On the edge density of bipartite 3-planar and bipartite gap-planar graphs

Aaron Büngener, Maximilian Pfister

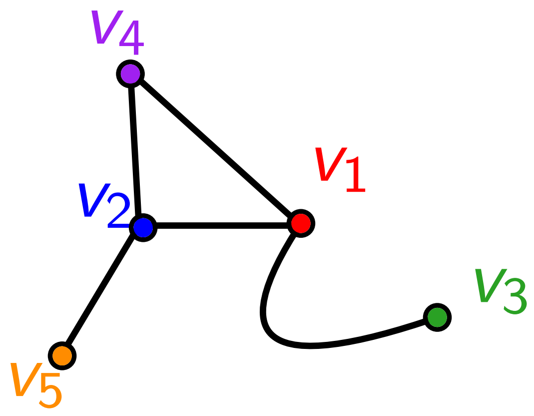
Preliminaries

Graphs

$$V = \{v_1, v_2, v_3, v_4, v_5\}$$

$$E = \{v_1 v_2, v_1 v_3, v_1 v_4, v_2 v_5, v_2 v_4\}$$

Drawings



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Bipartite: No odd cycles

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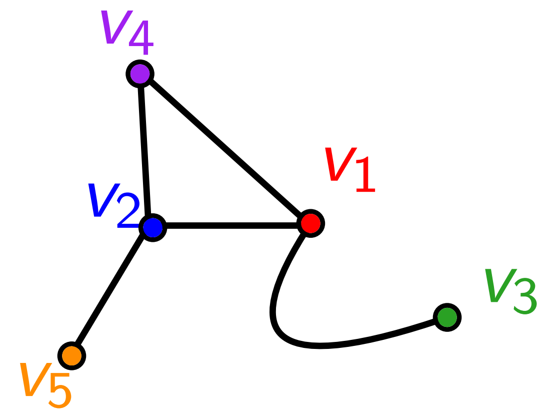

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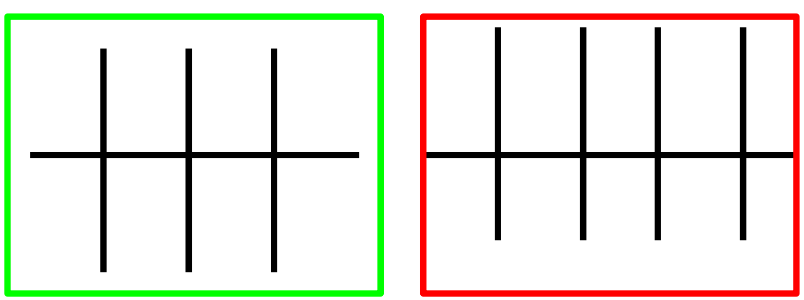

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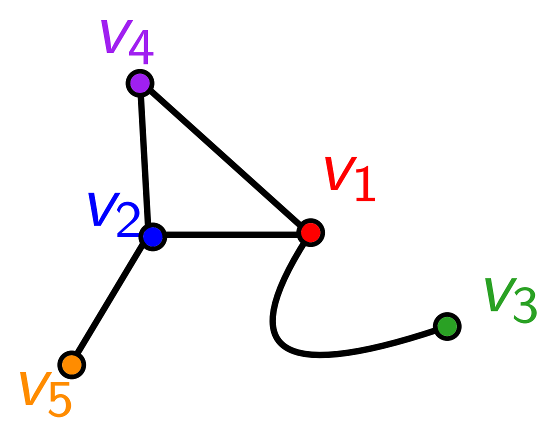

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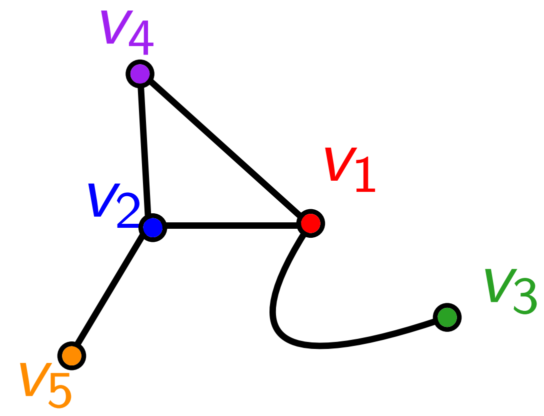

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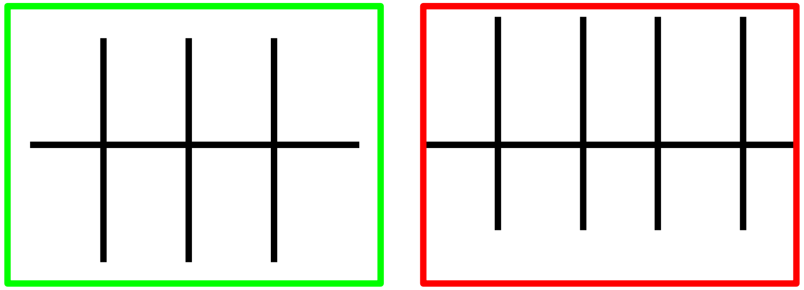

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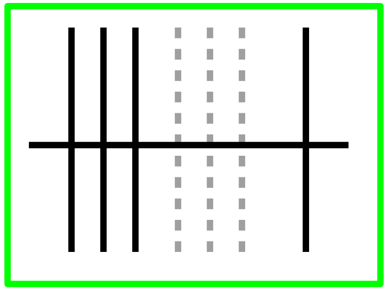

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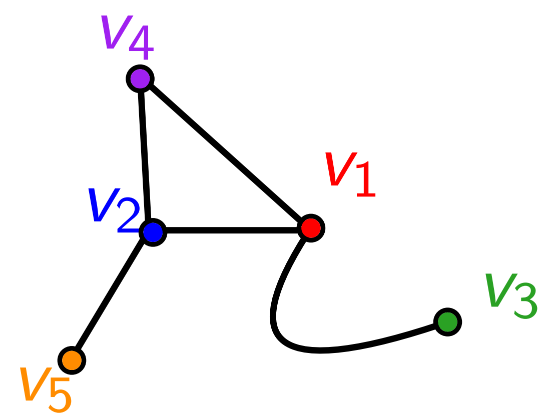

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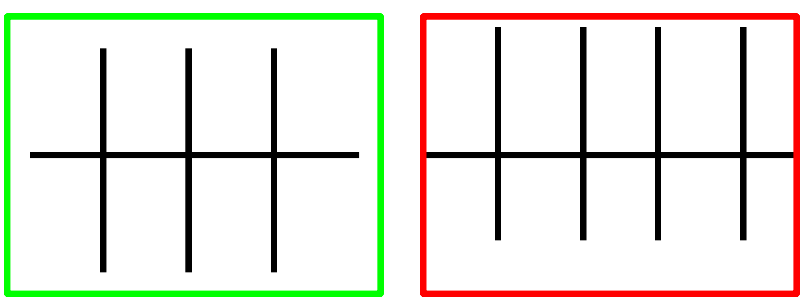

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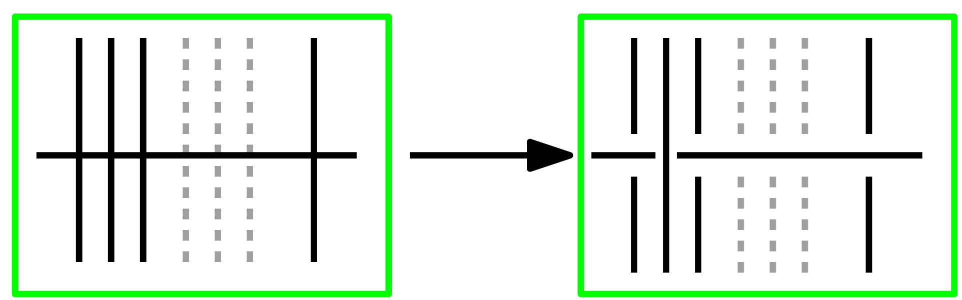

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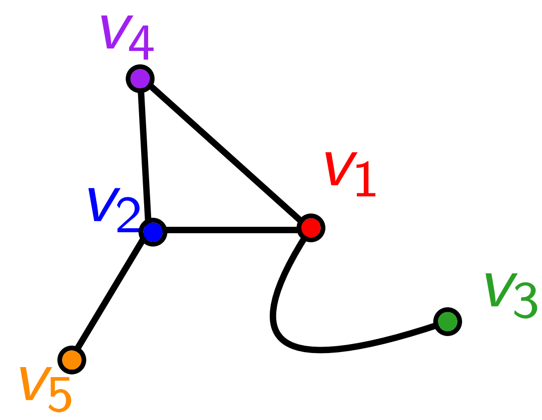

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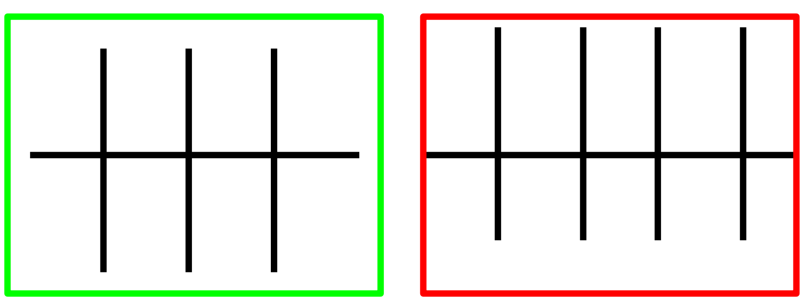

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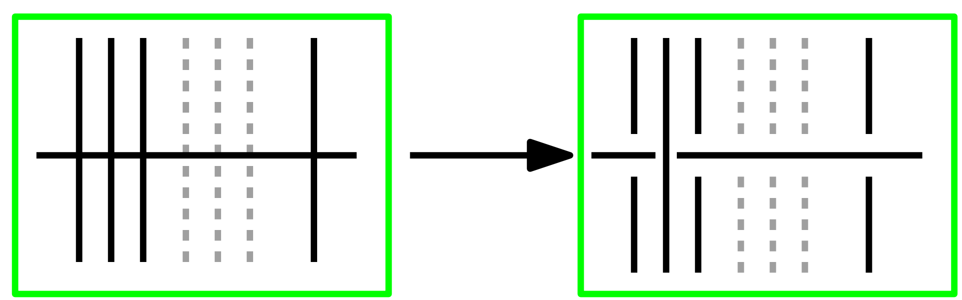

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Edge density: Maximum number of edges

Related work

3-planarity:

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- Bipartite 2-planar graphs [A1]

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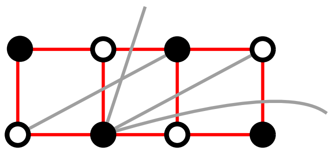

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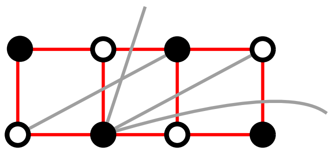

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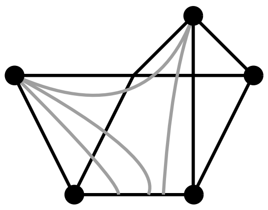

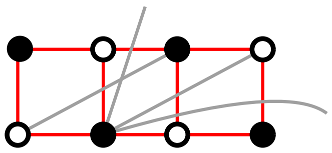

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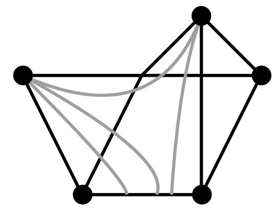

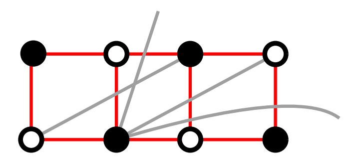

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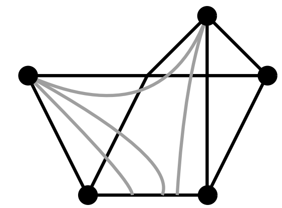

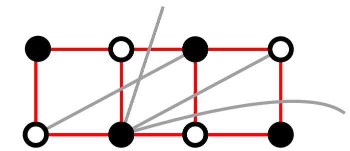

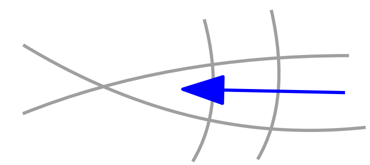

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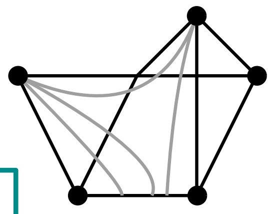

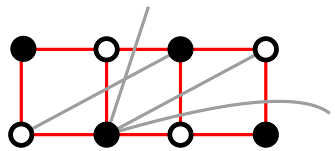

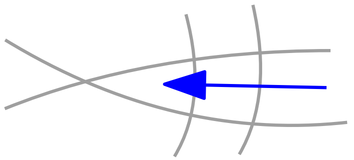

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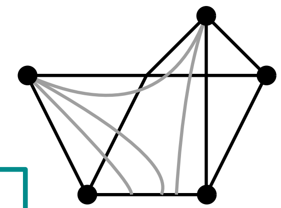

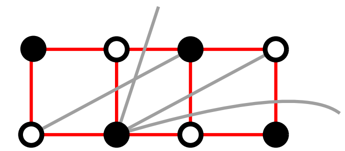

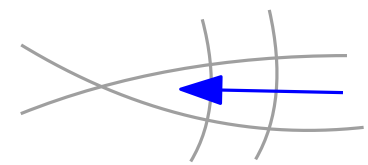

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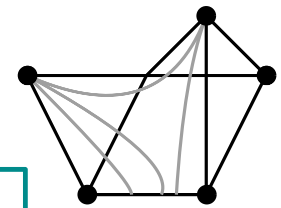

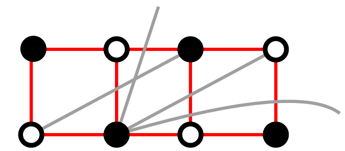

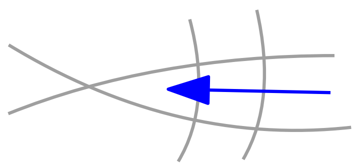

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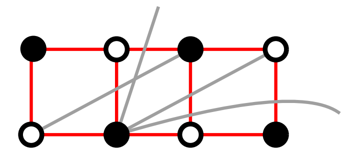

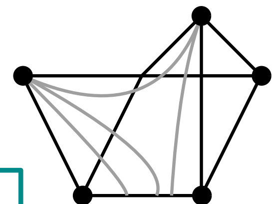

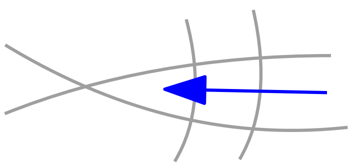

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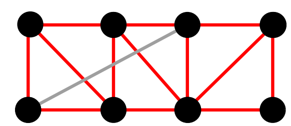

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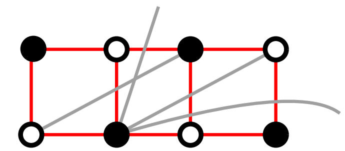

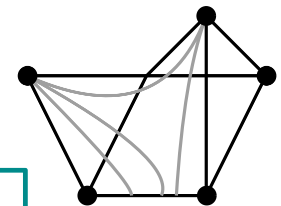

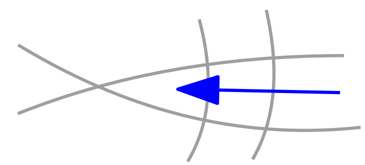

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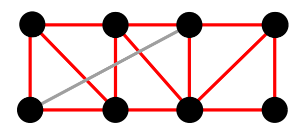

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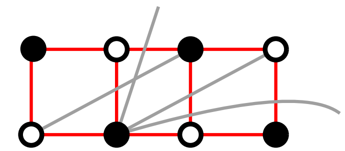

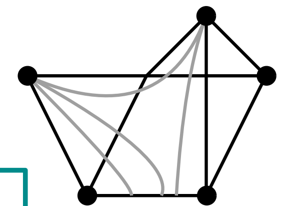

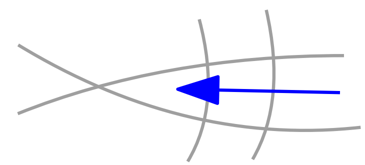

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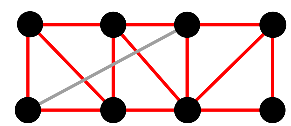

Bipartite gap-planar graphs have at most $4n - 8$ edges.

