Improving the Crossing Lemma by characterizing dense 2-planar and 3-planar graphs

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Motivation

Crossing Lemma is a classical result from combinatorial geometry, and proved by Leighton, and Ajtai, Chvatal, Newborn and Szemeredi 1983.

Made it into the 'Proofs from the Book'

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Proof:
Since $cr(G) \geq m - (3n - 6)$, we have $cr(G) - m + 3n \geq 0$

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We have $E(n_p) = pn, E(m_p) = p^2 m, E(X_p) = p^4 cr(G)$

By linearity of expectation
$$0 \leq E(X_p) - E(m_p) + 3E(n_p) \geq 0 = p^4 cr(G) - p^2 m + 3pn \; .$$

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Then $cr(G) \geq \frac{p^2 m - 3pn}{p^4} = \frac{m}{p^2} - \frac{3n}{p^3}$.

Choosing $p = \frac{4n}{m}$,
we get $cr(G) \geq \frac{1}{64}[\frac{4m}{(n/m)^2} - \frac{3n}{(n/m)^3}] = \frac{1}{64} m^3/n^2$

That's all !

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Remove from G one non-planar edge at a time,
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Now it goes like this:
Remove from G an edge with ≥ 2 crossings, as long as $m \geq 4n$,
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$\rightarrow cr(G) \geq 2(m - 4n) + n = 2m - 7n$ if $m \geq 4n$

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Pach, Radoicic, Tardos and Toth 06 roughly followed those lines:

$\rightarrow cr(G) \geq \frac{7}{3}m - \frac{25}{3}n$

$\rightarrow cr(G) \geq \frac{1}{31.08}\frac{m^3}{n^2} = 0.032\frac{m^3}{n^2}$

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<u>or even Ackerman 2014 (bound for 4-planarity):</u>

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One way for improvement: Density of 5-planar graphs !

A closer look to dense 2/3-planar graphs

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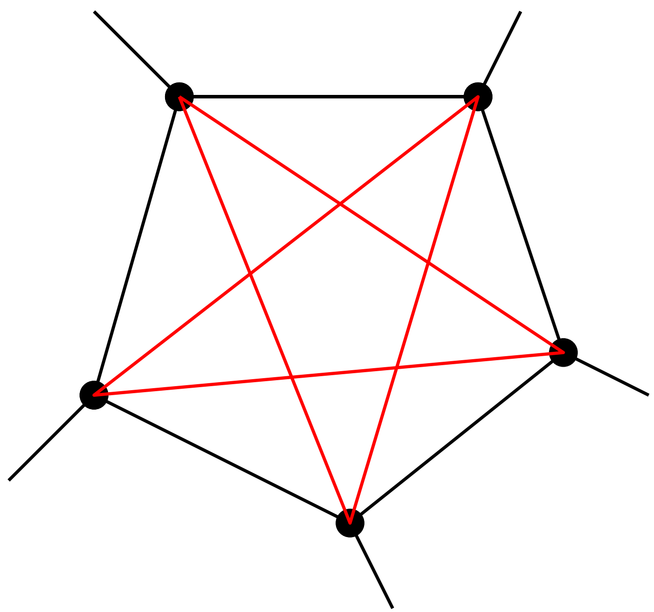

