# On the Uncrossed Number of Graphs

Martin Balko, Petr Hliněný, Tomáš Masařík,
Joachim Orthaber, Birgit Vogtenhuber, Mirko H. Wagner

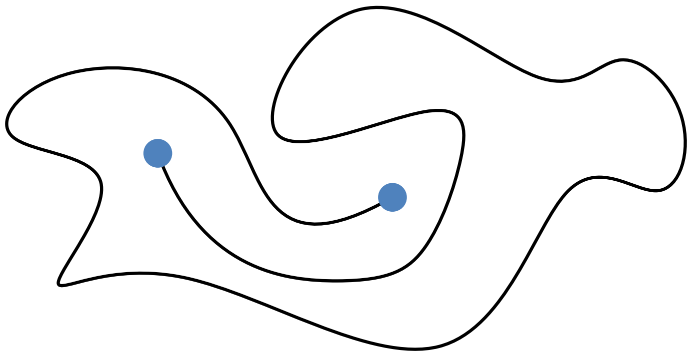
September 18, 2024

# Drawings of Graphs

**Drawing** $D$ of a graph $G$ in the plane:

*vertices*: pairwise distinct points

*edges*: Jordan arcs connecting end-vertices



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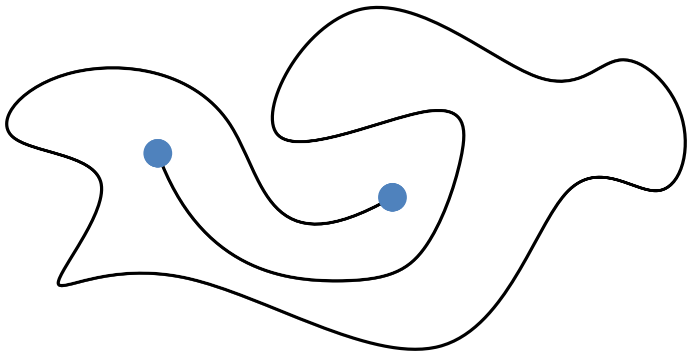

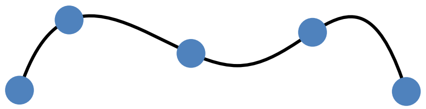

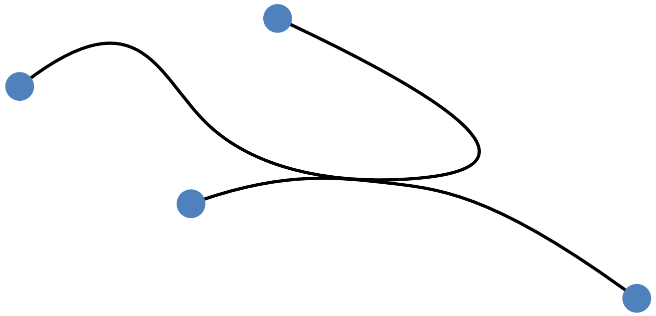

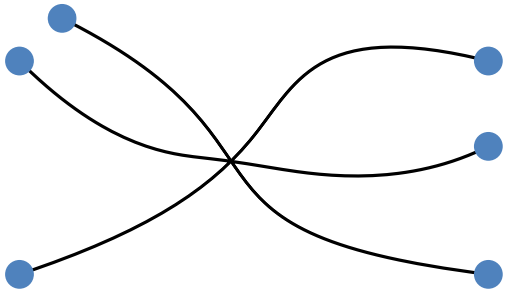

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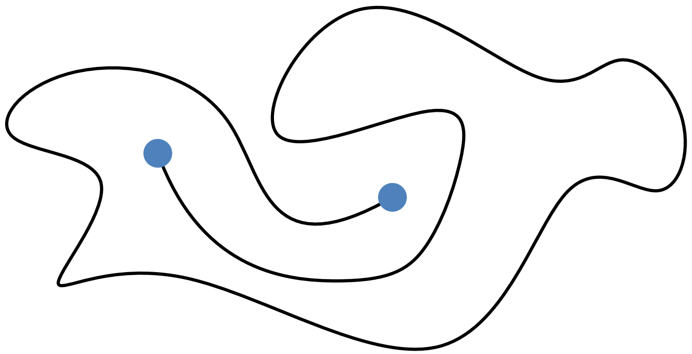