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Drawing Γ

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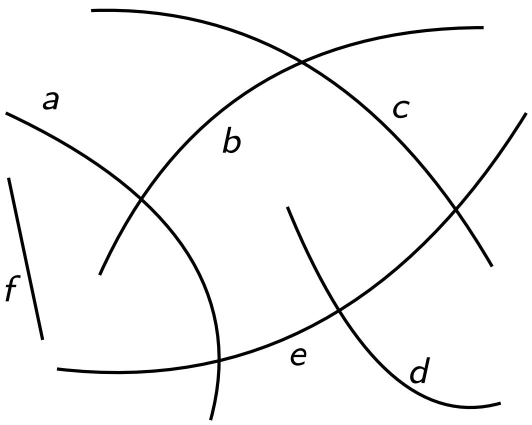

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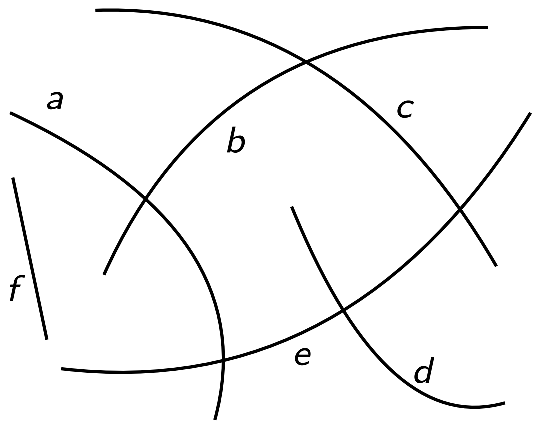

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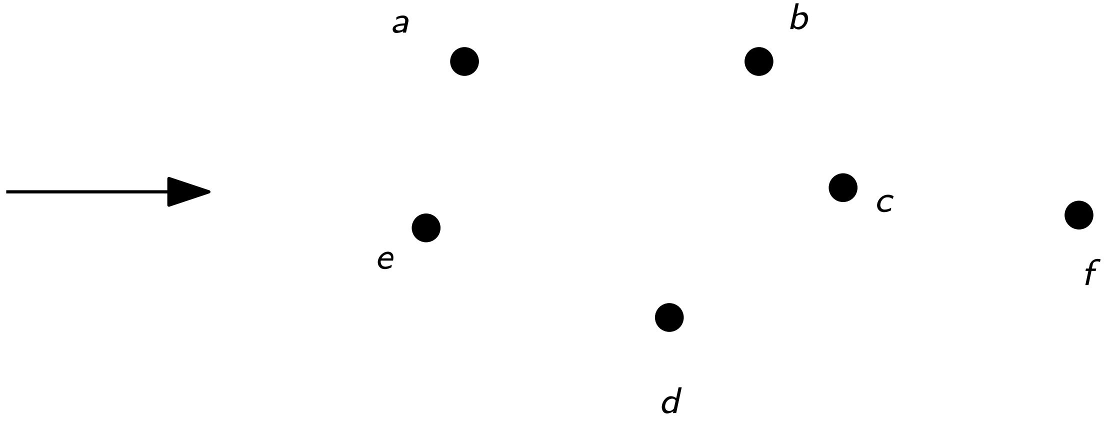

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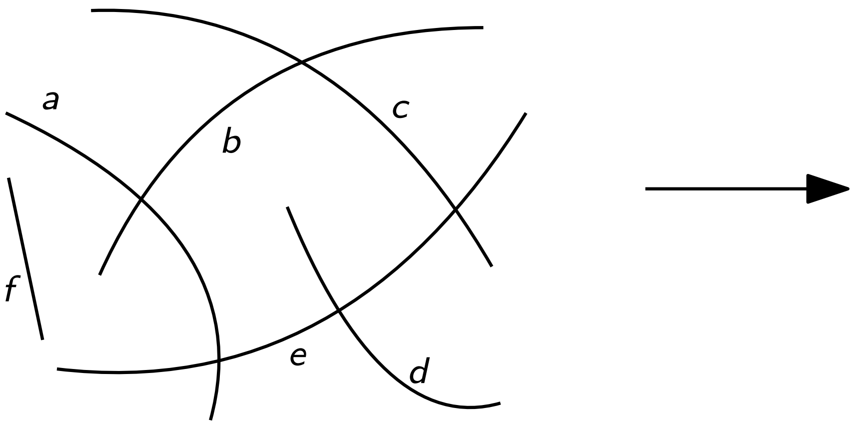

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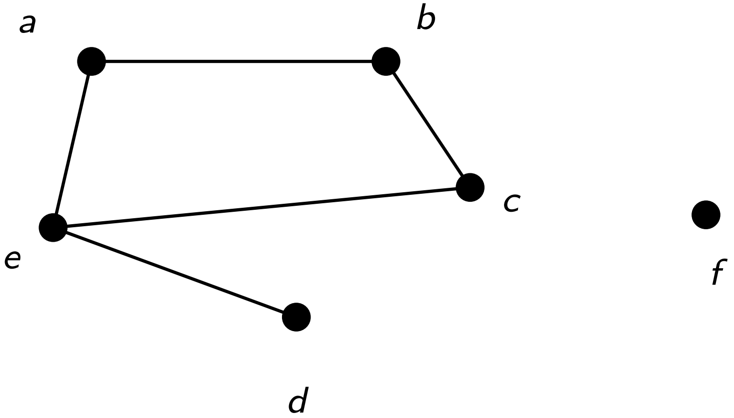

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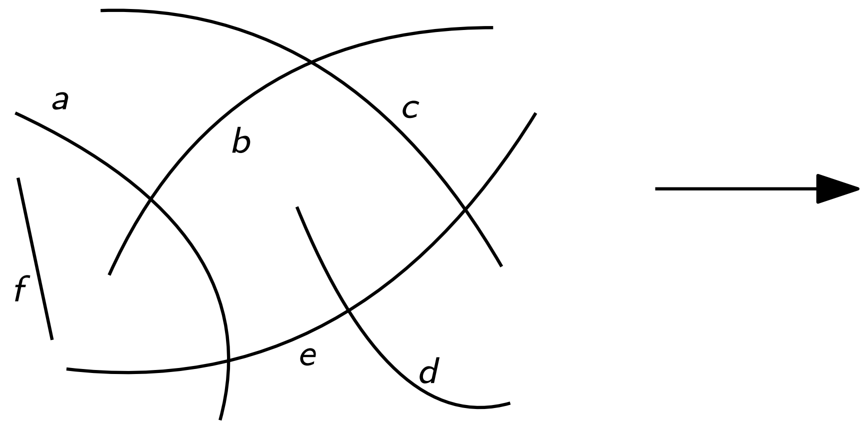

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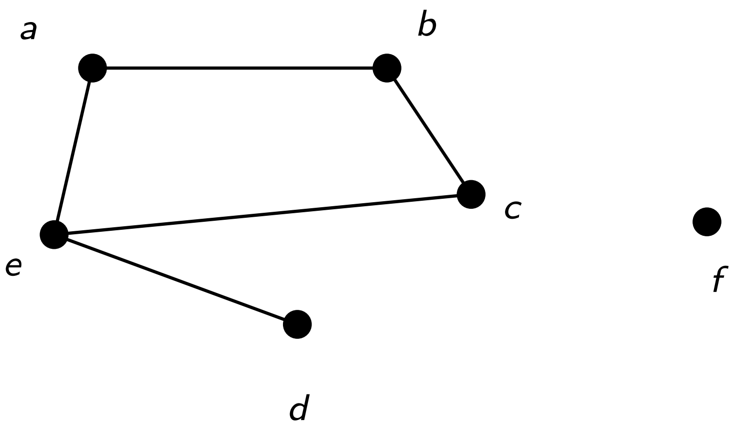

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[B]: Bae et. al.: Gap-planar graphs, TCS 745

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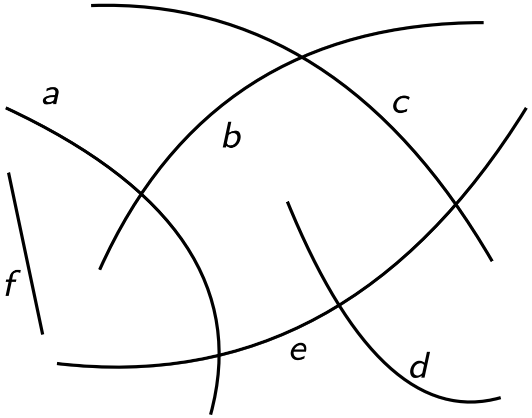

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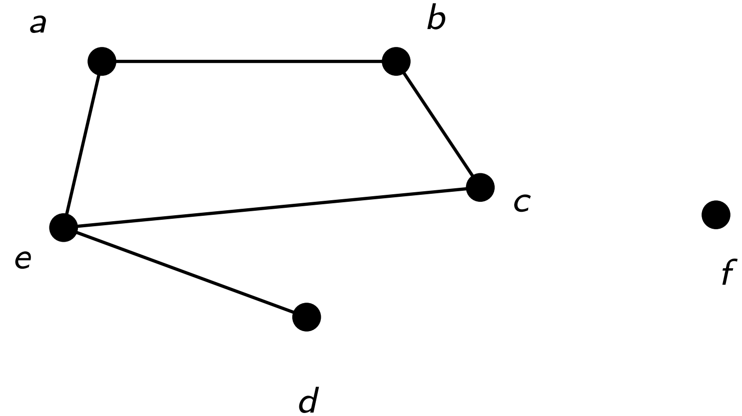

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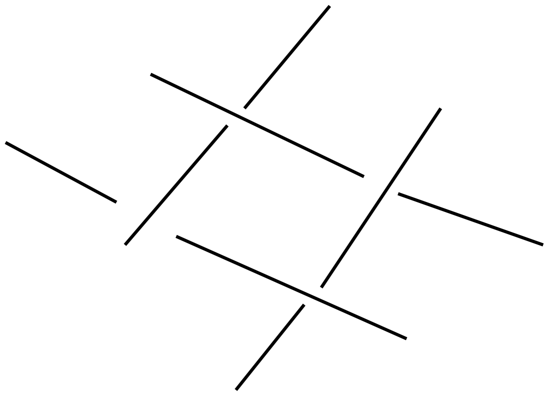
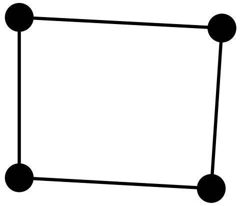
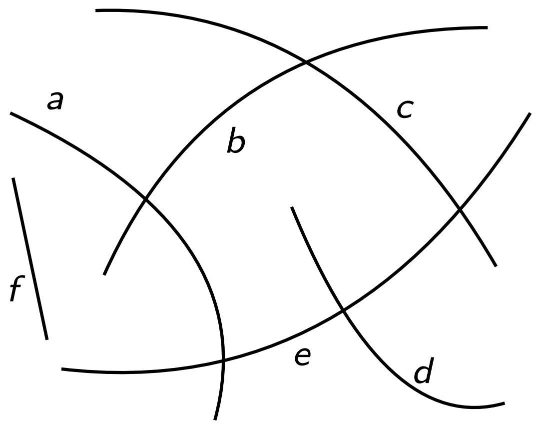




[B]: Bae et. al.: Gap-planar graphs, TCS 745

Gap-planarity

Drawing Γ



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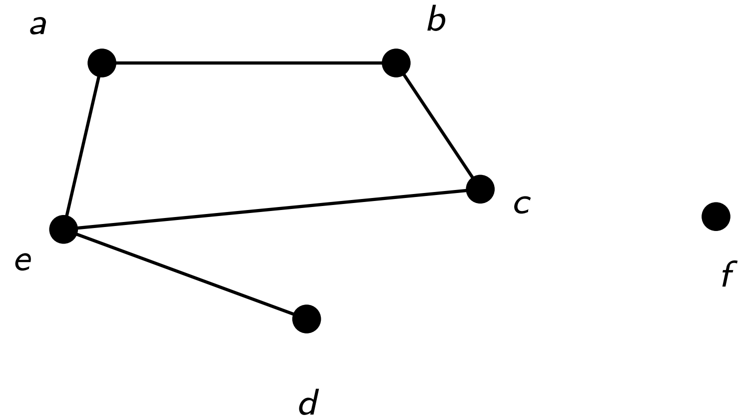

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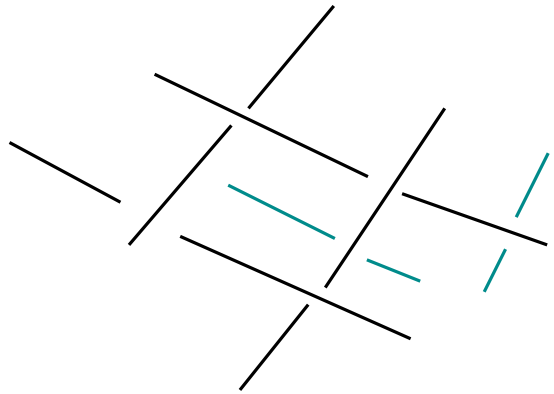

