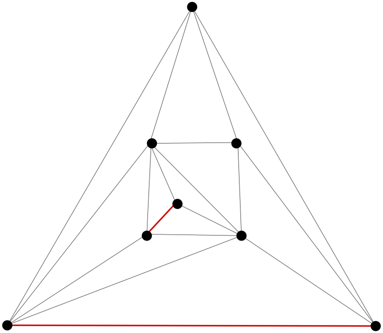

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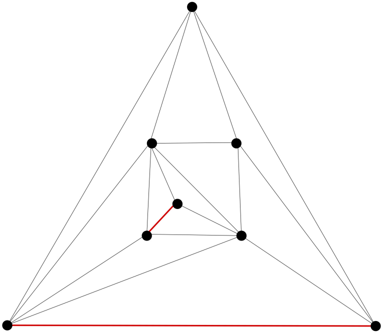

Definition

A matchstick graph is a plane graph with unit length edges.

Matchstick graphs

Construction:

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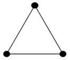

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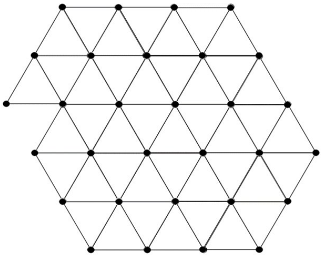

Theorem (Swanepoel, Lavollée, 2022)

The maximum number of edges a matchstick graph on n vertices can have, is

$$\lfloor 3n - \sqrt{12n-3} \rfloor.$$

1-planar unit distance graphs

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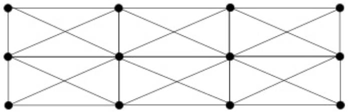
A drawn graph is 1-plane if each edge is crossed at most once.



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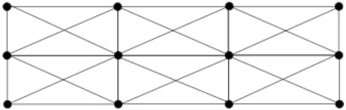

Theorem (Pach, Tóth, 1997)

The maximum number of edges of a 1-planar graph is

$$4n - 8.$$

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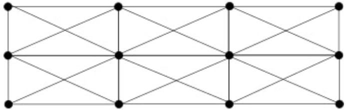

Theorem (Pach, Tóth, 1997)

The maximum number of edges of a 1-planar graph is

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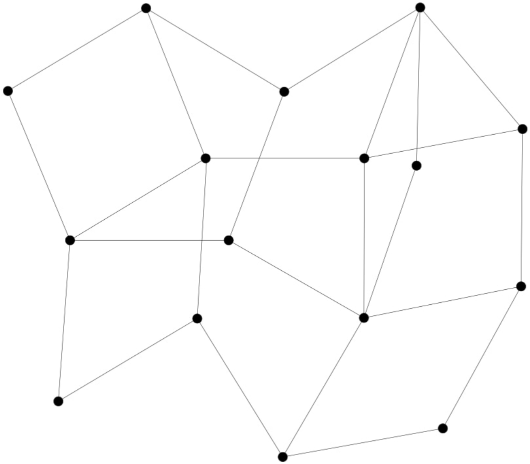
Theorem (G., Tóth, 2023)

For the maximum number of edges a 1-planar unit distance graph can have, $u_1(n)$, we have

$$\lfloor 3n - \sqrt{12n-3} \rfloor \leq u_1(n) \leq 3n - \sqrt[4]{n}/10.$$

Sketch of the proof



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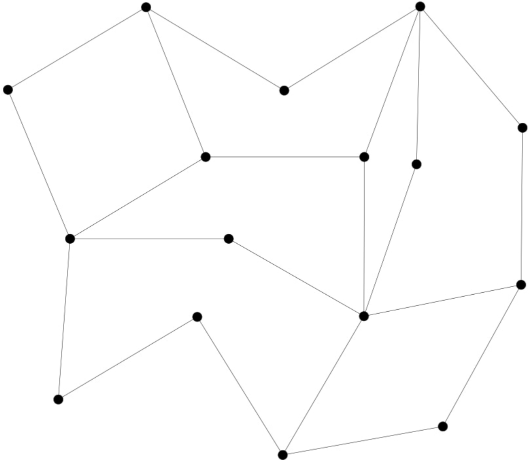