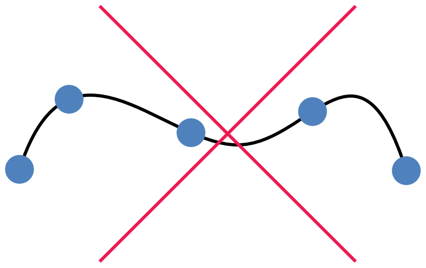

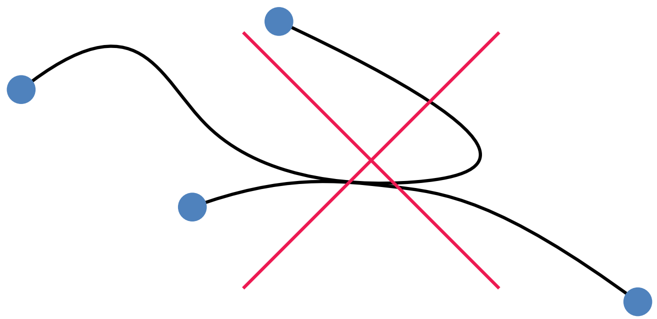

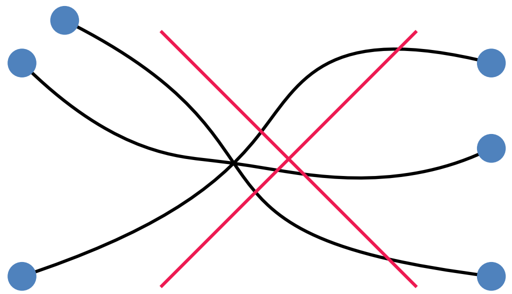

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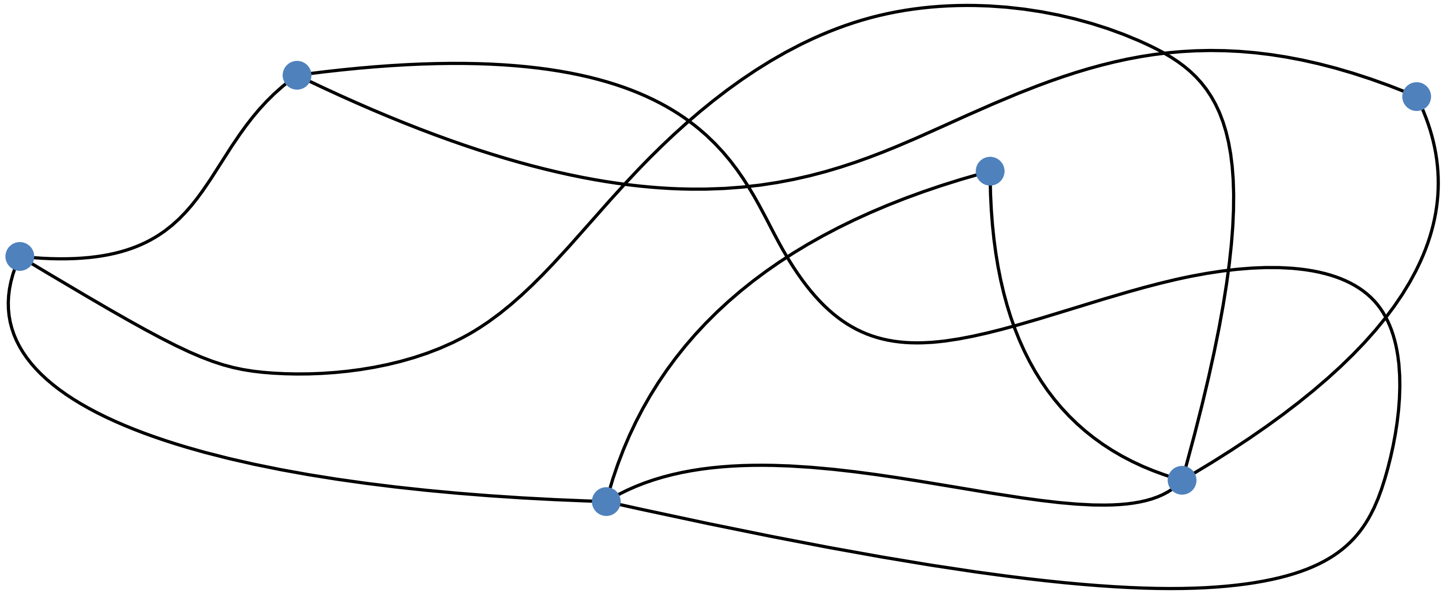

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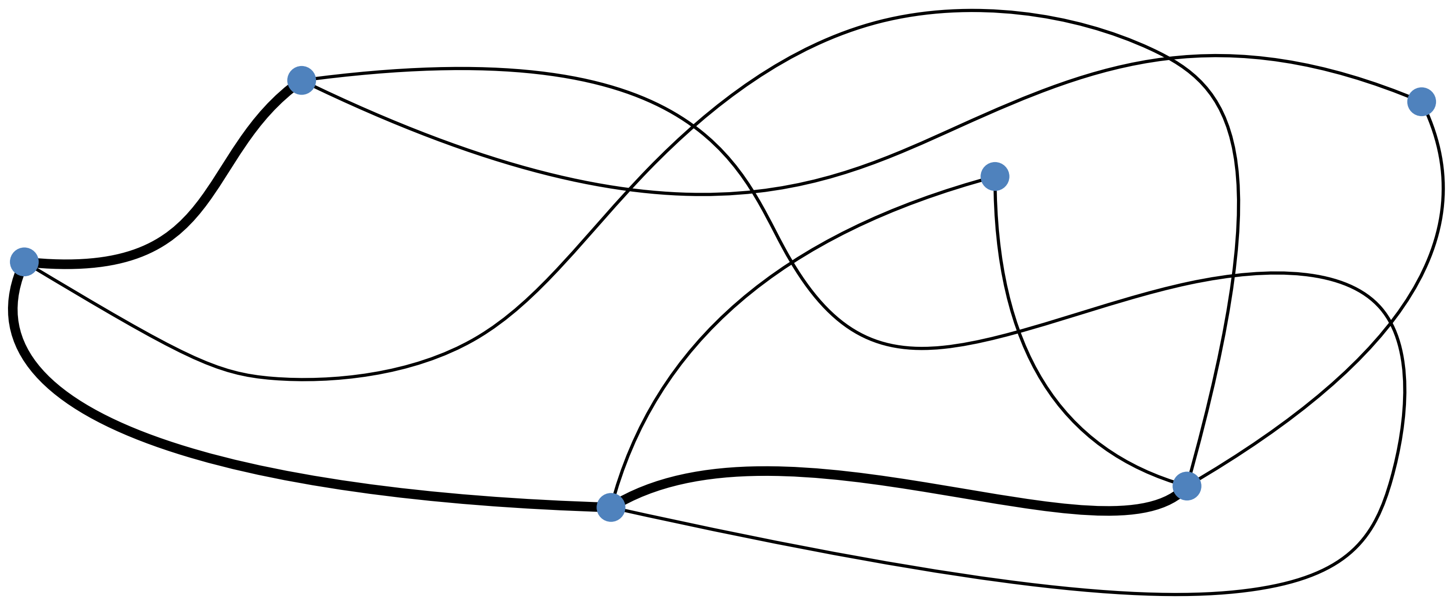

*uncrossed edge*: has no crossings

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**Plane drawing** $D$: all edges are uncrossed.