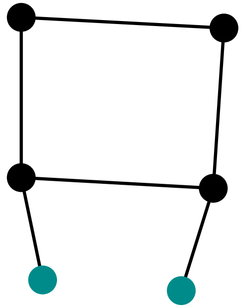

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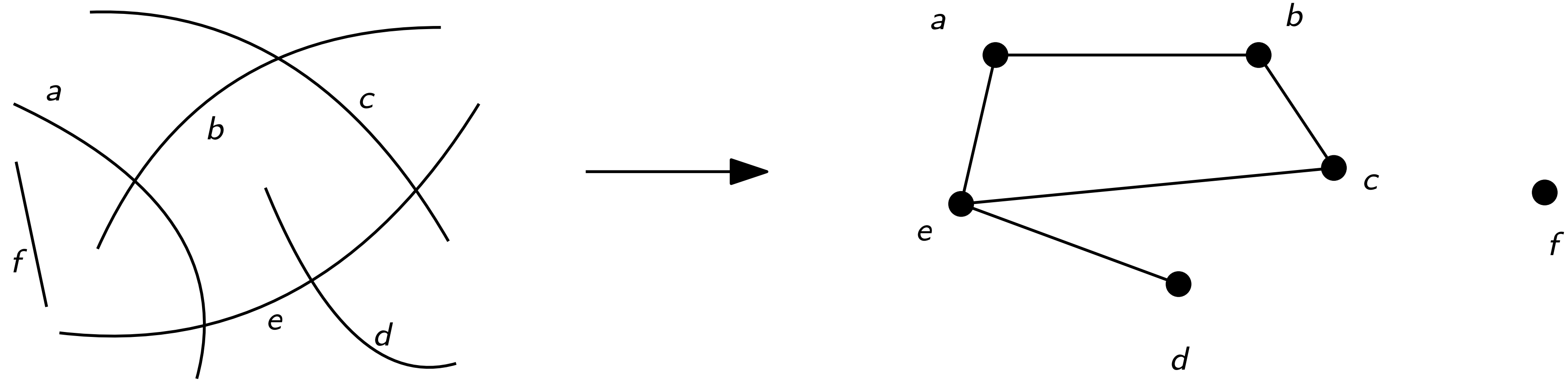

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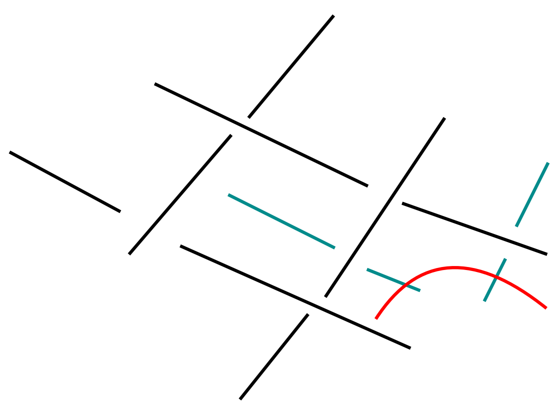

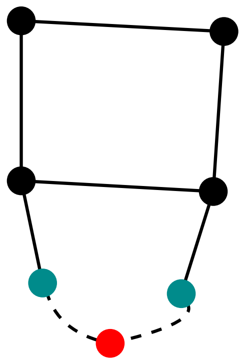

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A bipartite planar subgraph has at most $2n - 4$ edges, thus the desired bound of $4n - 8$ follows.

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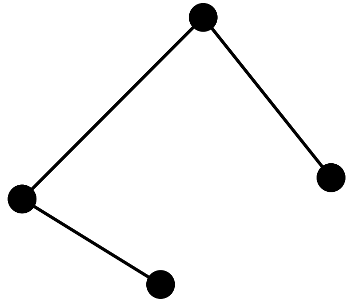

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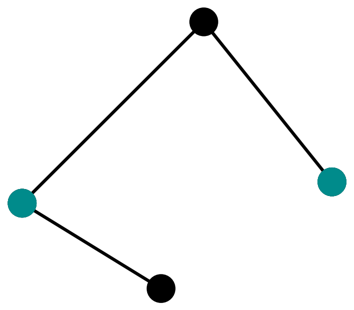

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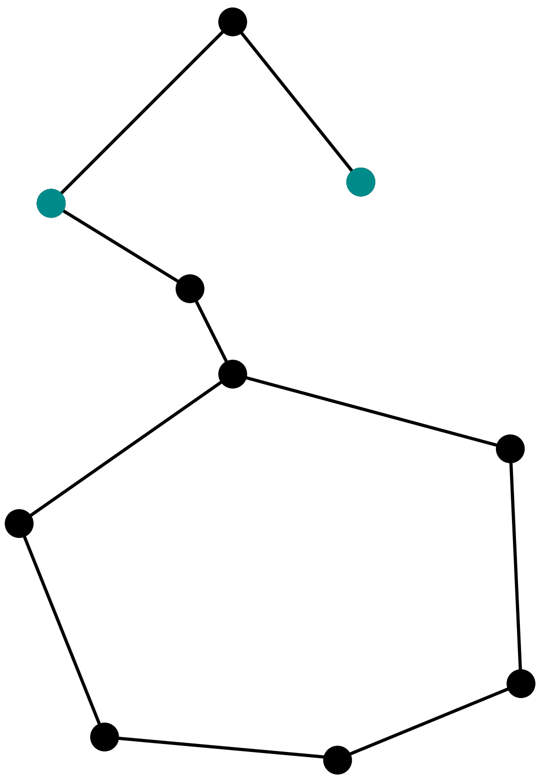

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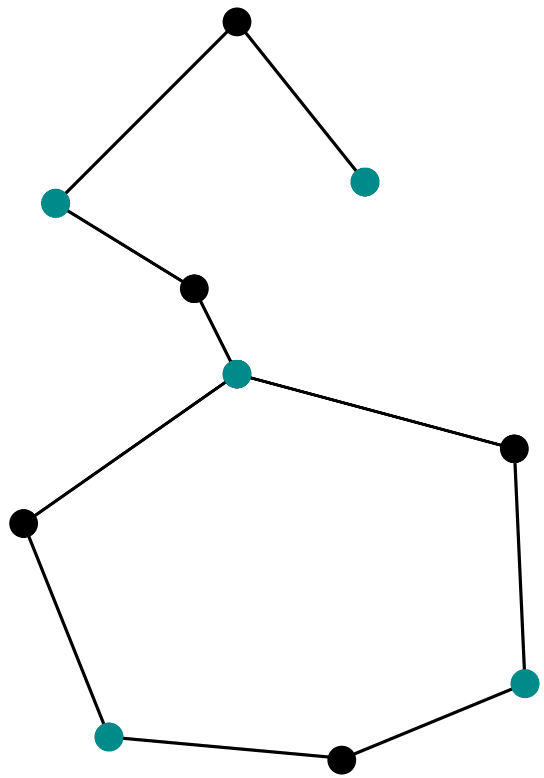

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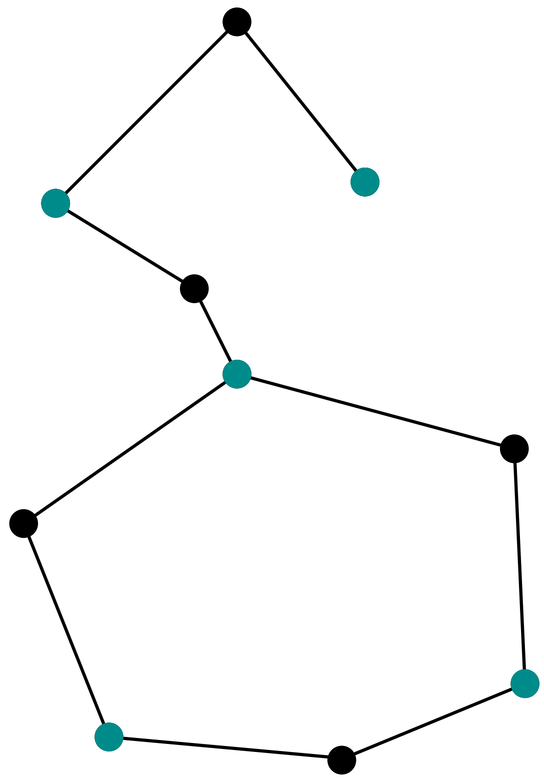

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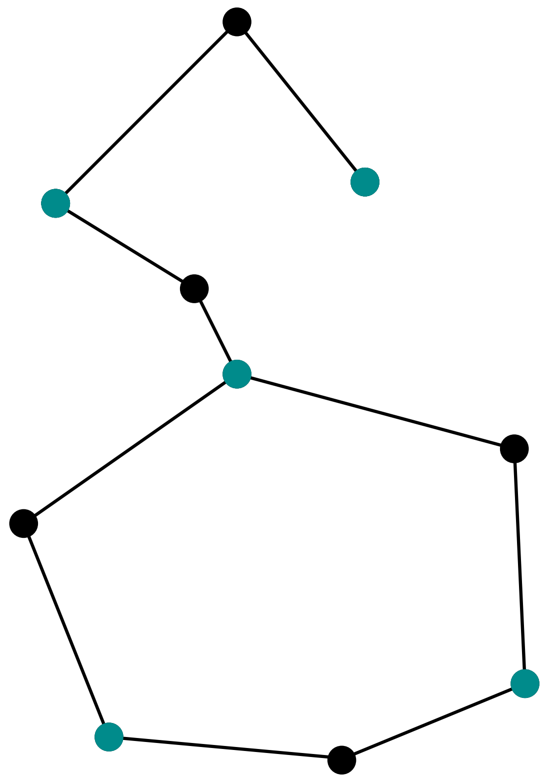

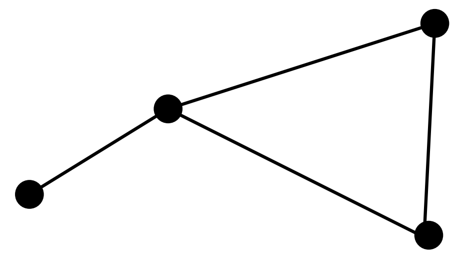

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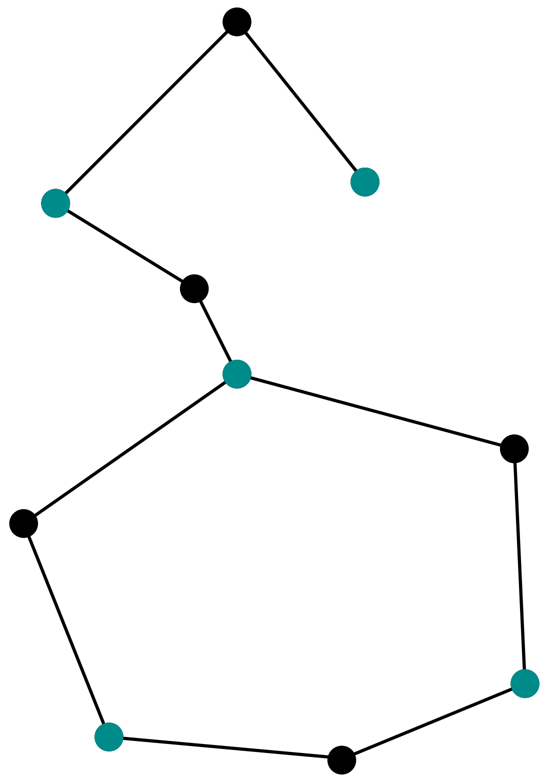

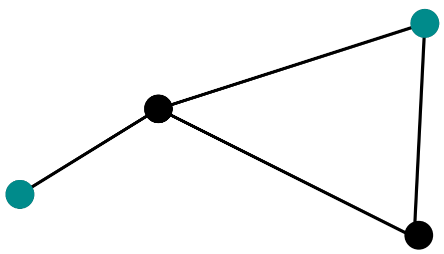

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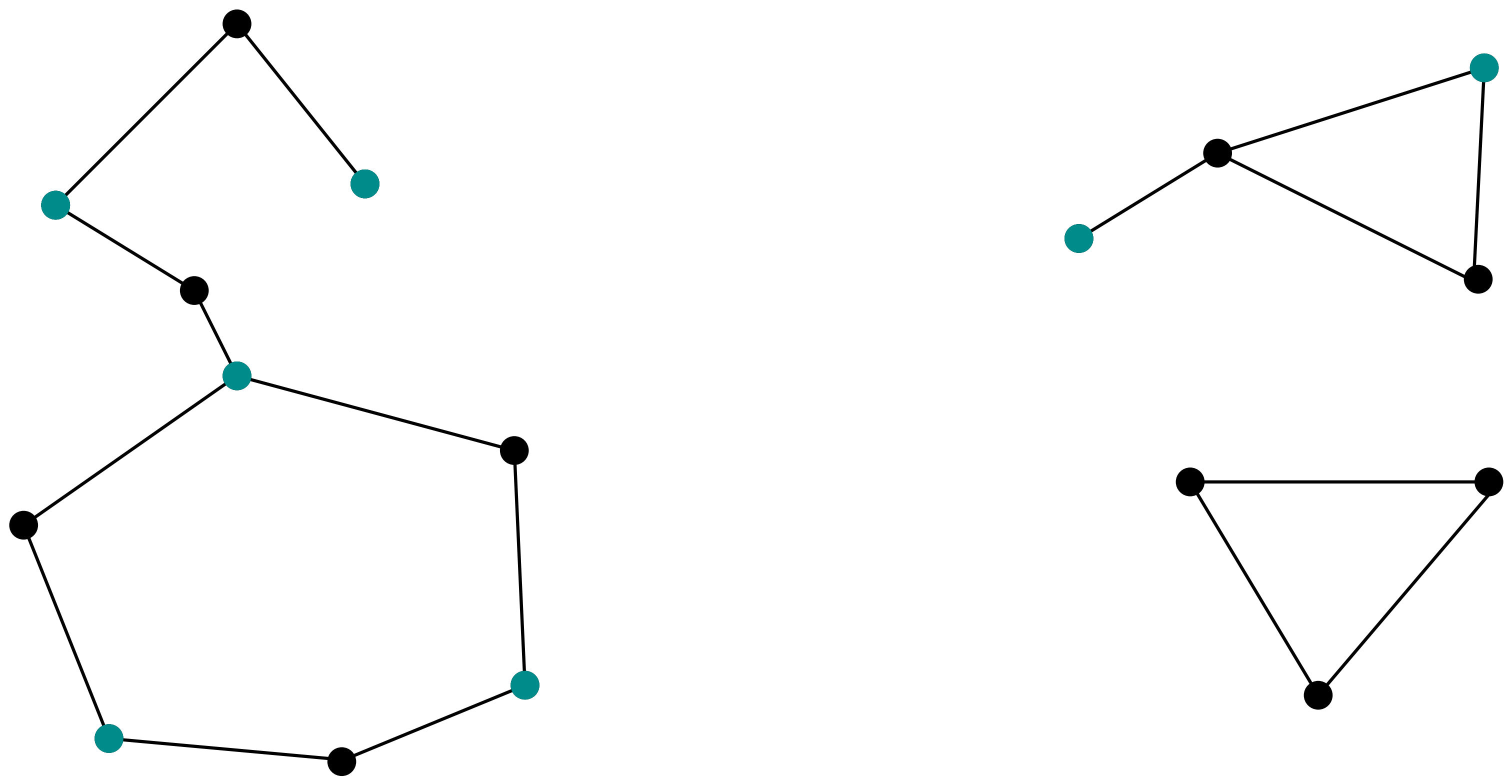