1. Take a maximal plane subgraph G

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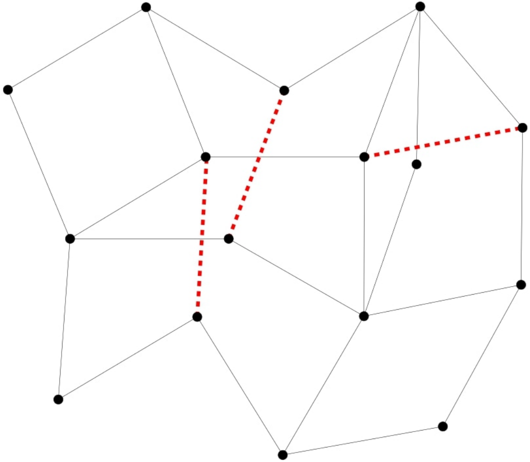

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2. Add the remaining edges in red

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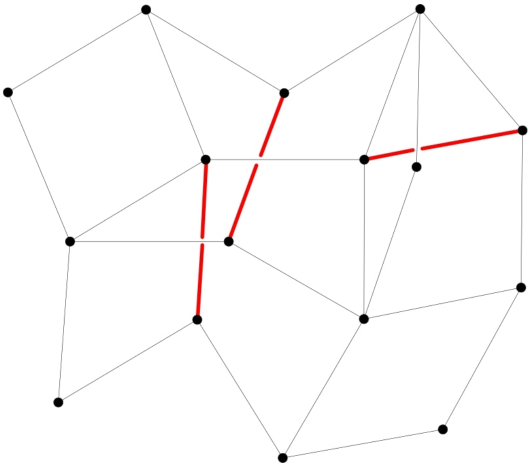

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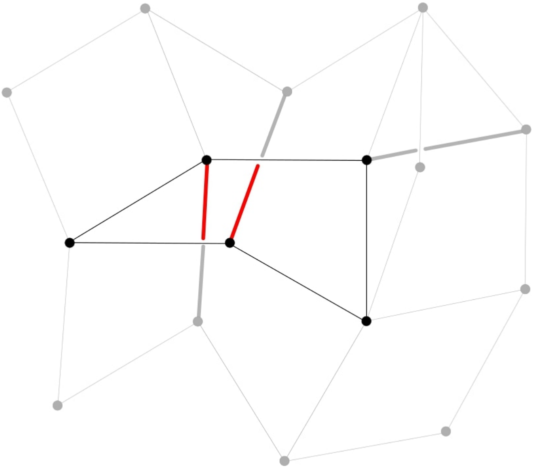

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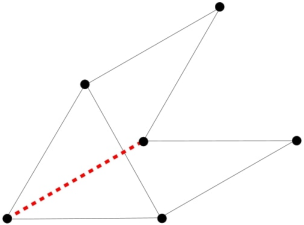
Counting halfedges

Triangles



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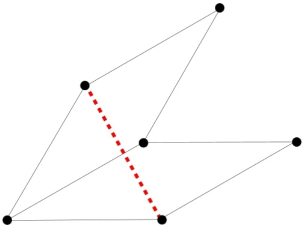

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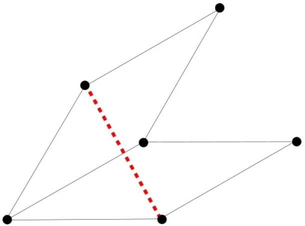

Halfedges: 0

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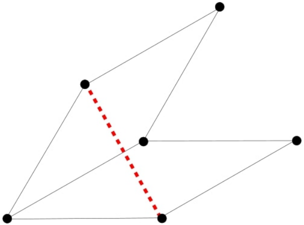

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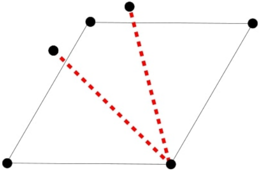

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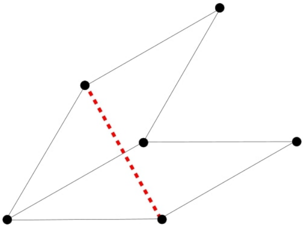

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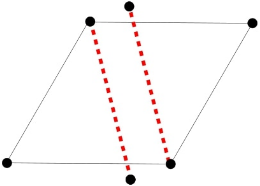