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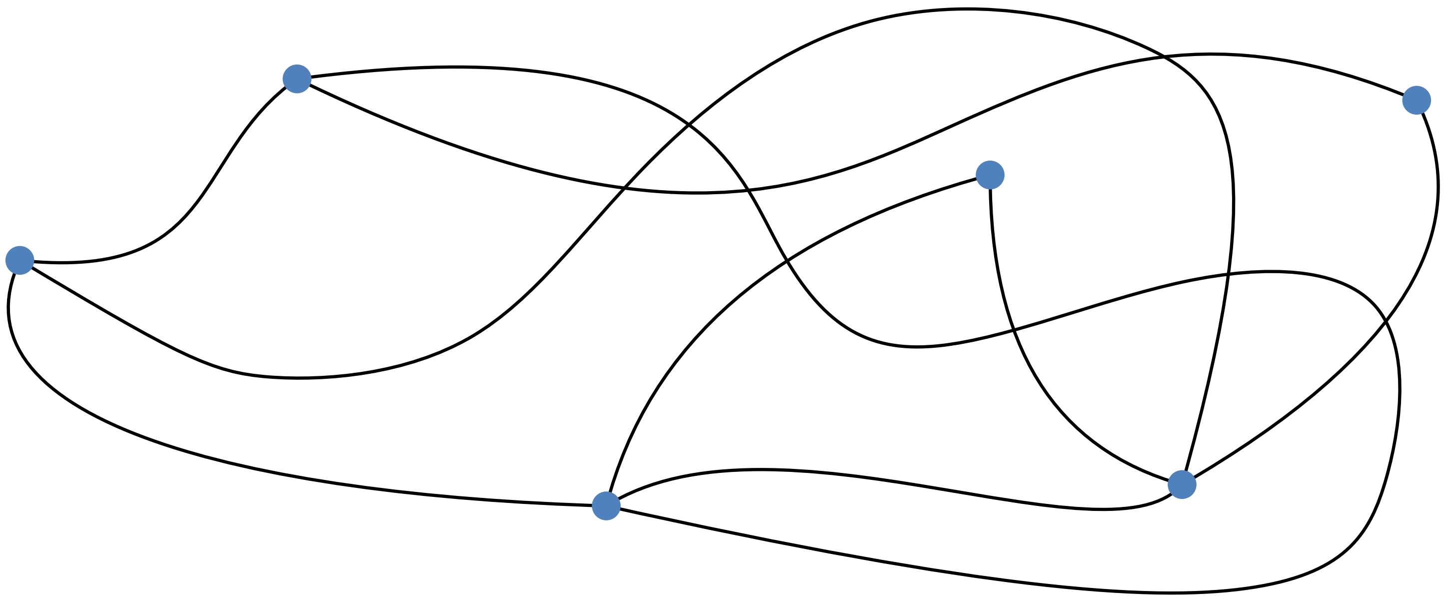

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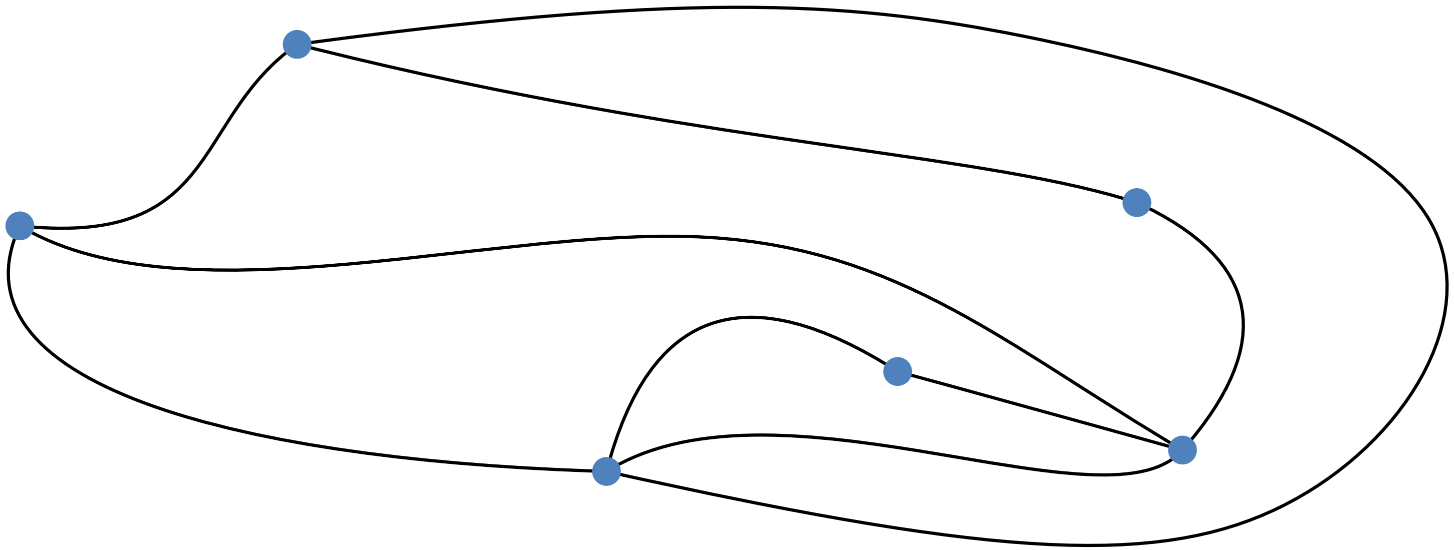

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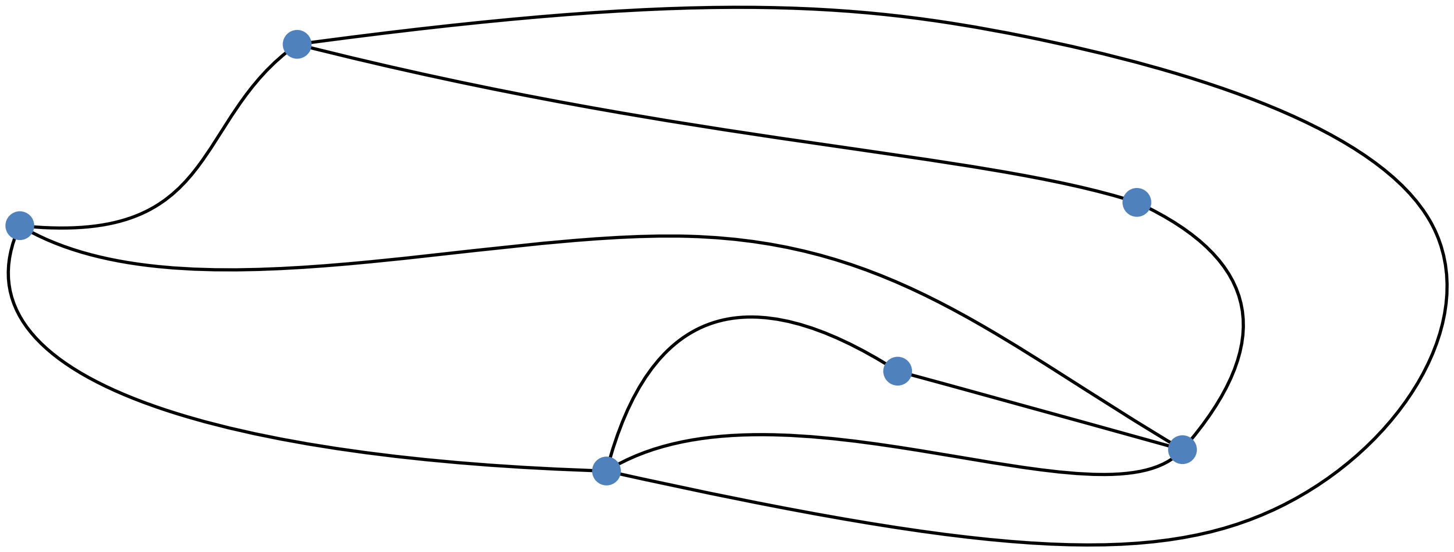