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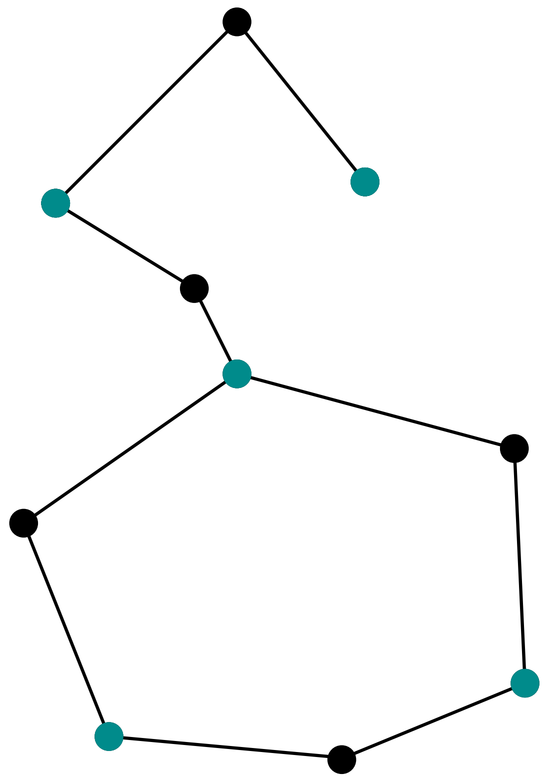

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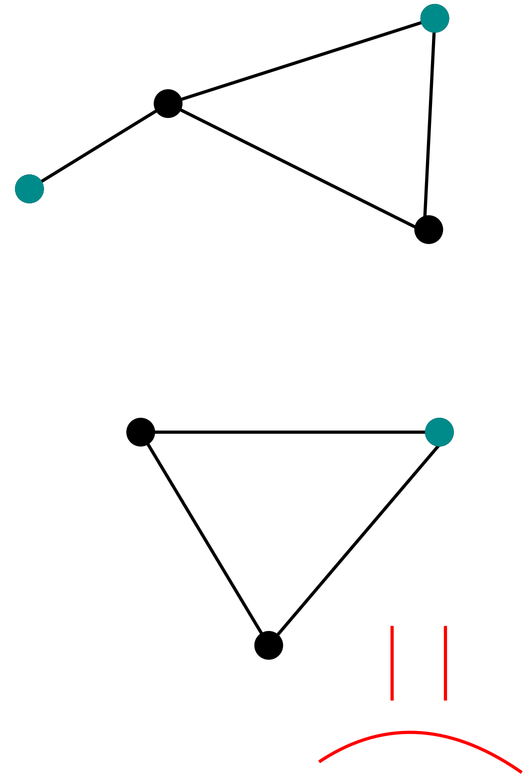
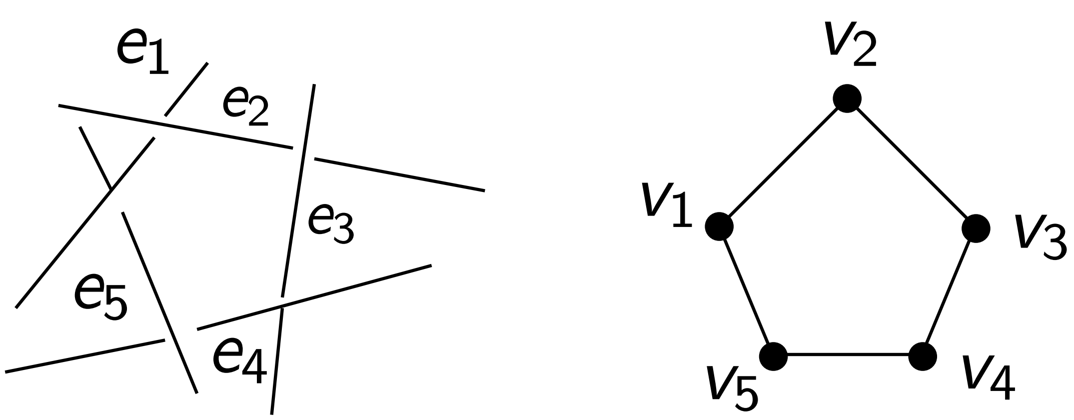


Augment the critical components

Idea: Merge singletons (uncrossed edges in Γ) with the bad components.



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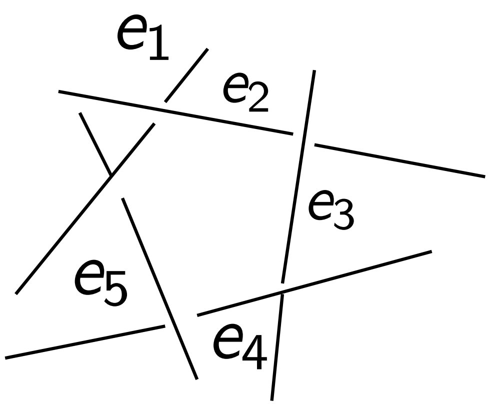

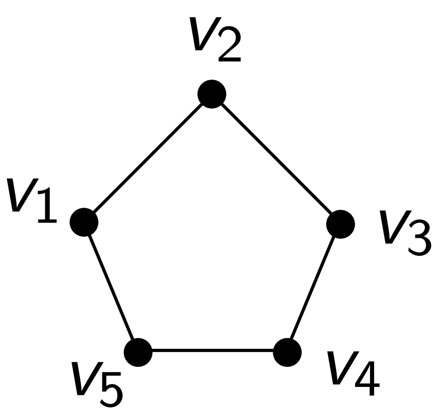

Lemma: For any two consecutive edges $e_i = a_i b_i$ and $e_{i+1} = a_{i+1} b_{i+1}$, either $a_i b_{i+1}$ or $a_{i+1} b_i$ exists.

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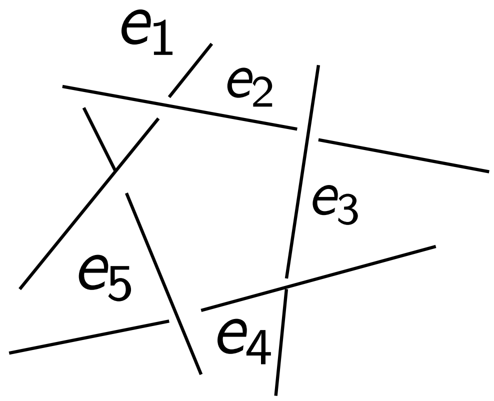

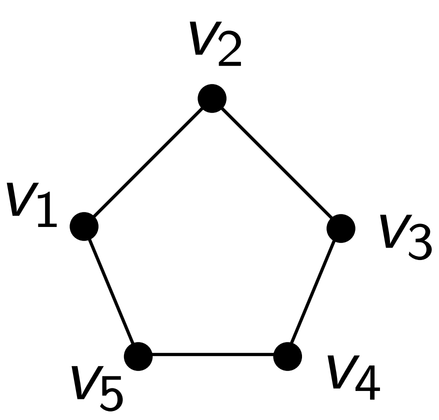

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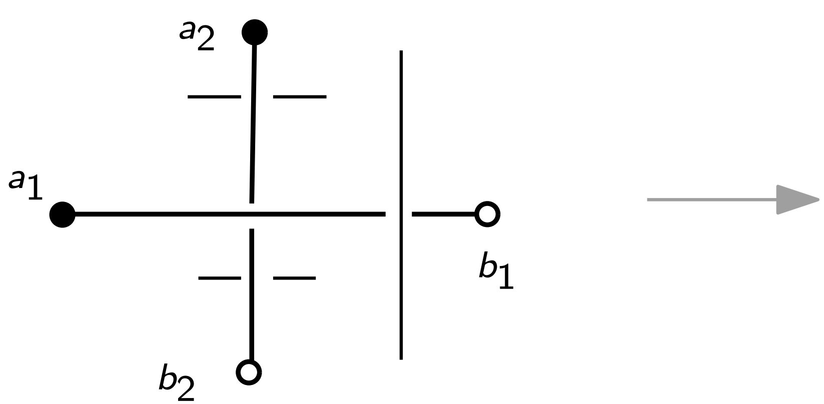

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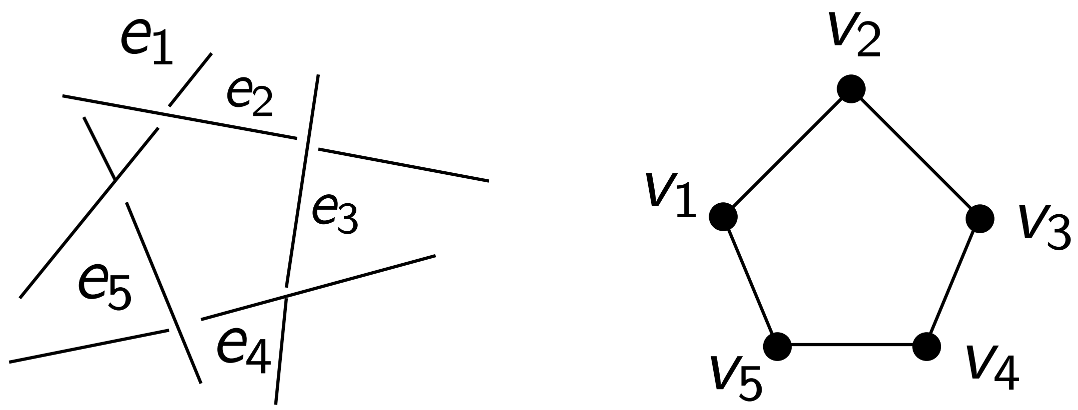

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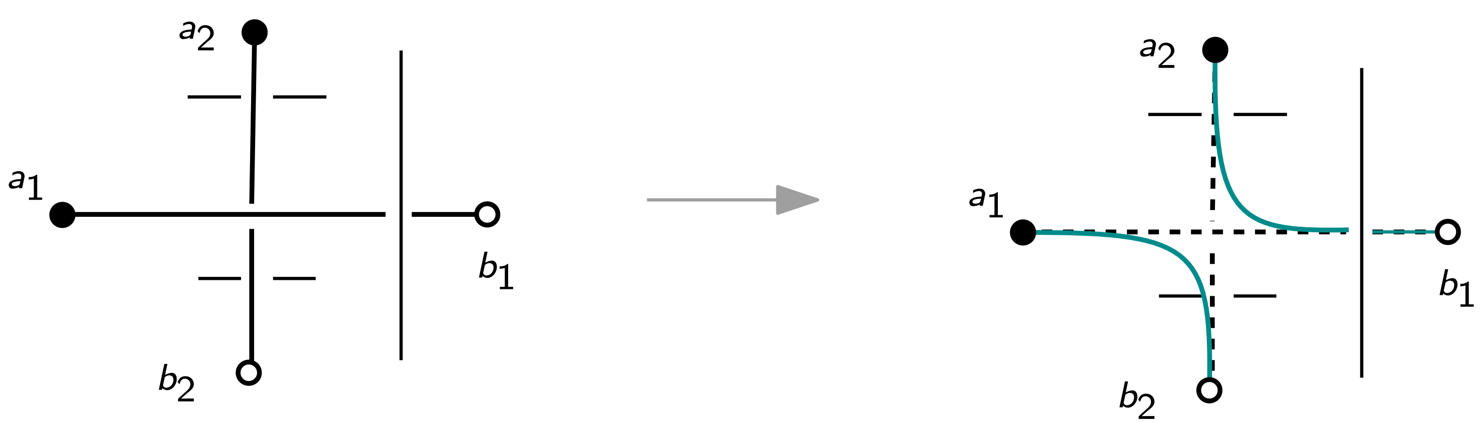
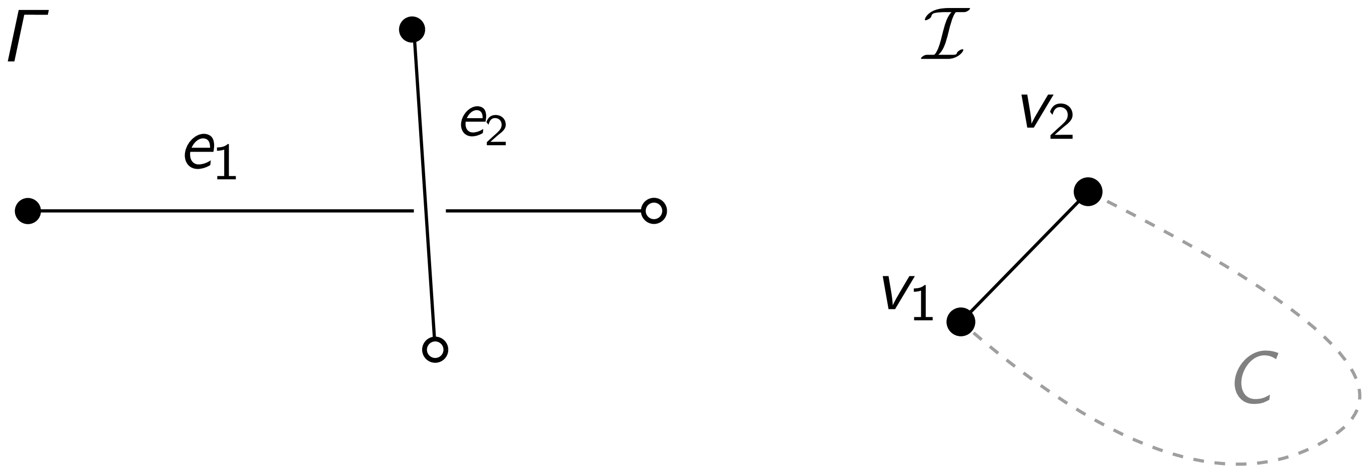


Case analysis I



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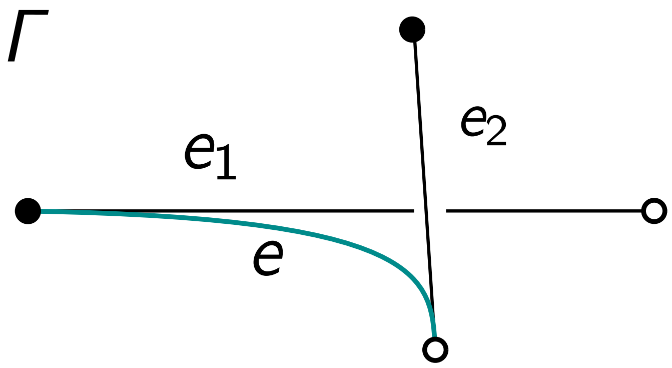