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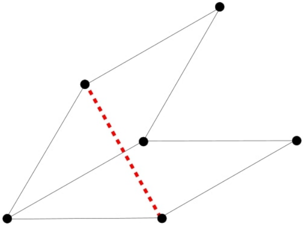

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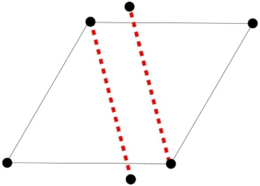
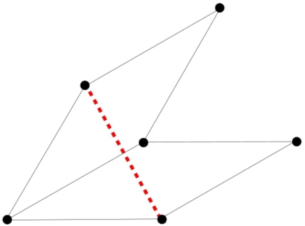


Halfedges: ≤ 2

Counting halfedges

Triangles



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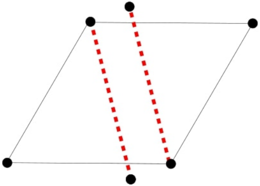


$\#$ Halfedges: ≤ 2

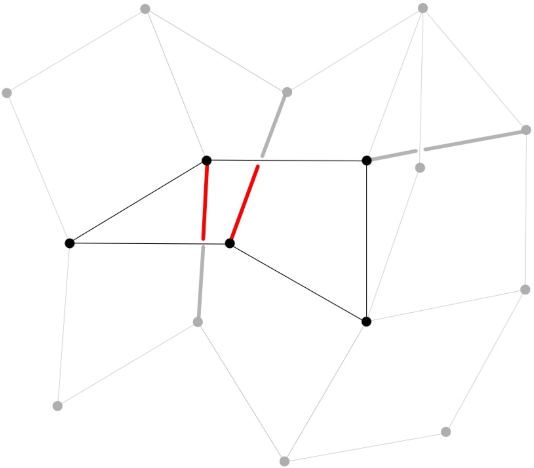
Key Lemma

A face ϕ with $m \geq 5$ edges has at most

$$2\,(t(\phi) - m/10)$$

halfedges, where $t(\phi)$ is the number of edges needed to triangulate ϕ.

Meaning of the Key Lemma



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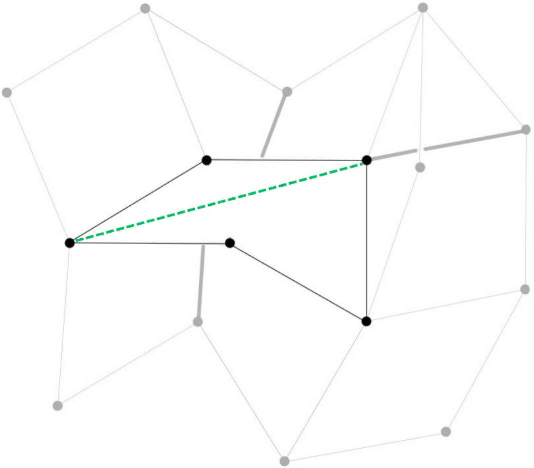

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End of proof

We call an edge good if it bounds a face with $m \geq 5$ edges.

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There are a lot of...

1. good edges:

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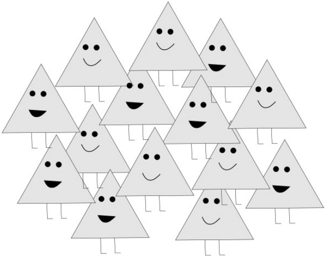

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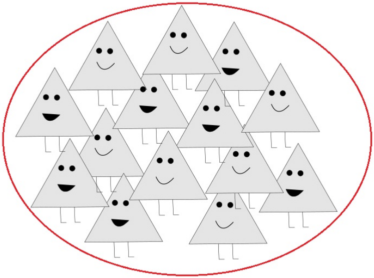

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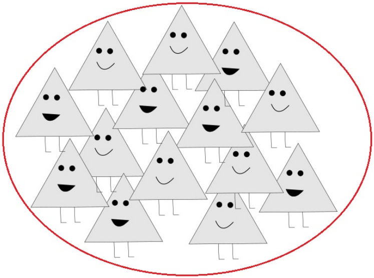