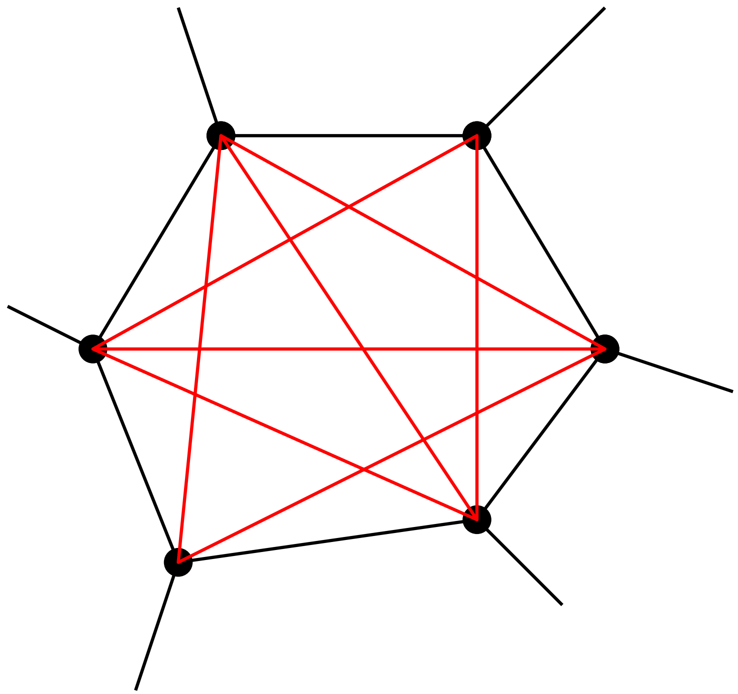

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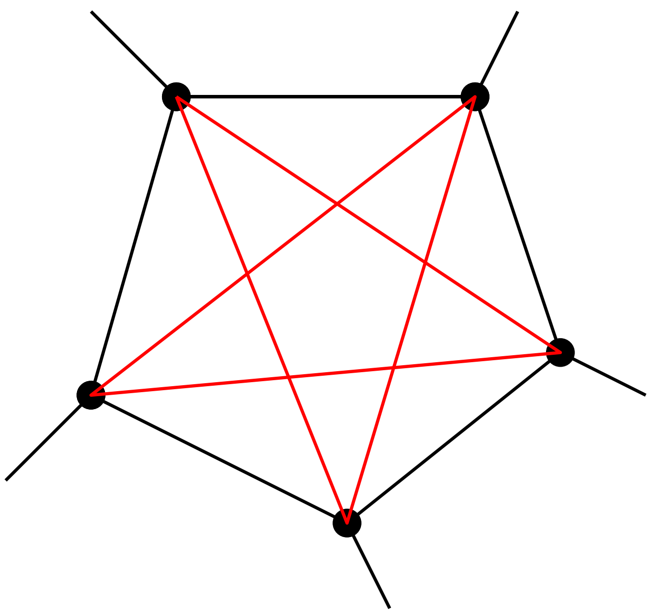

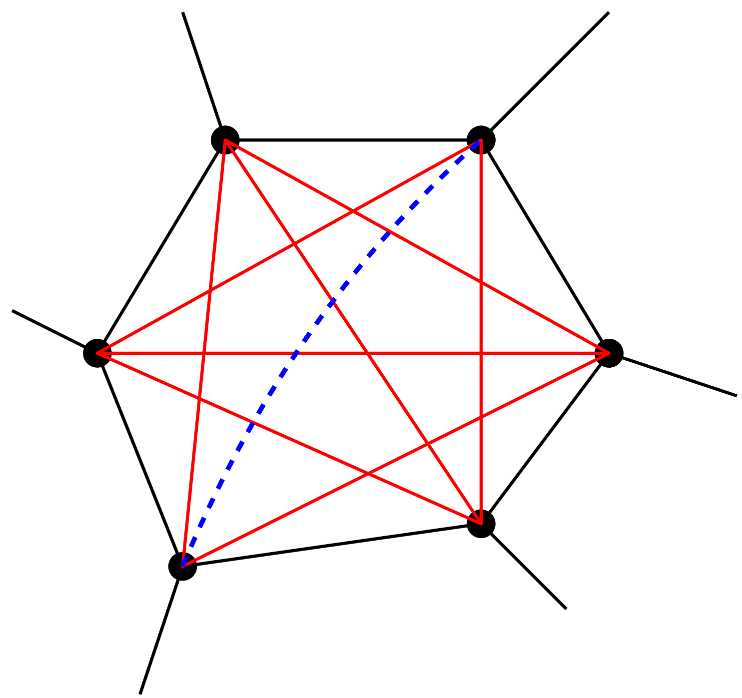

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Theorem:
Any graph G that admits a 2-planar F_5^2-free drawing has $\leq 4.5n$ edges.
If additionally, the drawing is even F_6^2-free, then $m \leq \frac{13}{3}n$

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Any graph G that admits a 3-planar F_6^3-free drawing has $\leq 5n$ edges.