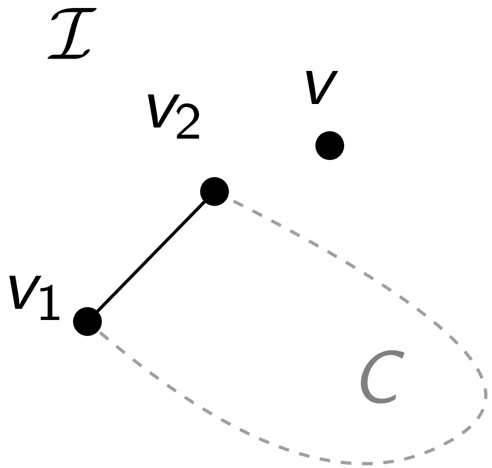

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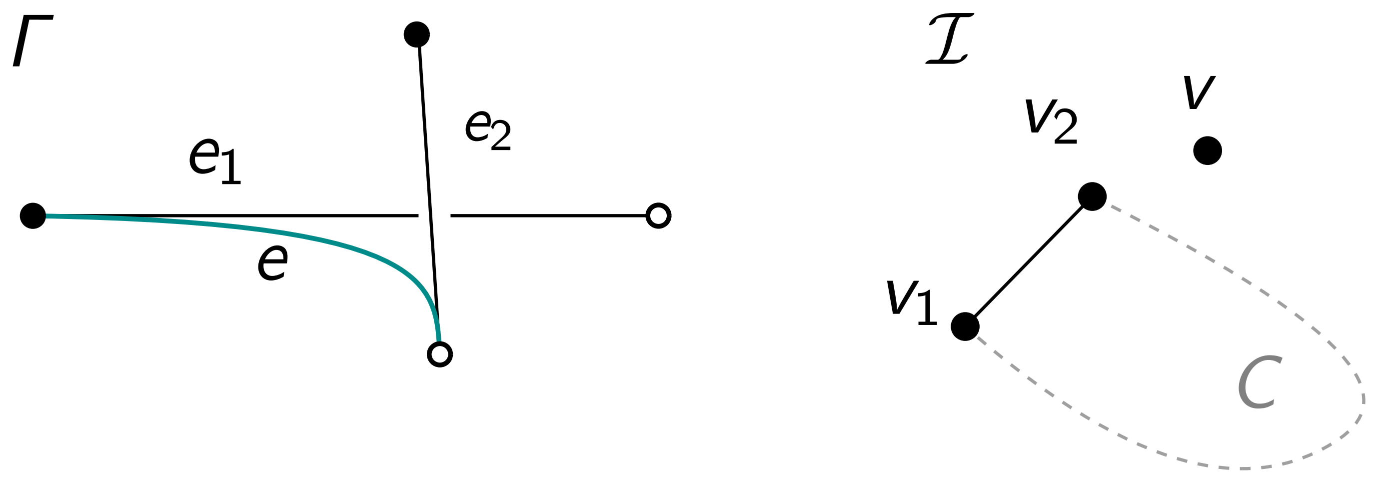

C1 : e crosses an edge of odd cycle C

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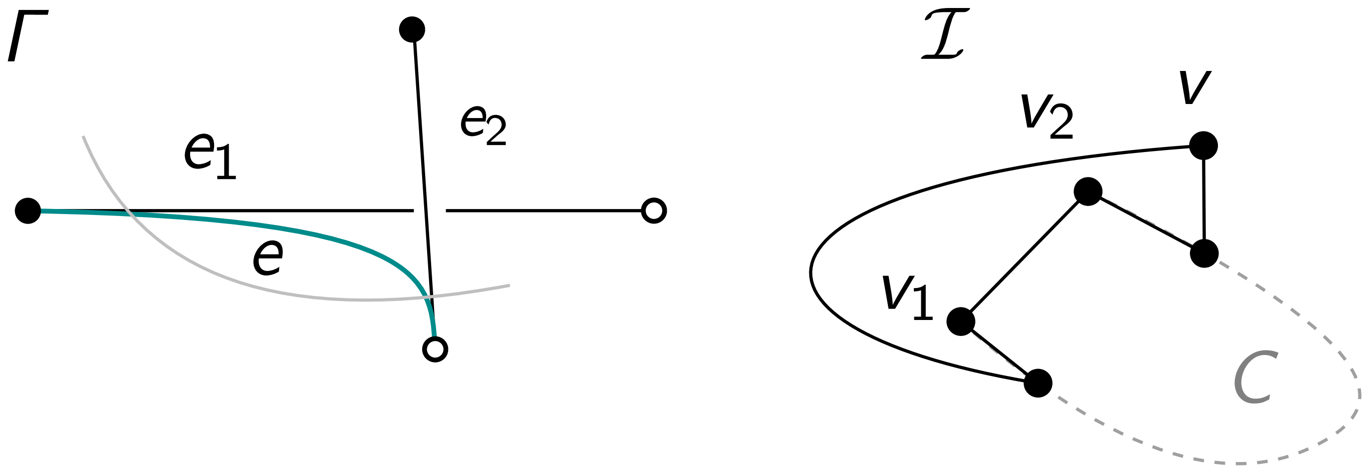

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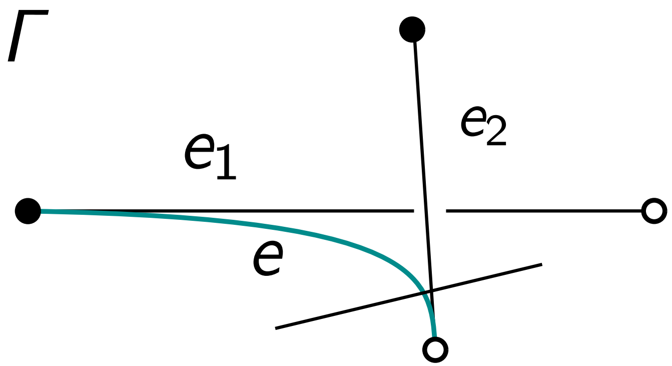

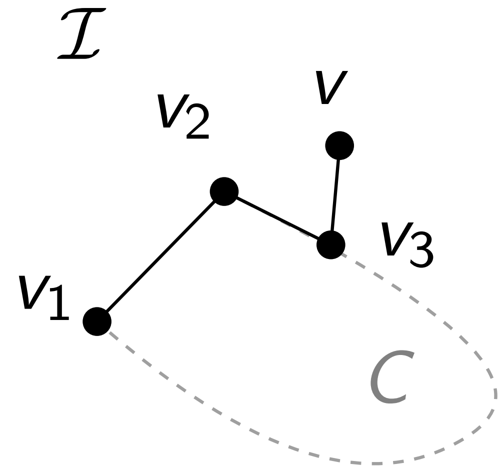

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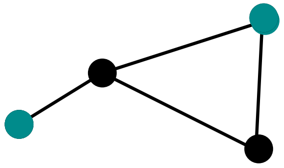

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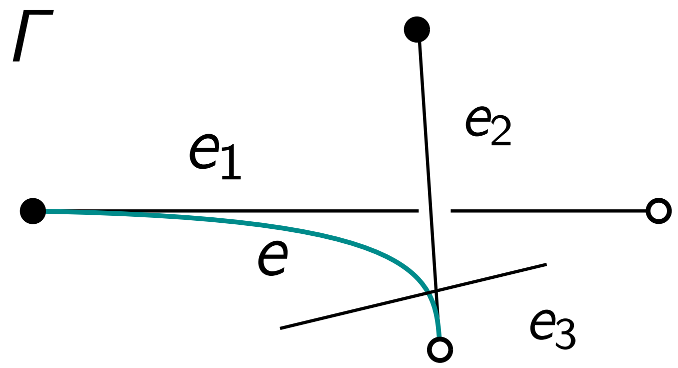

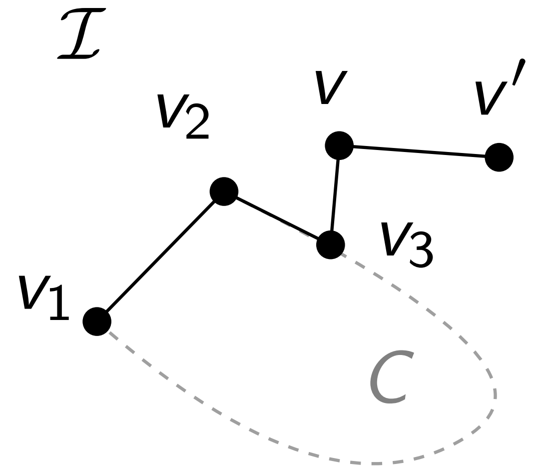

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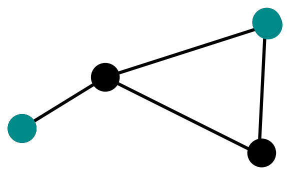

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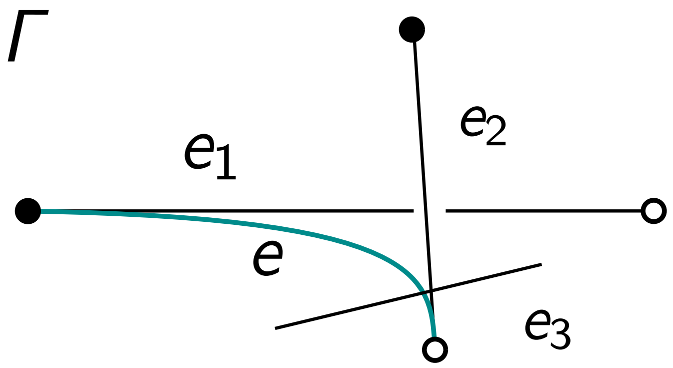

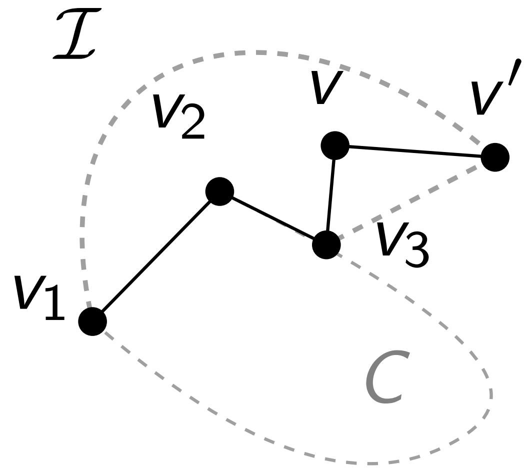

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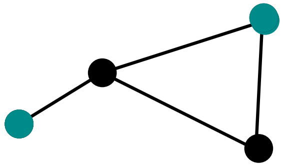

Edge corresponding to v' also crosses e_1 or e_2 $\rightarrow$ X is not pseudoforrest

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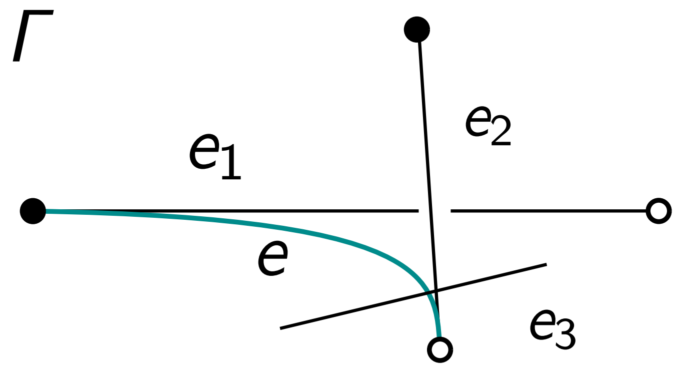

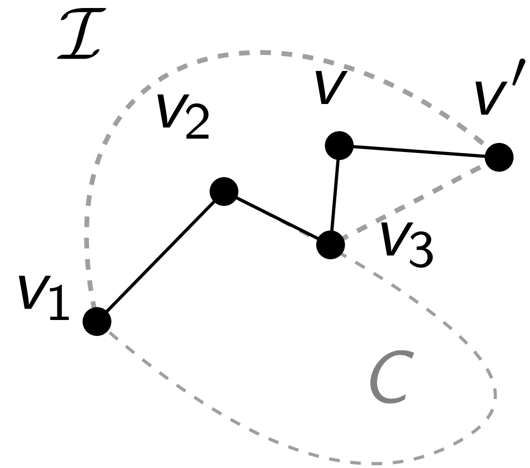

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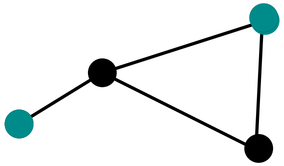

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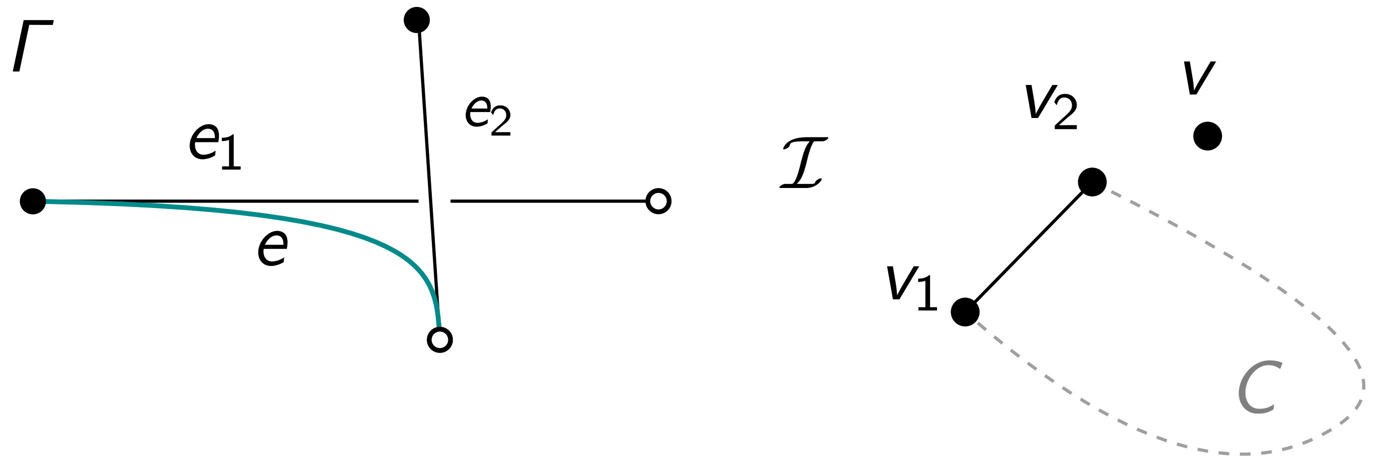
$\Rightarrow$ C1 cannot occur

Part II



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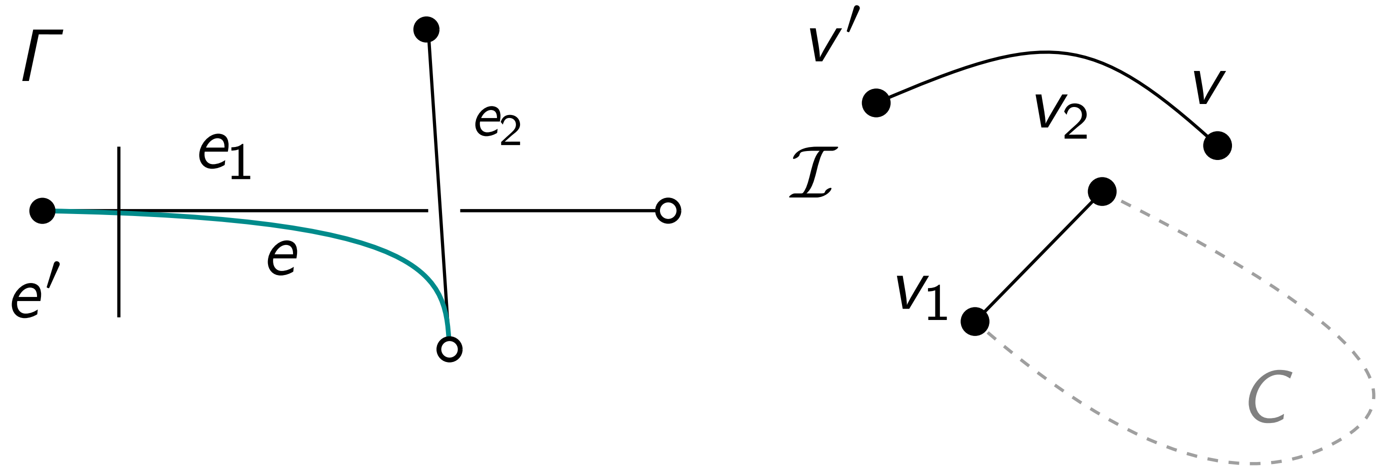