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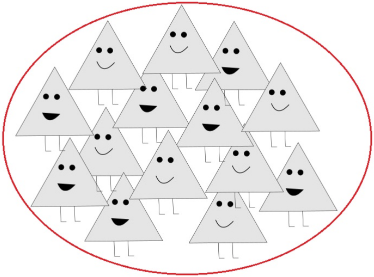

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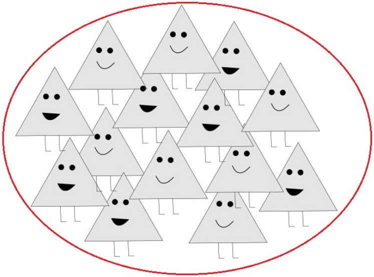

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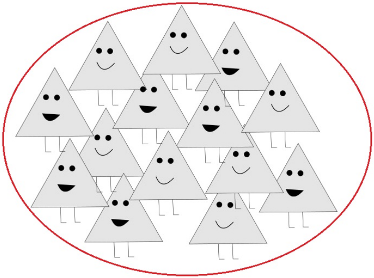

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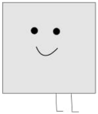

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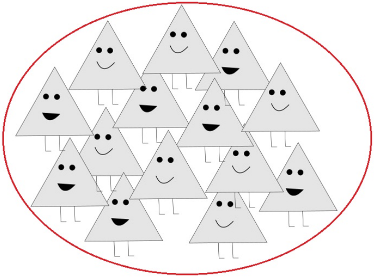

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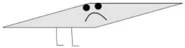

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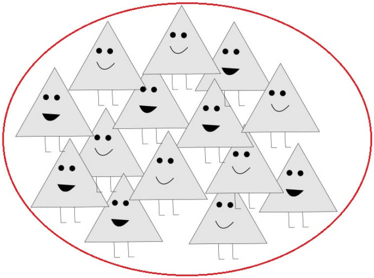

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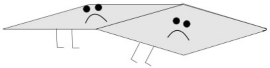

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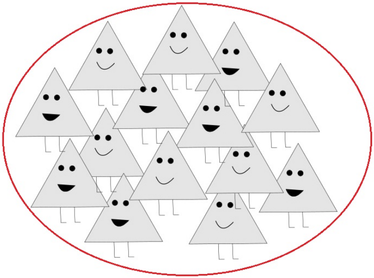

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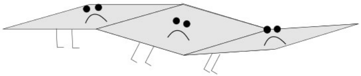

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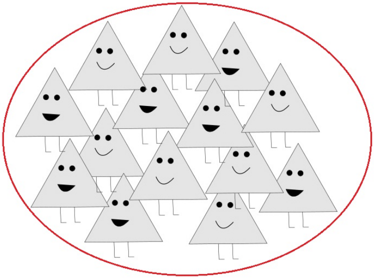

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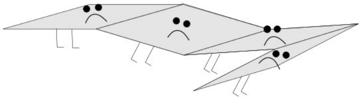

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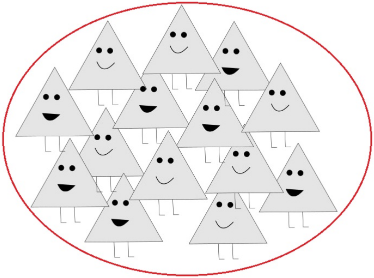

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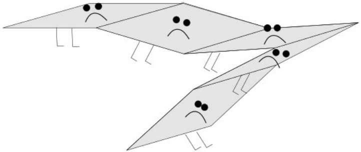

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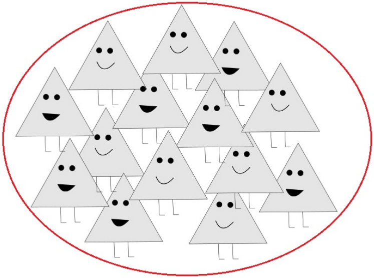

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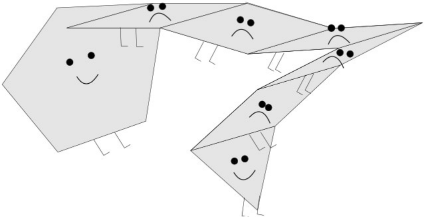