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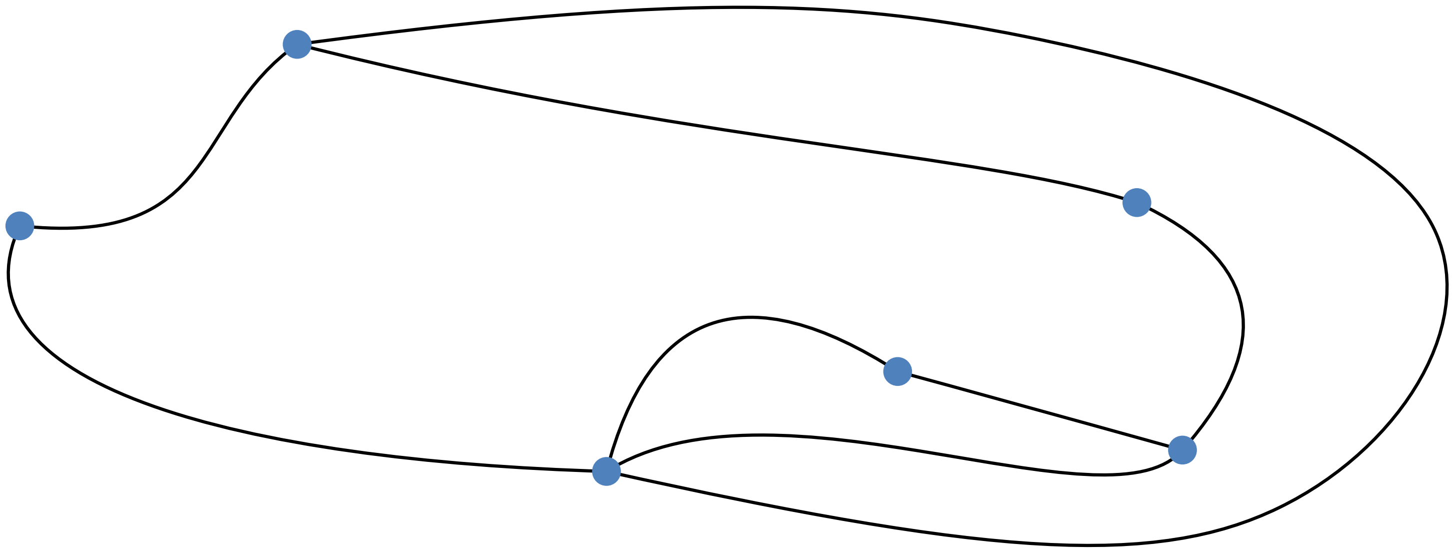

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Ideal: *All graphs are planar*

Reality: *Most graphs are non-planar*

Obvious questions: *How to draw/visualize non-planar graphs?*
*How to measure non-planarity?*

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*draw $G$ with few crossings*

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*all edges uncrossed*
*no drawing of the whole graph*

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$\theta_o(K_n)$ and $\theta_o(K_{n,m})$ known [Guy, Nowakowski, 1990]

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maybe many crossed edges
drawing of the whole graph

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*different drawings of the whole graph*
*every edge uncrossed in at least one drawing*

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*draw few copies of $G$ such that every edge is uncrossed in some copy*

**Uncrossed Number** $\operatorname{unc}(G)$ of a graph $G$:

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complete graph $K_7$: $\mathrm{unc}(K_7) = \theta_o(K_7) = 3$

Conjectures:
1. $\mathrm{unc}(K_n) = \theta_o(K_n)$ if $n \neq 4$
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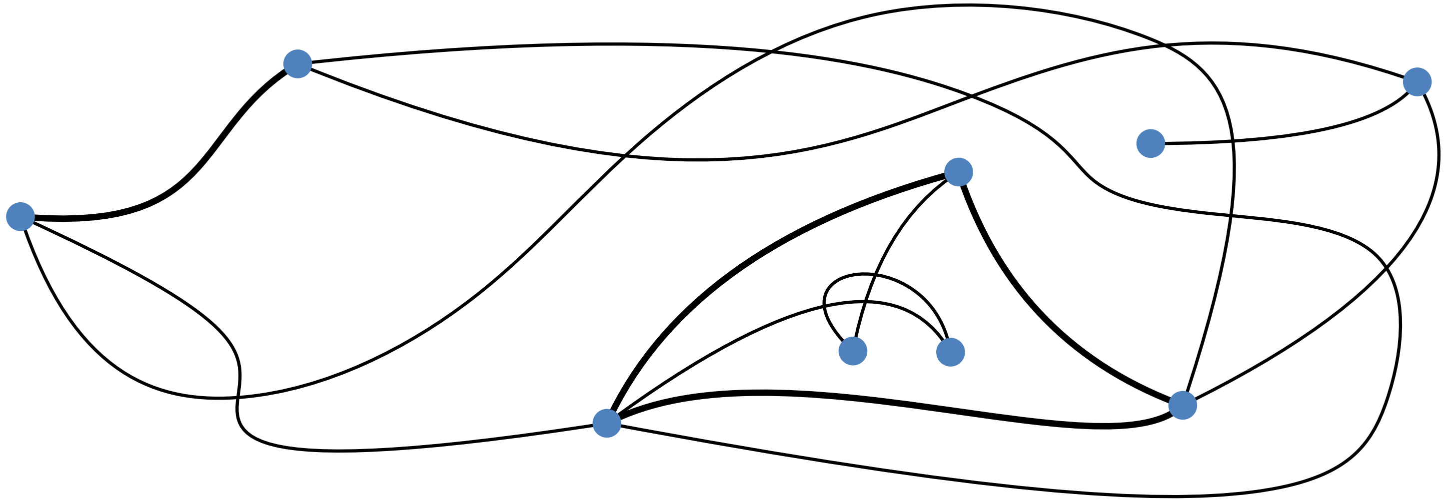
*our starting point for this work*

# Uncrossed Subdrawings

*uncrossed subdrawing* $D'$ *of a drawing* $D$ *of a graph* $G$:

(sub)set of uncrossed edges of $D$, all vertices of $D$

*represents* a plane subgraph of $G$



*uncrossed subdrawing* $D'$ *of a graph* $G$:

$D'$ uncrossed in some drawing $D$ of $G$

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**Lemma.** Let $D'$ be an uncrossed subdrawing of a connected graph $G$.