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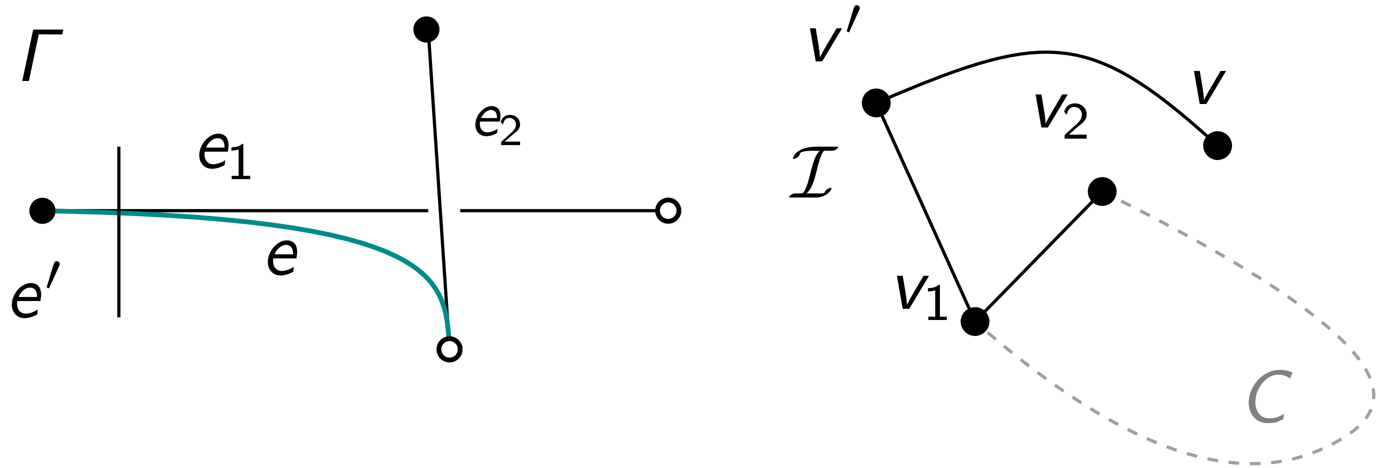

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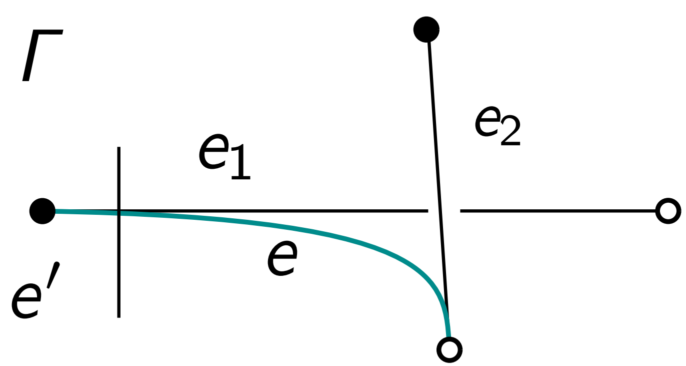

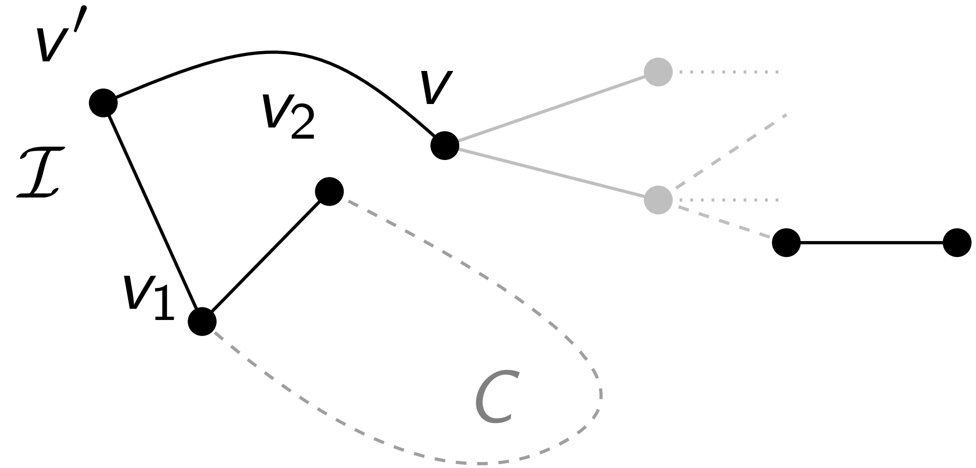

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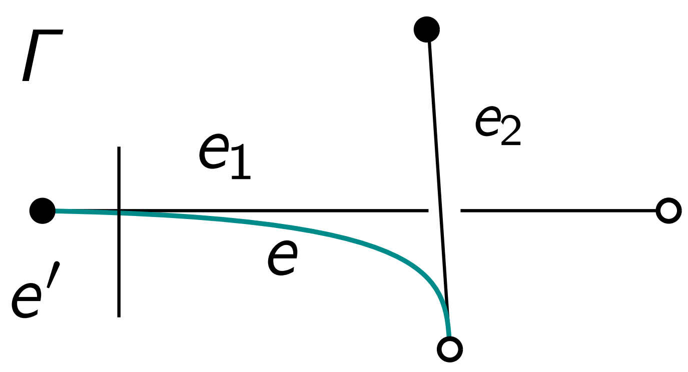

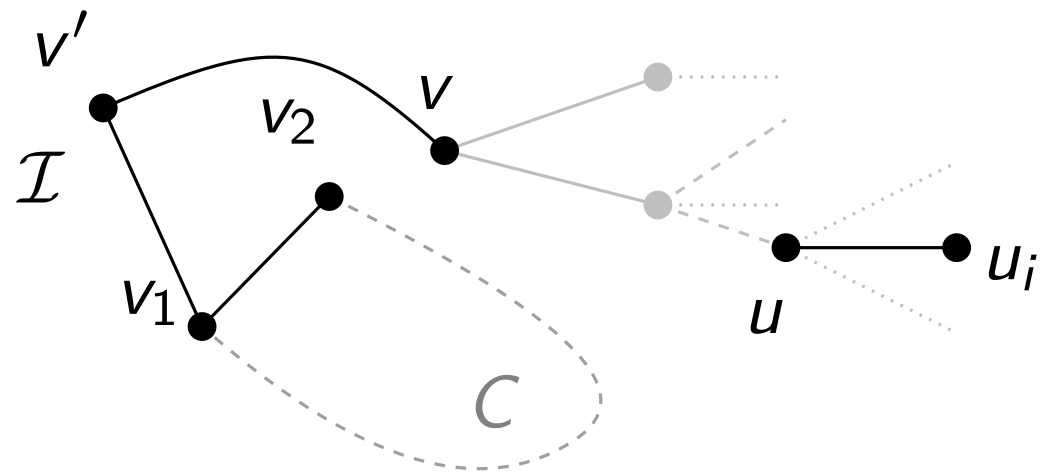

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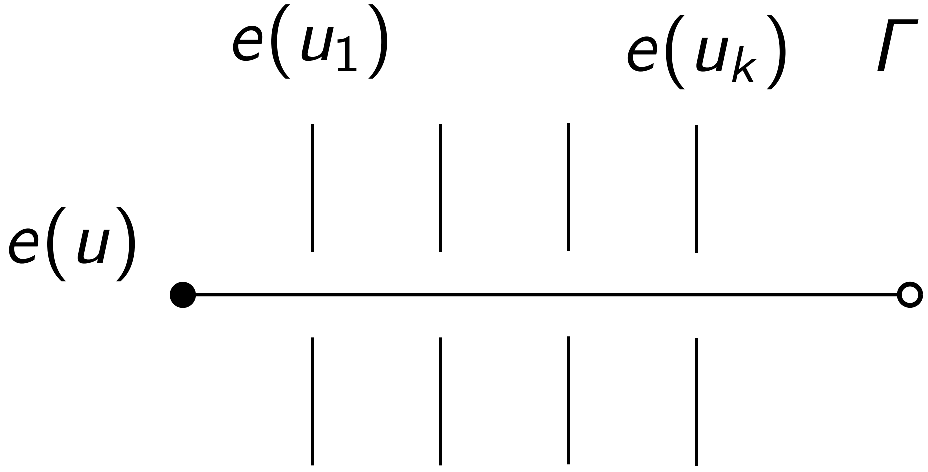
Let u be vertex on second lowest level with children $u_1, \ldots, u_k$.

Part III



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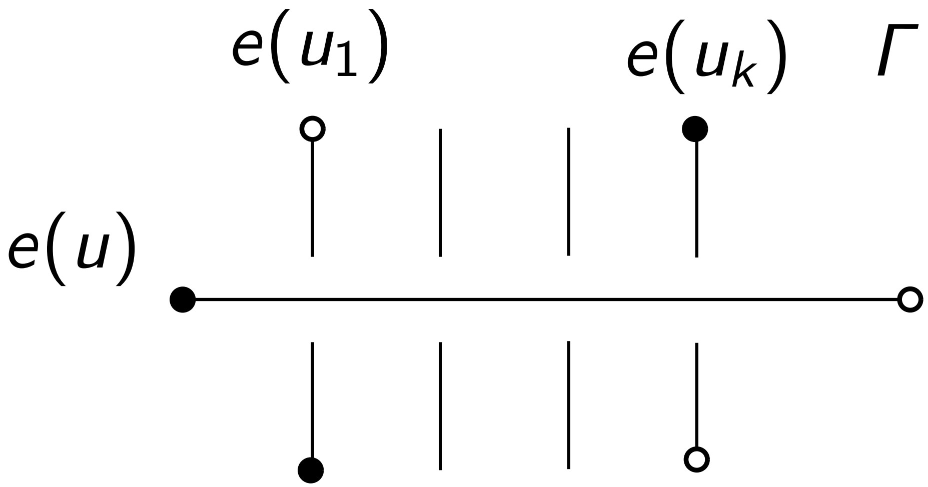

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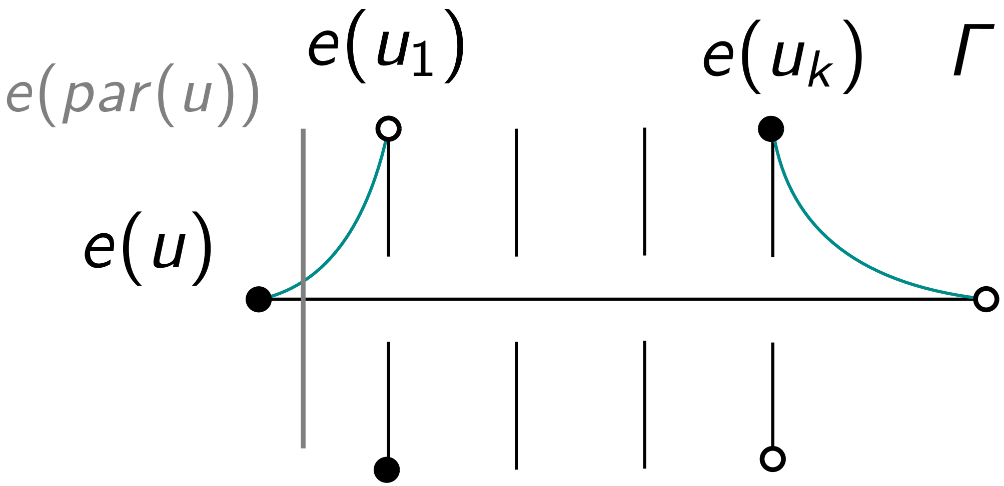

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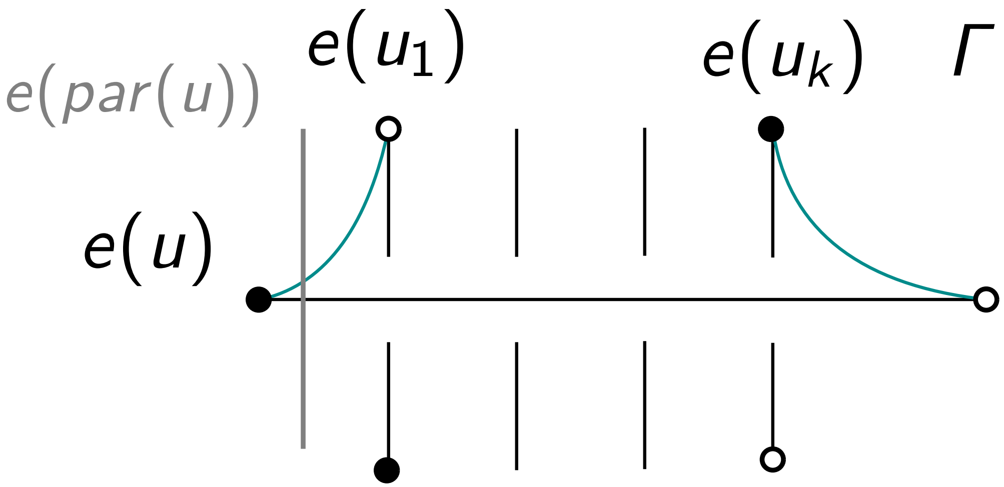

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$\Rightarrow$ In both C2 and C3, we can identify an uncrossed edge for any two consecutive edges of cycle.