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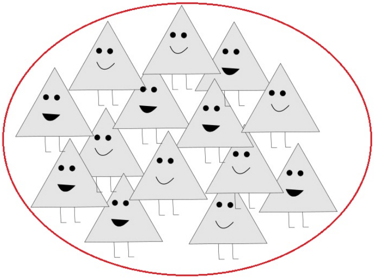

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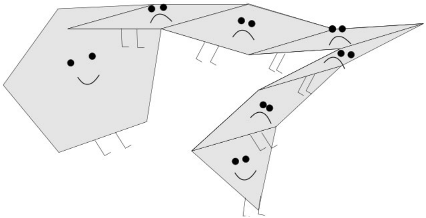

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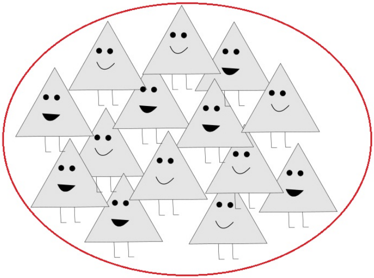

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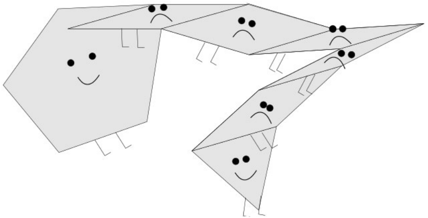

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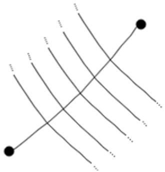

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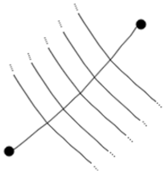

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The maximum number of edges of a k-planar graph is: $c \cdot n\sqrt{k}$.

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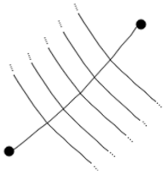

Theorem (Pach, Tóth, 1997)

The maximum number of edges of a k-planar graph is: $c \cdot n\sqrt{k}$.

Theorem (Rote, 2023 and G., Tóth, 2023)

For the maximum number of edges of a k-planar unit distance graph, $u_k(n)$, we have

$$2^{\Omega(\log k/\log\log k)} n \le u_k(n) \le c\sqrt[4]{k}n.$$

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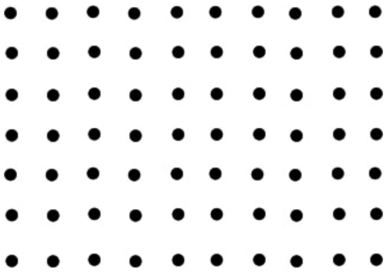

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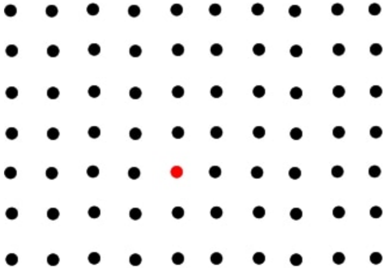

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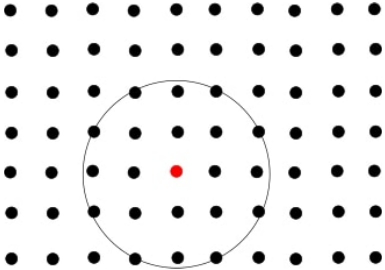

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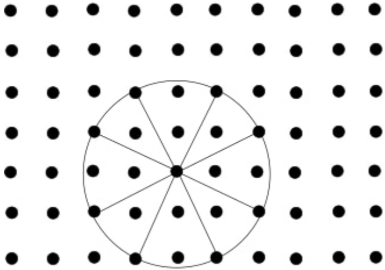

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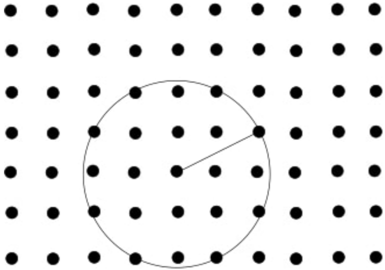

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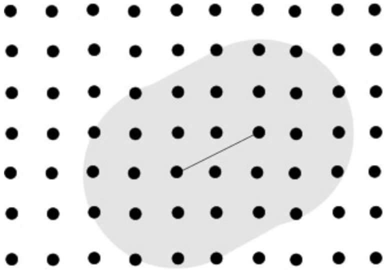