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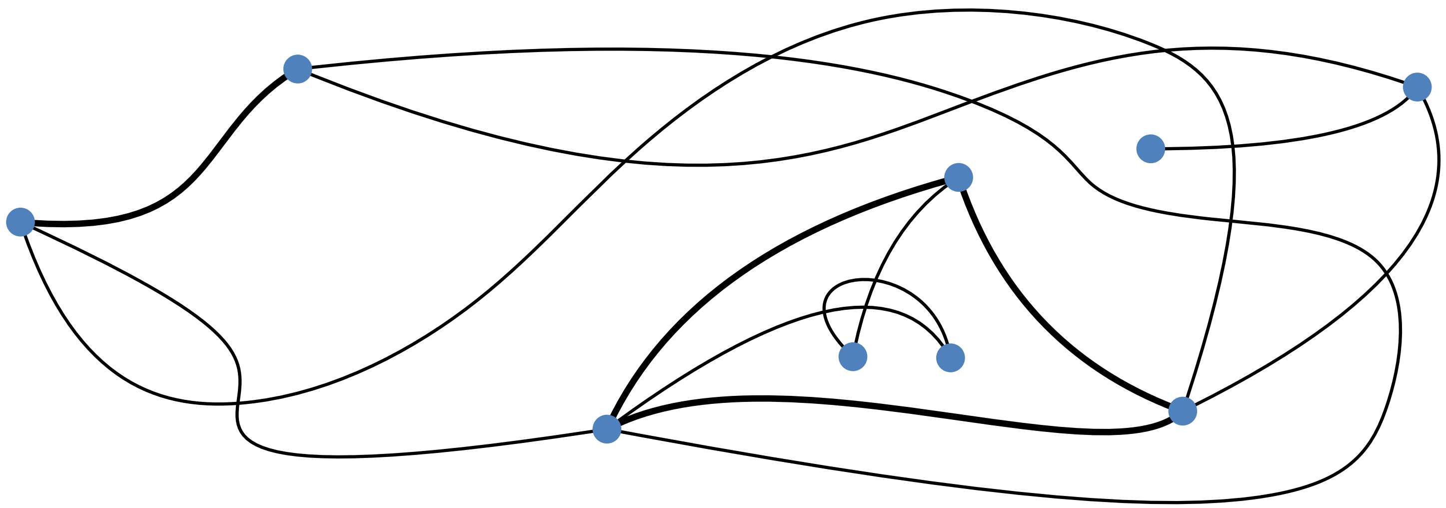

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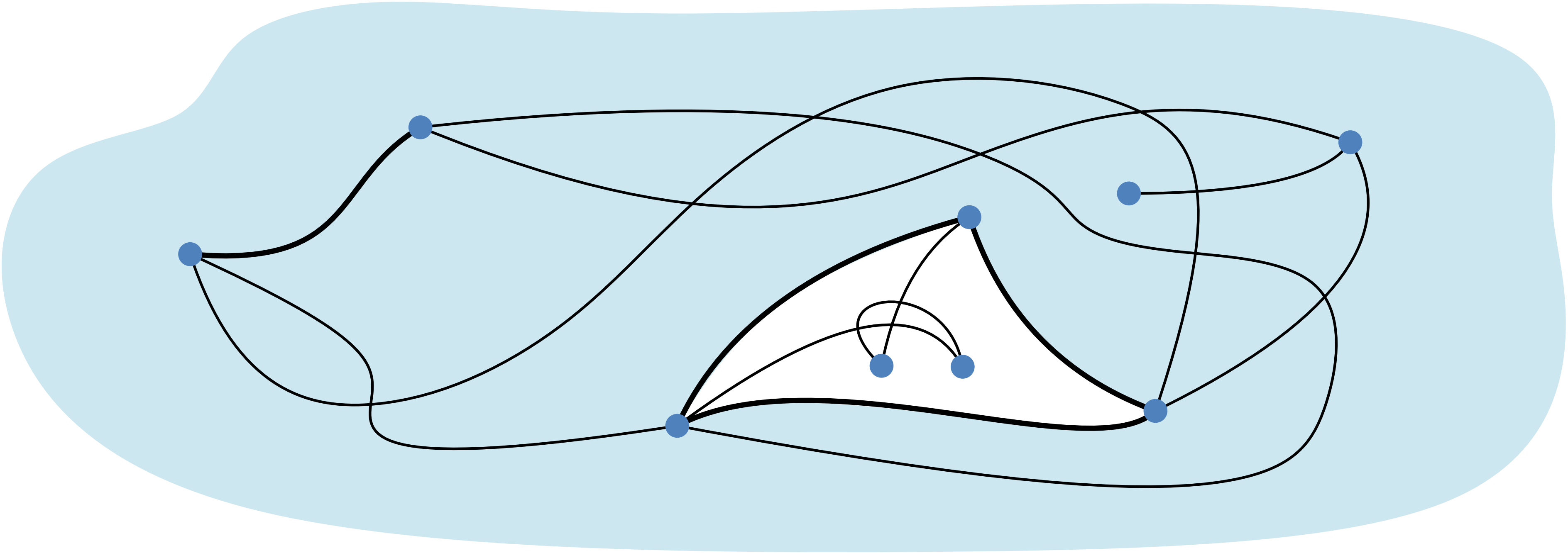