Proof: Discharging!

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Proof: Discharging!

Corollary:
For every 2-planar drawing of any graph with $n \geq 3$ vertices and $\frac{13}{3}(n-2) + x$ edges for $x \in [0, \frac{2}{3}(n-2)]$, the number of F_5^2 and F_6^2 configurations is at least x.

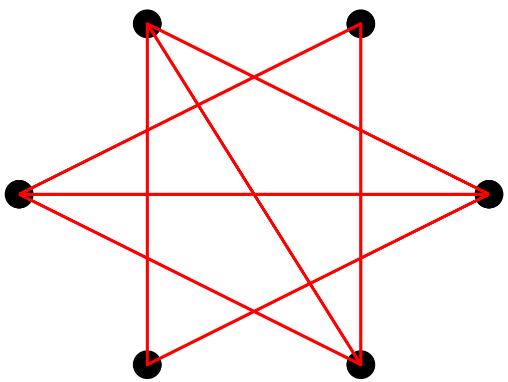
Consequences for the Crossing Lemma

Counting:

So assume $m > 5n$ and let D be a crossing-minimal drawing of G.
From D, we iteratively remove the edge with the most crossings until $5n$ edges are left.
So, first ≥ 5 crossings for m_5 edges $\rightarrow 5m_5$
Then 4 crossings for $4m_4$ edges $\rightarrow 4m_4$
Then remove m_3 edges from m_3 F_6^3 configurations $\rightarrow 3m_3$
Then $m \leq 5n$.

In total, this is $\geq 5m_4 + 4m_4 + 3m_3$ crossings.

Now, from each of the m_3 F_6^3 configs we can delete 2 more edges with 3 cross. $\rightarrow 6m_3$



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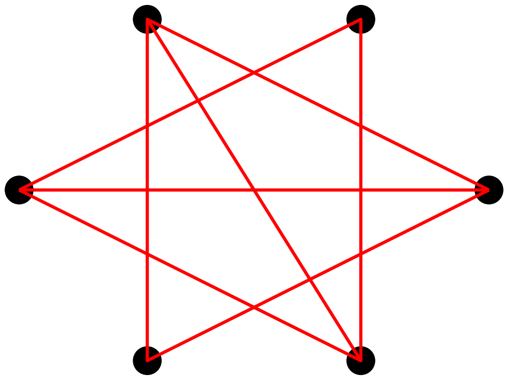

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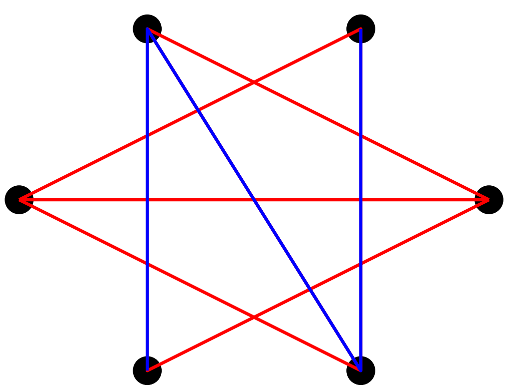
Possibly, we can delete more m_3^- edges with 3 cross. $\rightarrow 3m_3^-$
Then 2-planarity has been reached !



New bound for the Crossing Lemma

From here, we have $\geq \frac{7}{3}(5n - 2m_3 - m_3^-) - \frac{25}{3}n$ crossings [PRTT06]

In total, it can be bounded by $\geq 4m - \frac{50}{3}n$.

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This can be refined to $cr(G) \geq \frac{73}{18}m - \frac{503}{18}n$

There, the probabilistic argument can be applied, then

$$\boxed{cr(G) \geq \frac{1}{27.4}\frac{m^3}{n^2}}$$

Corollary

In k-planar graphs, we have at most $km/2$ crossings.

Hence $\frac{1}{27.4} m^3/n^2 \leq cr(G) \leq km/2$

Theorem:
For k-planar graphs with n vertices and m edges, we have $m \leq 3.72\sqrt{kn}$. Previously, we had 3.81

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THANK YOU !