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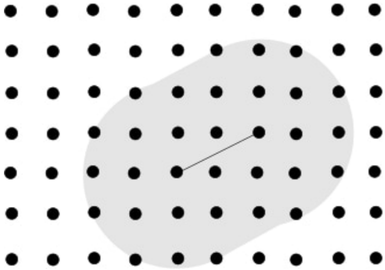

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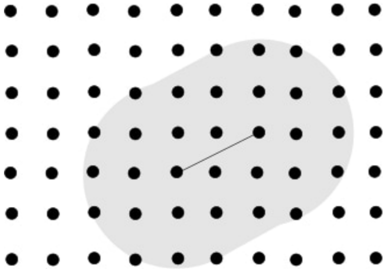

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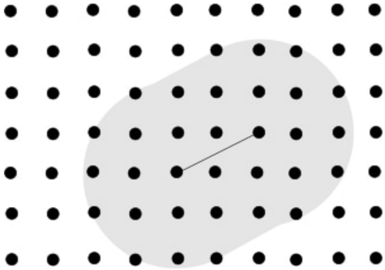

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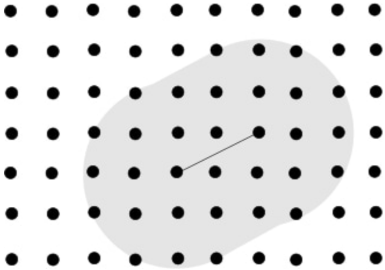

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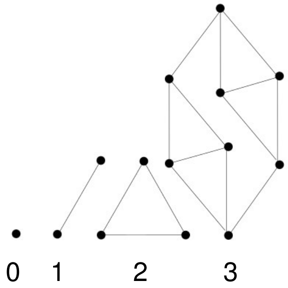

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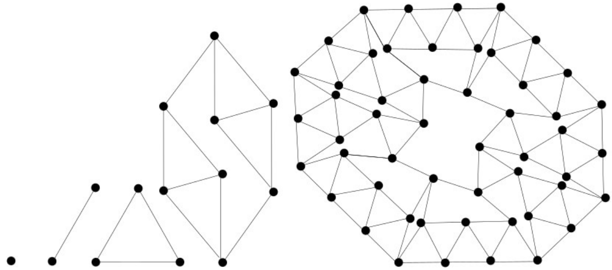

0 1 2 3 4

(Harborth)

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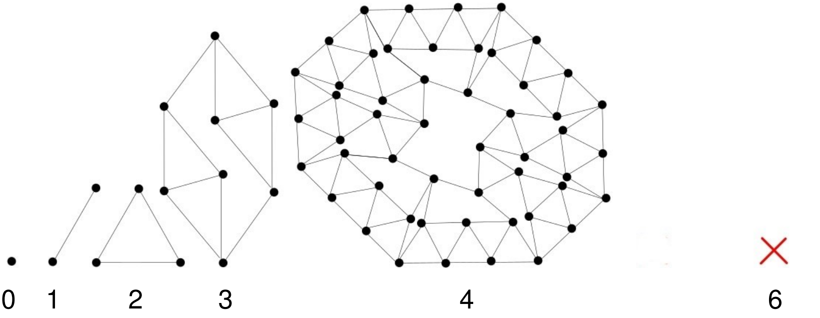

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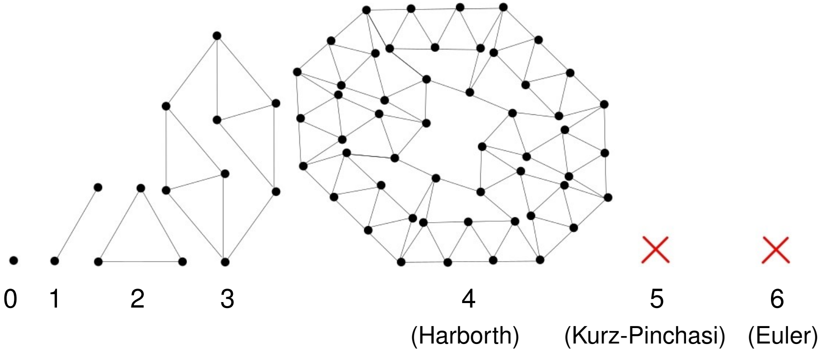