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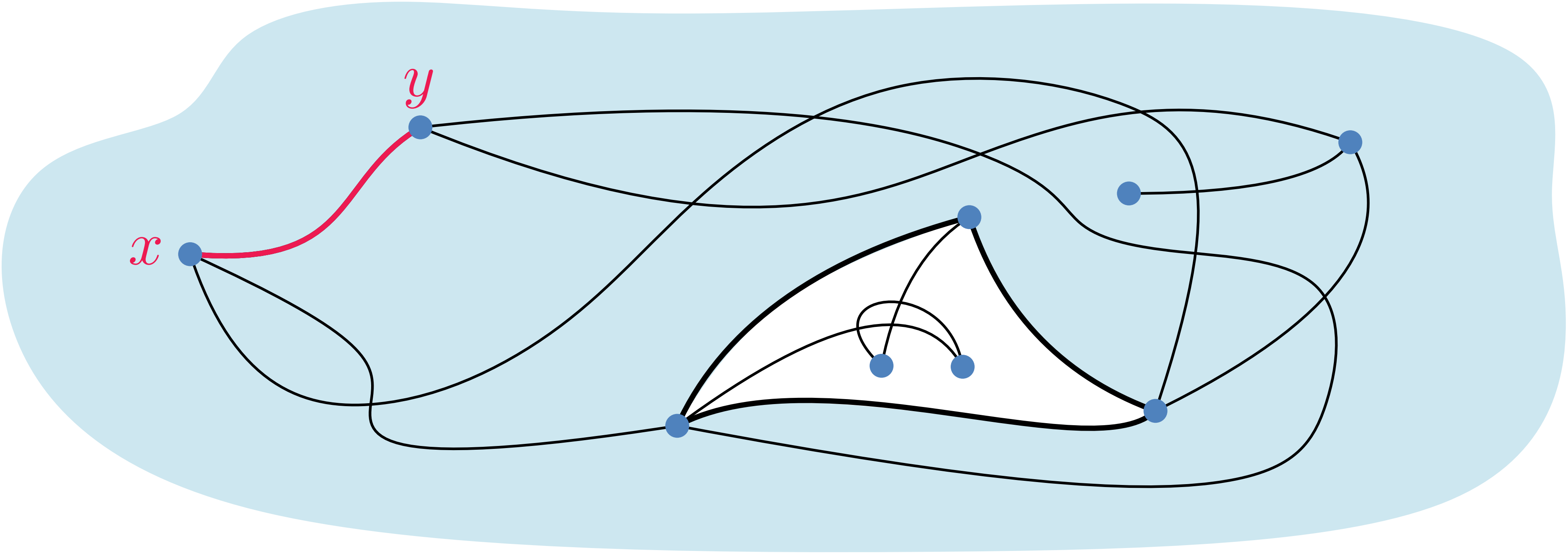

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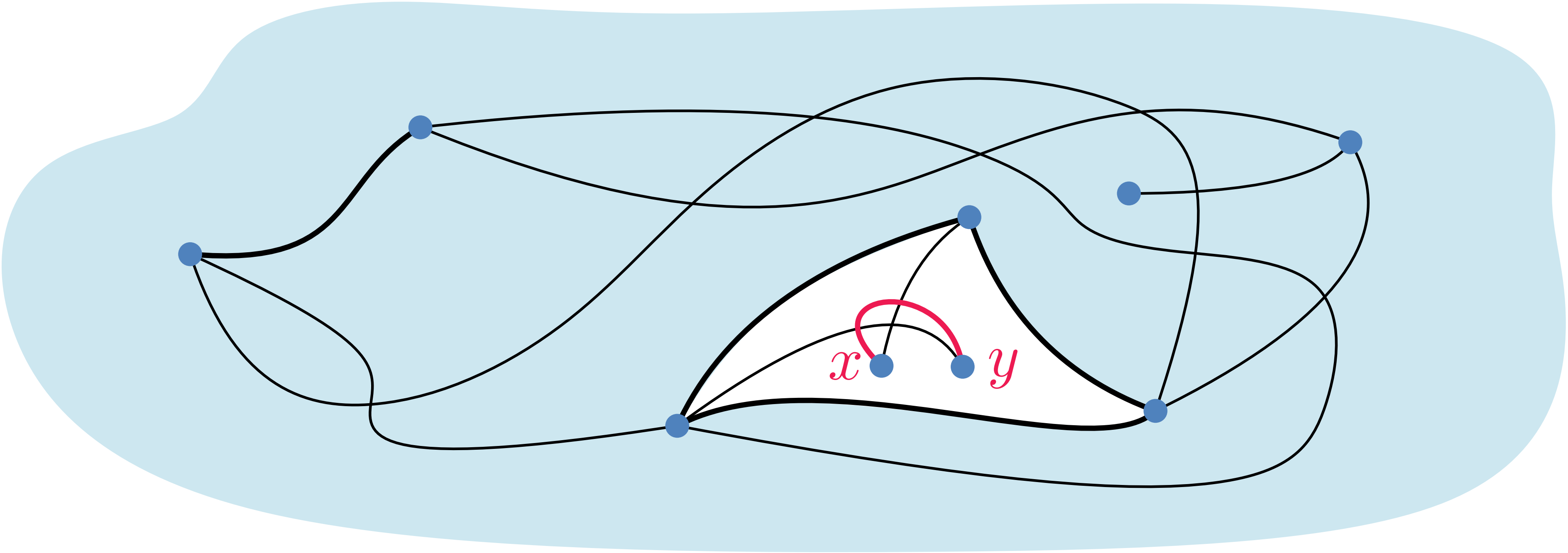