Part IV

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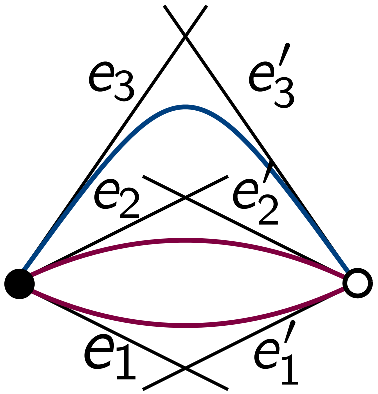

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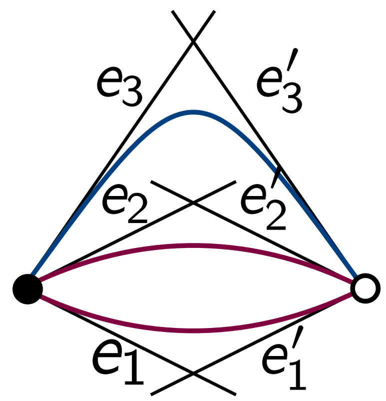

$\Rightarrow$ We have sufficiently many edges such that we can uniquely assign one to every critical component

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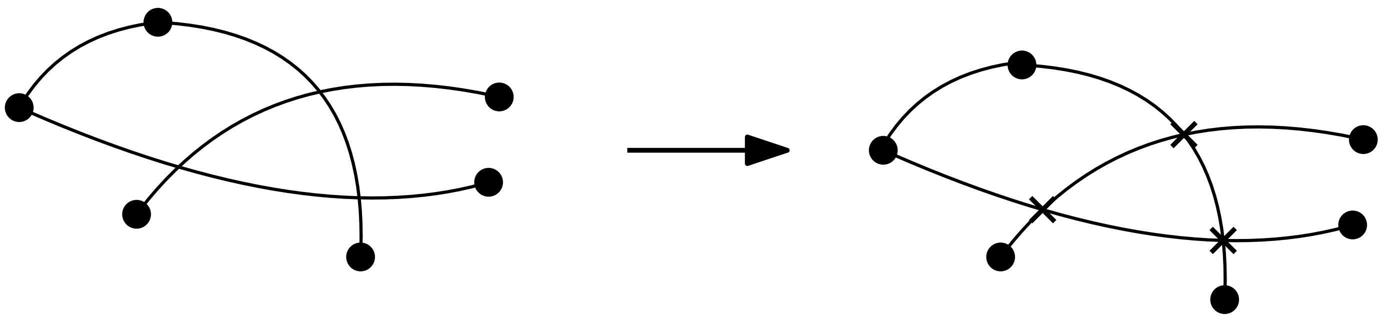

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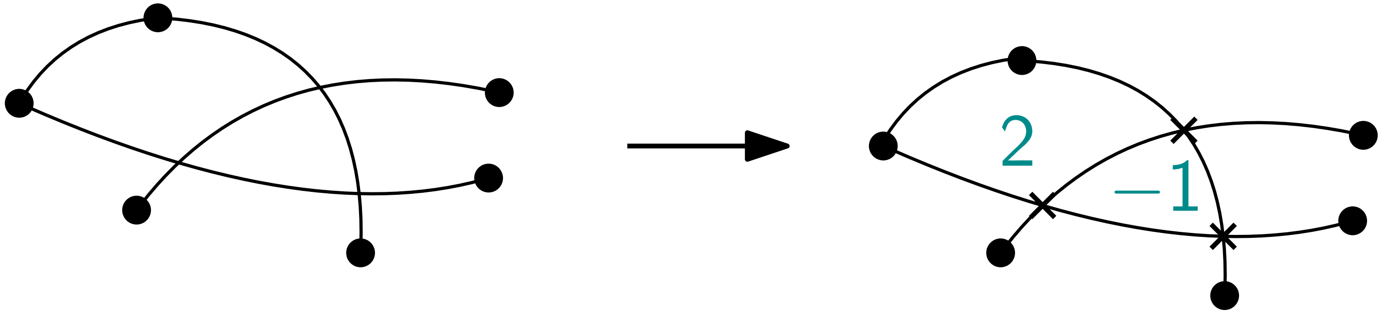

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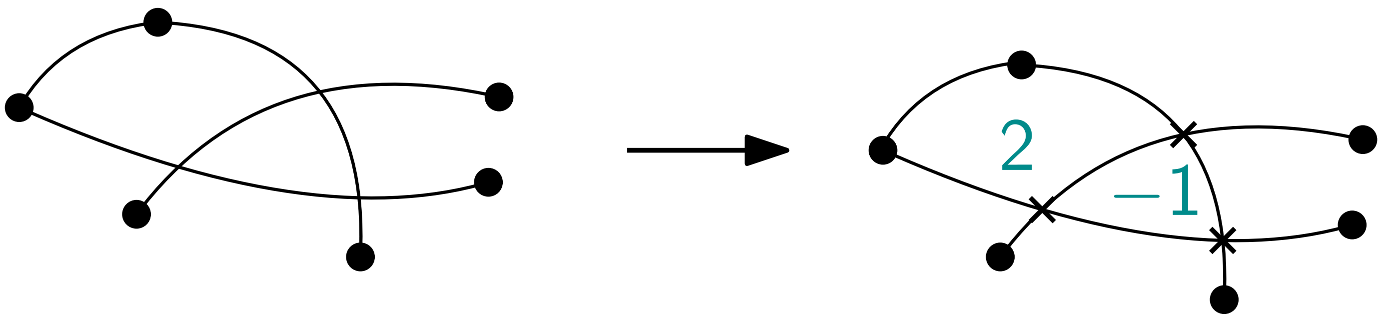

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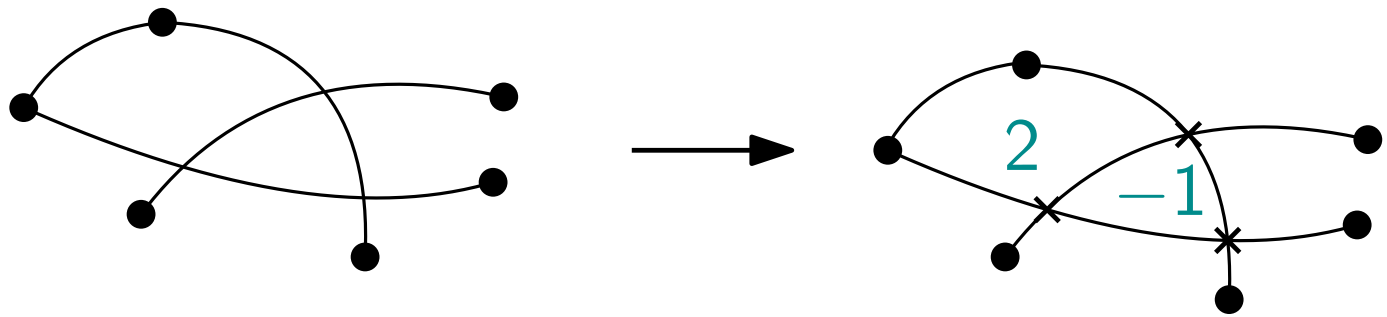

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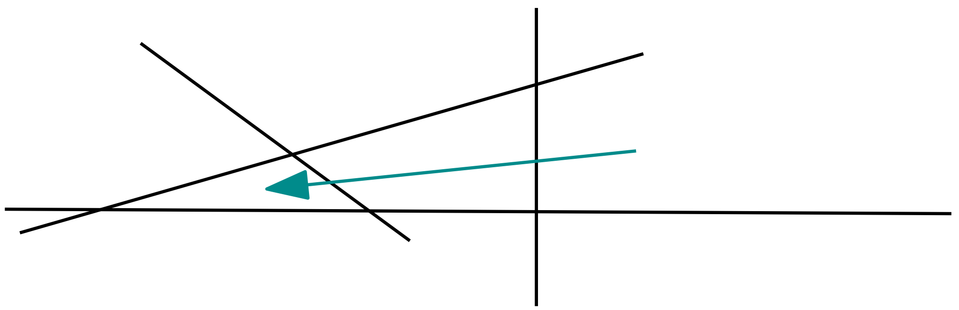
Problematic faces:

$-1 \not\geq 0$ $\quad$ $0 \not\geq 0.5$

Key Ideas

Local charging exchange:



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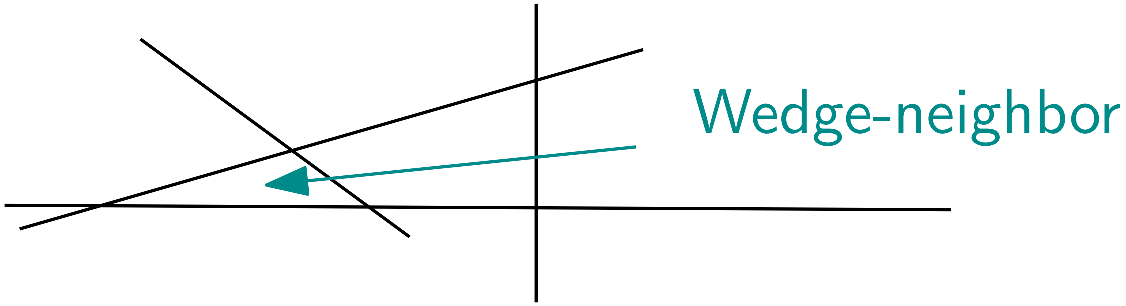

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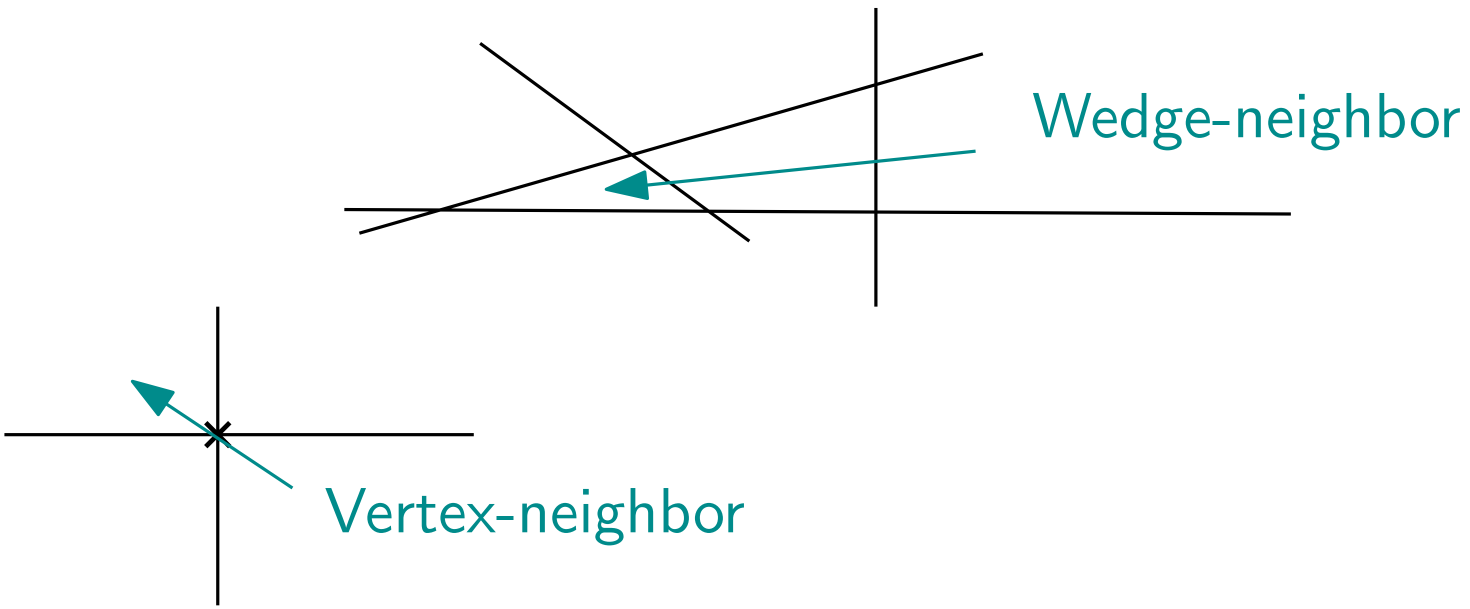

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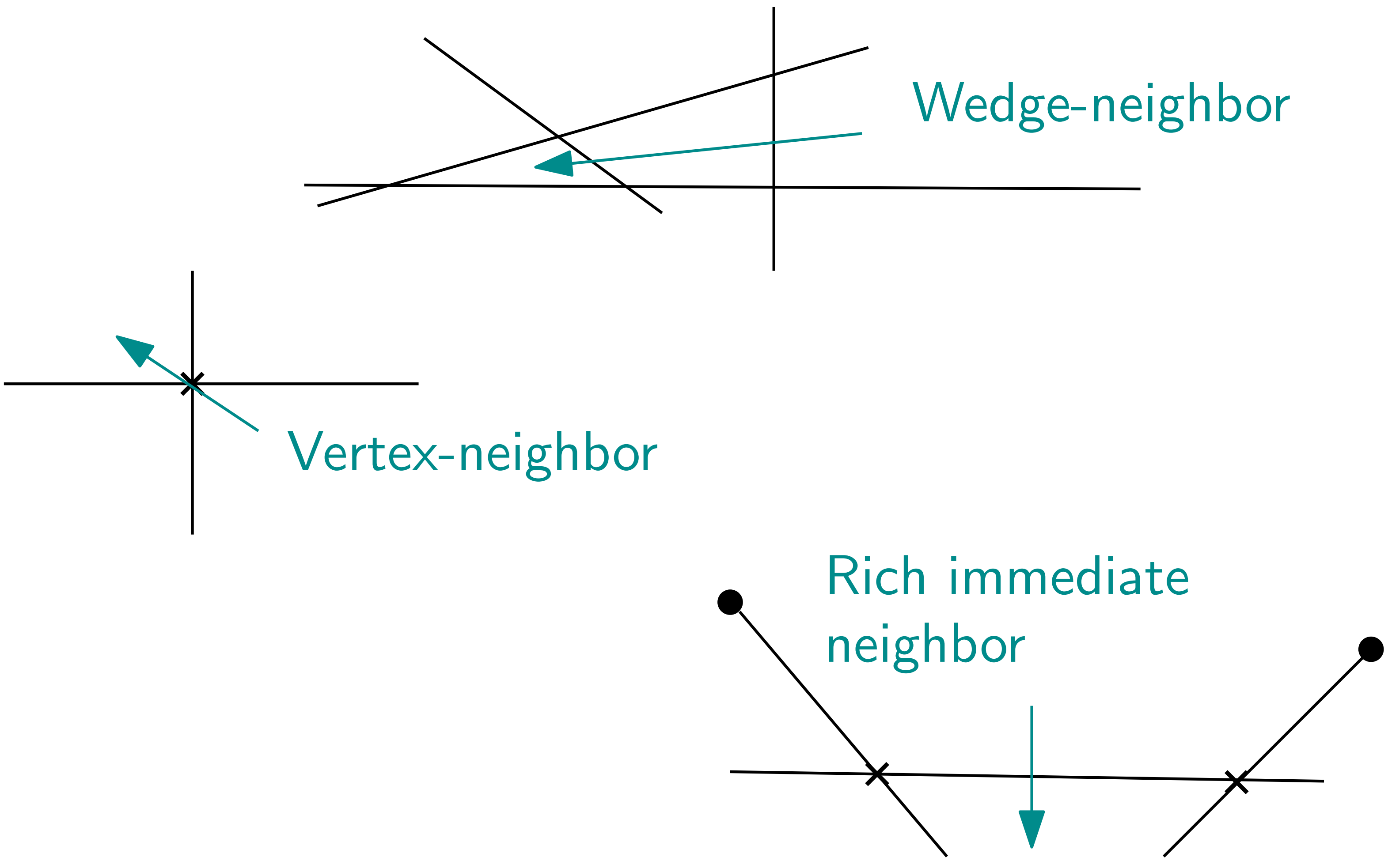

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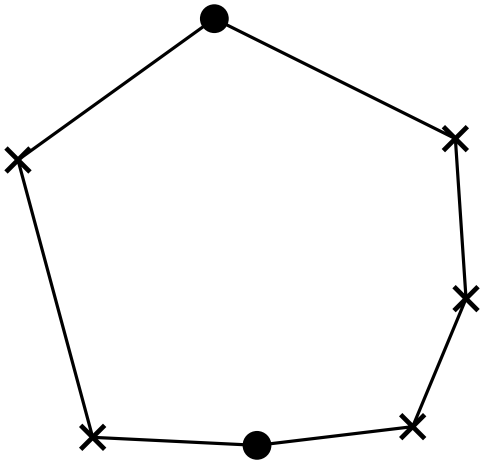