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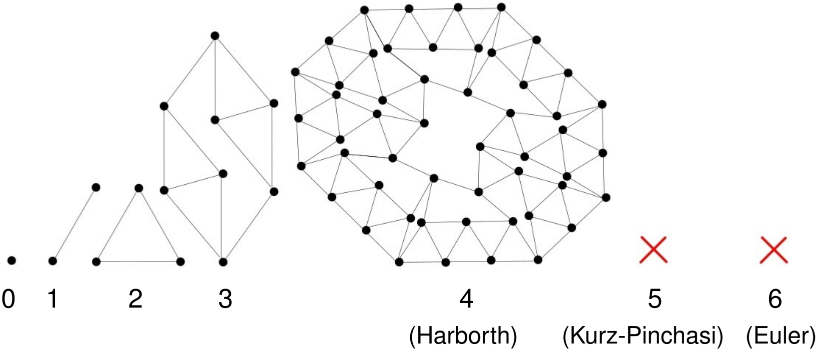

Question:

Is there a 5-regular 1-planar unit distance graph?

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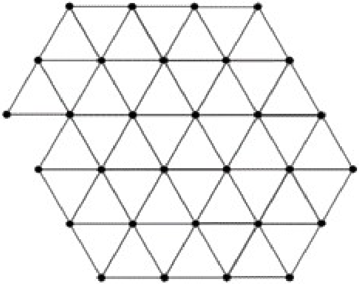

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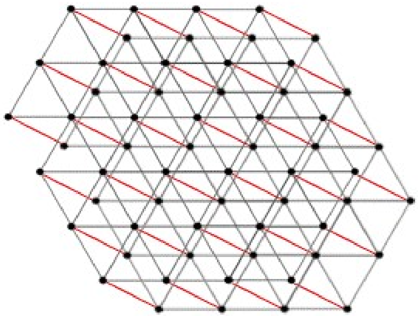

$\Rightarrow$ 3-planar unit distance graph with $\approx 3.5n$ edges

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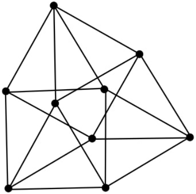

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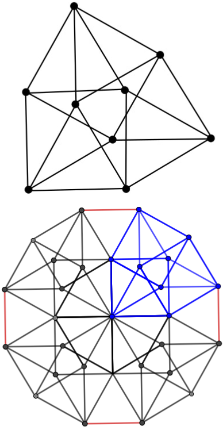

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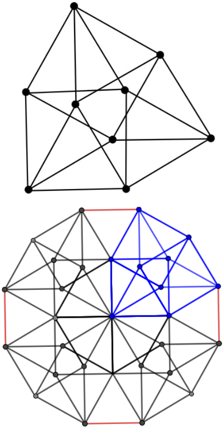

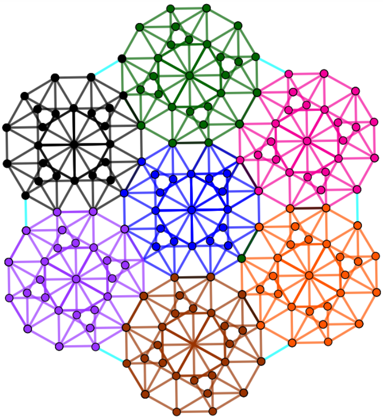

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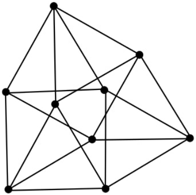

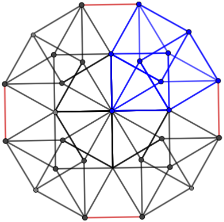

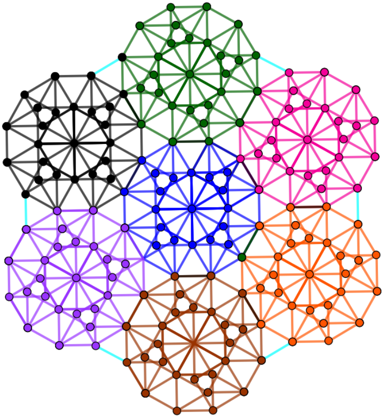

   - $k = 1$: ?

Thank you for your attention!