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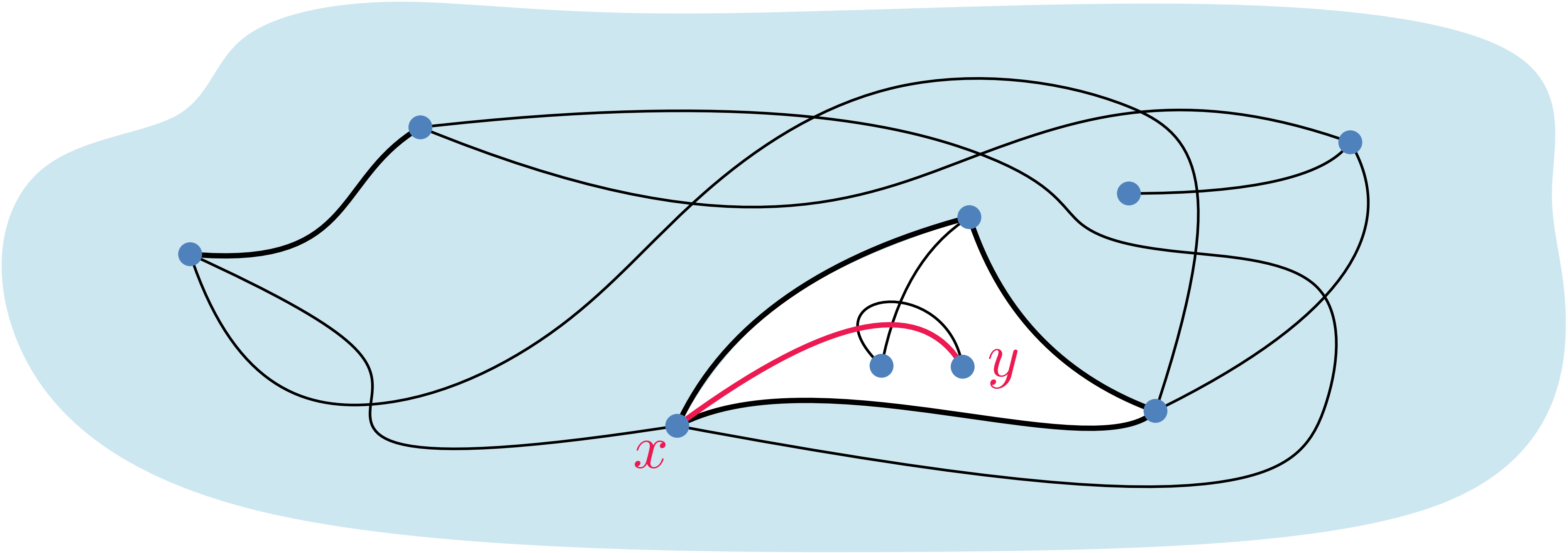


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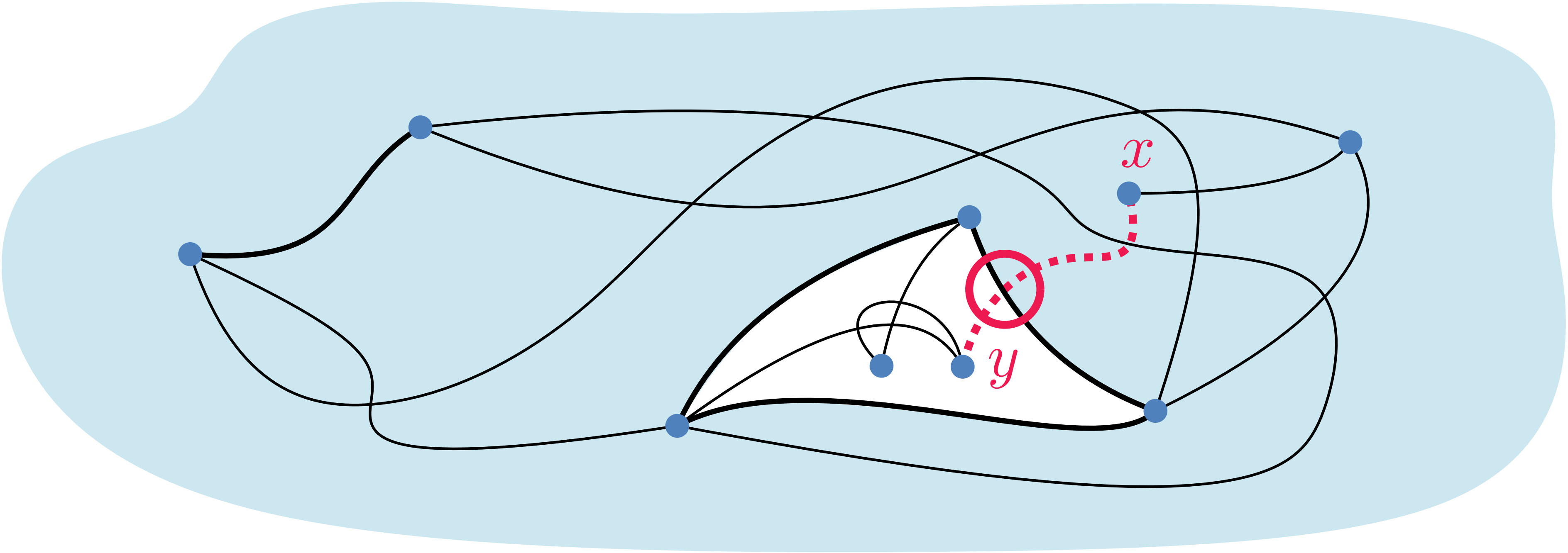


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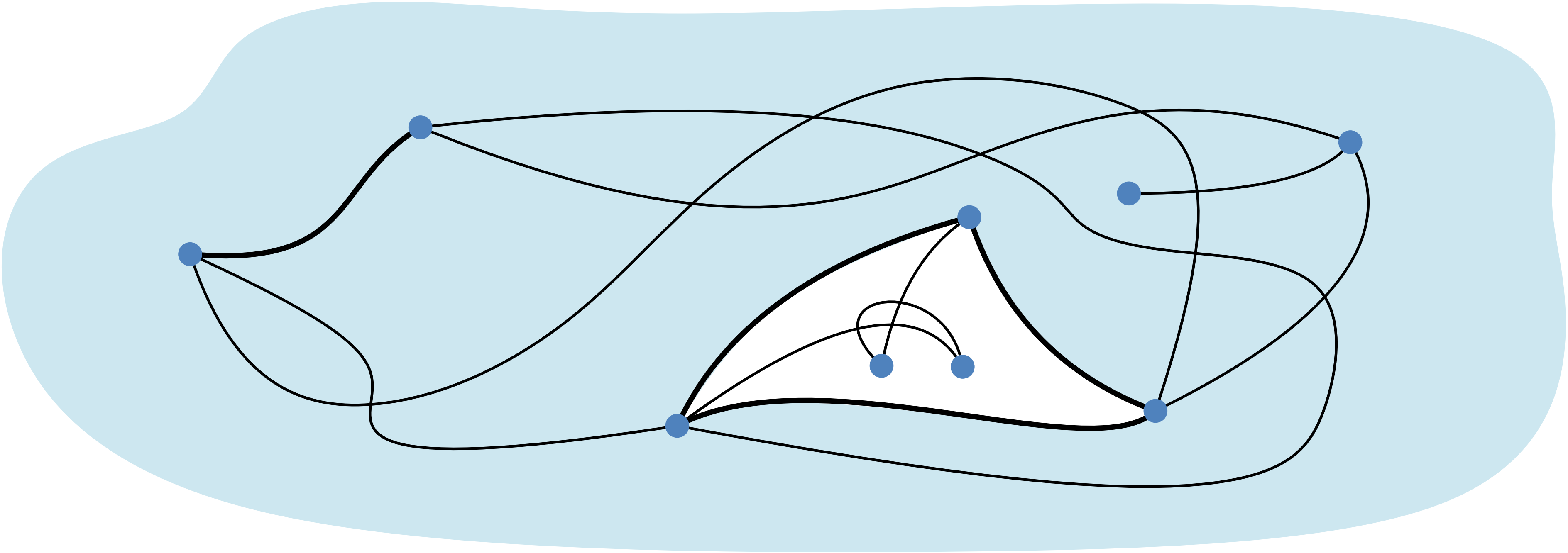