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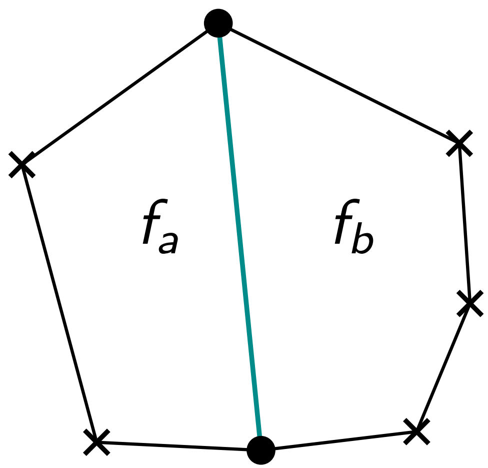

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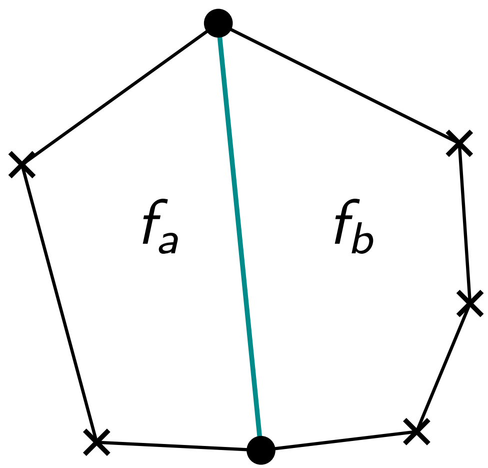

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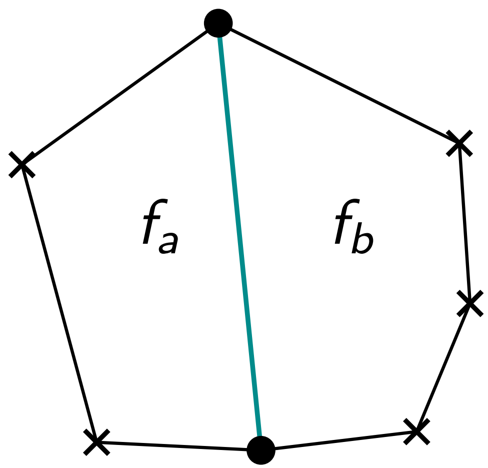

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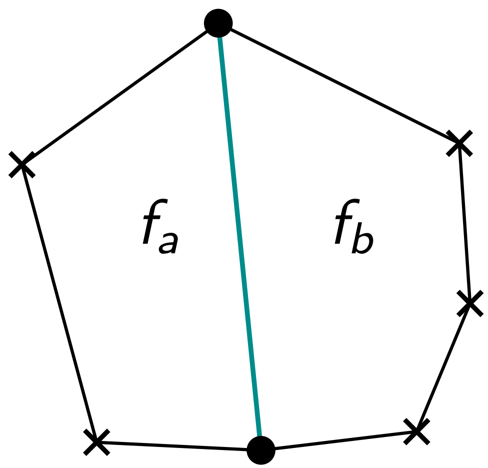

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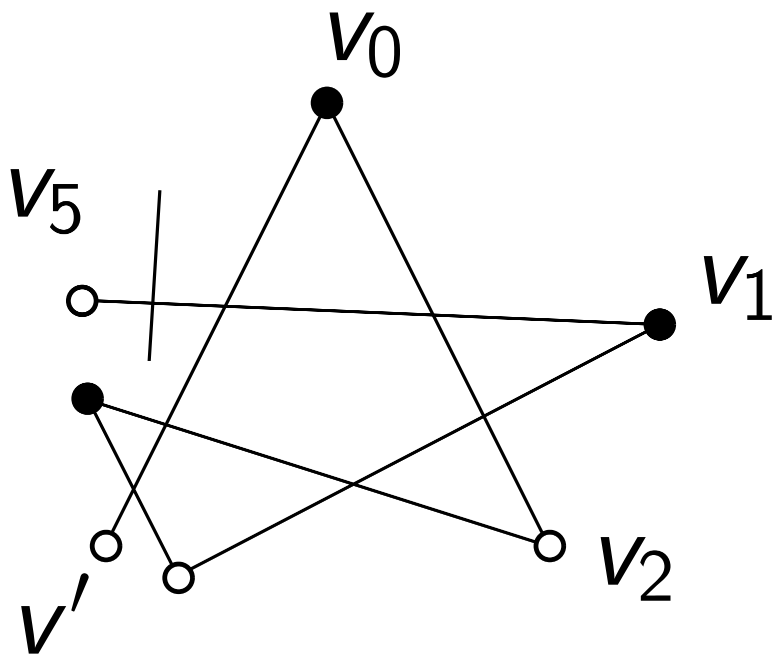
Can redistribute this local charge w.o. arguing about the remainder of a face

Example

At some point in our charging analysis we arrive at such a case:



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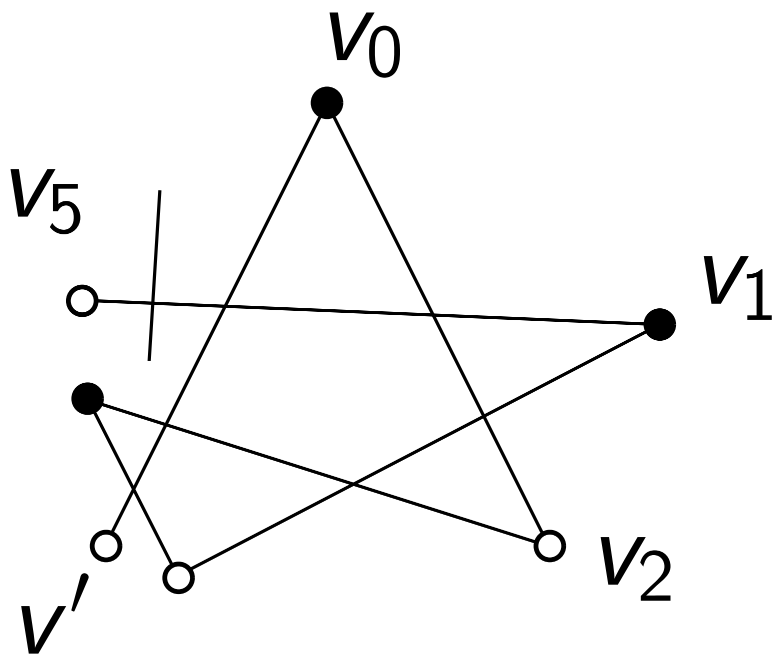

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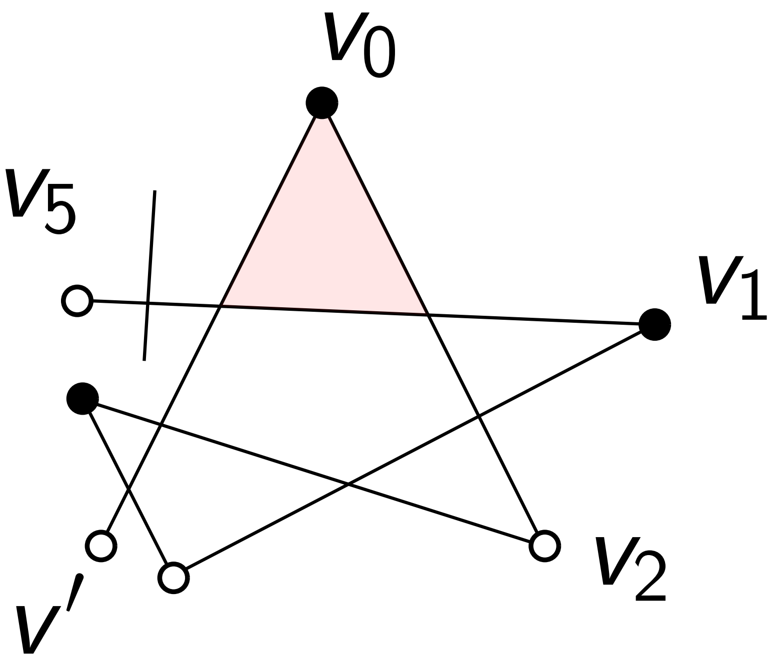

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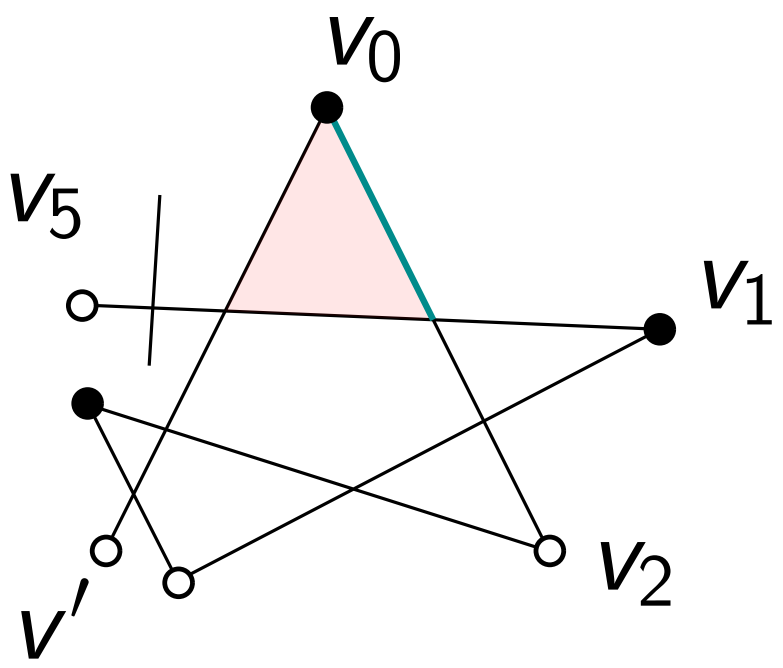

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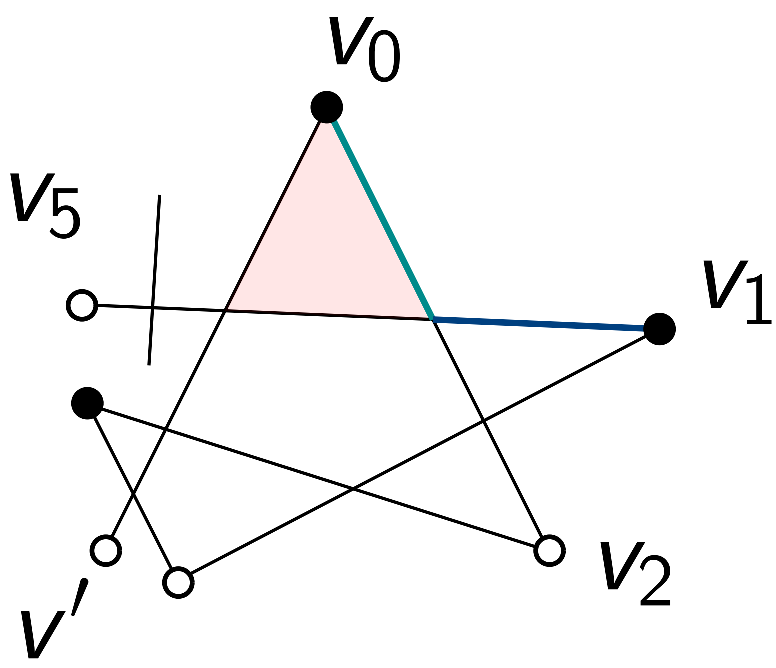

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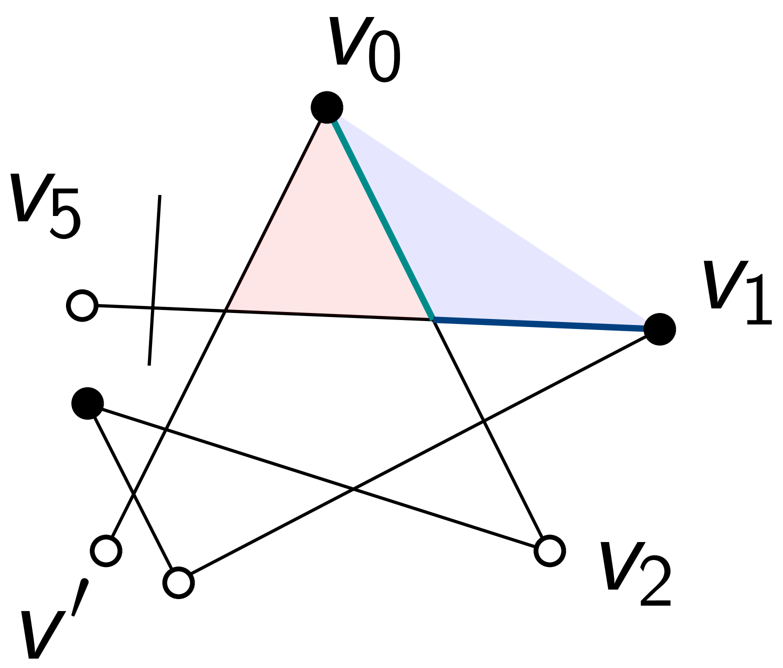

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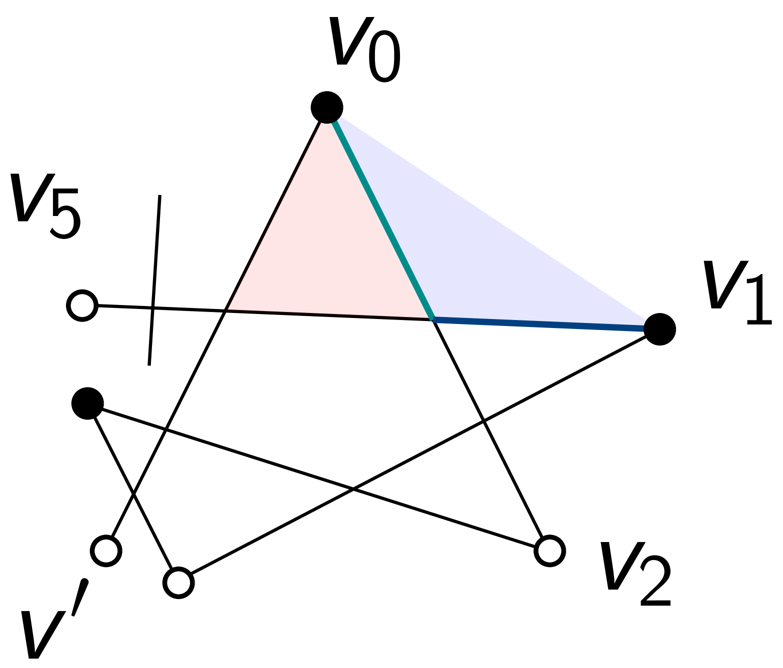

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Subdivide this face along the edge $(v_0, v_1) \rightarrow$ we obtain a 2-triangle which can distribute 0.5 charge to the pentagon.

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