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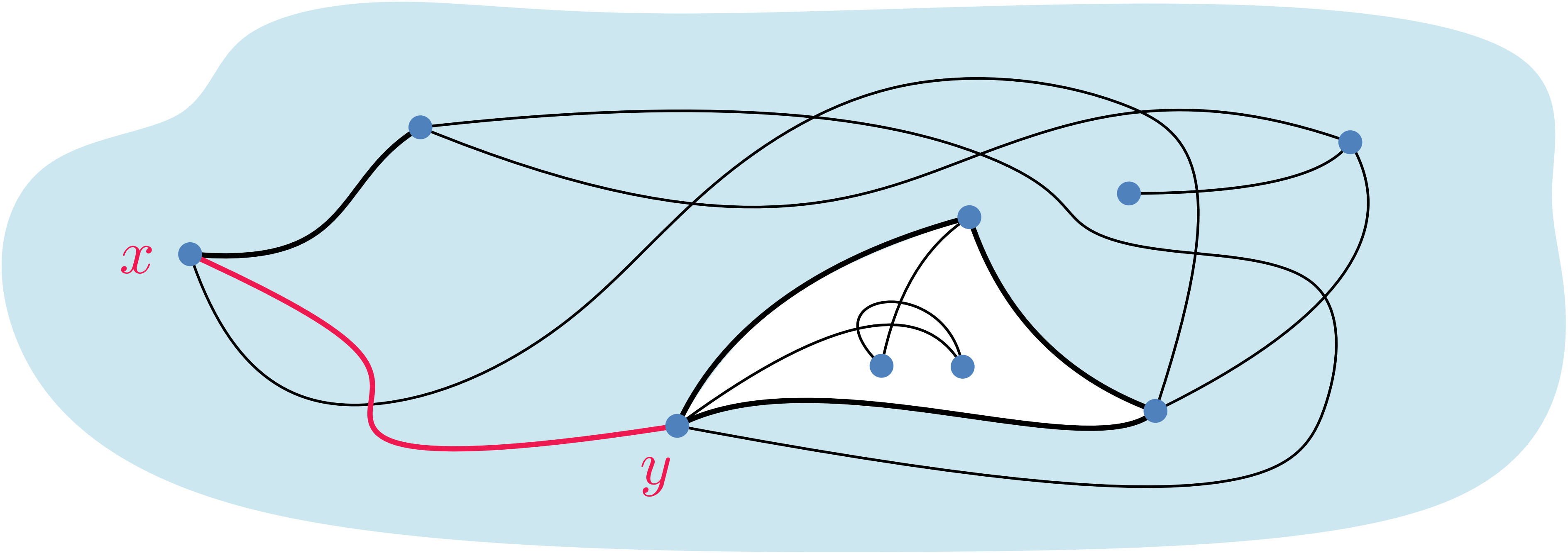


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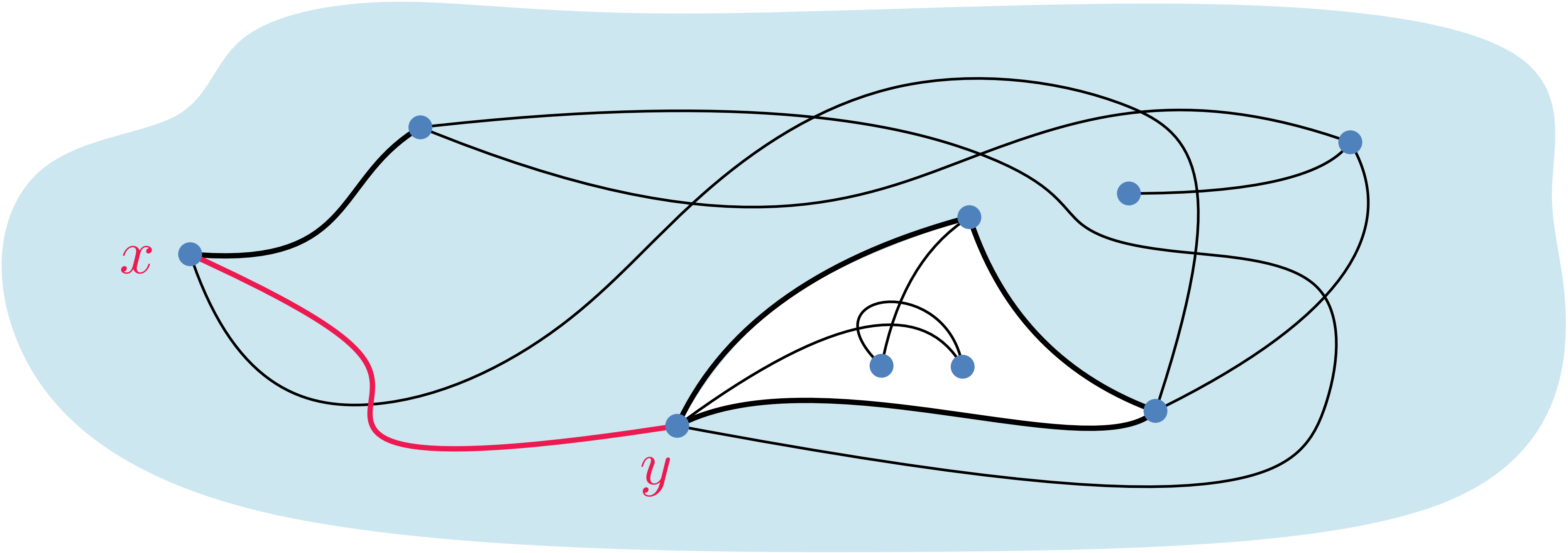


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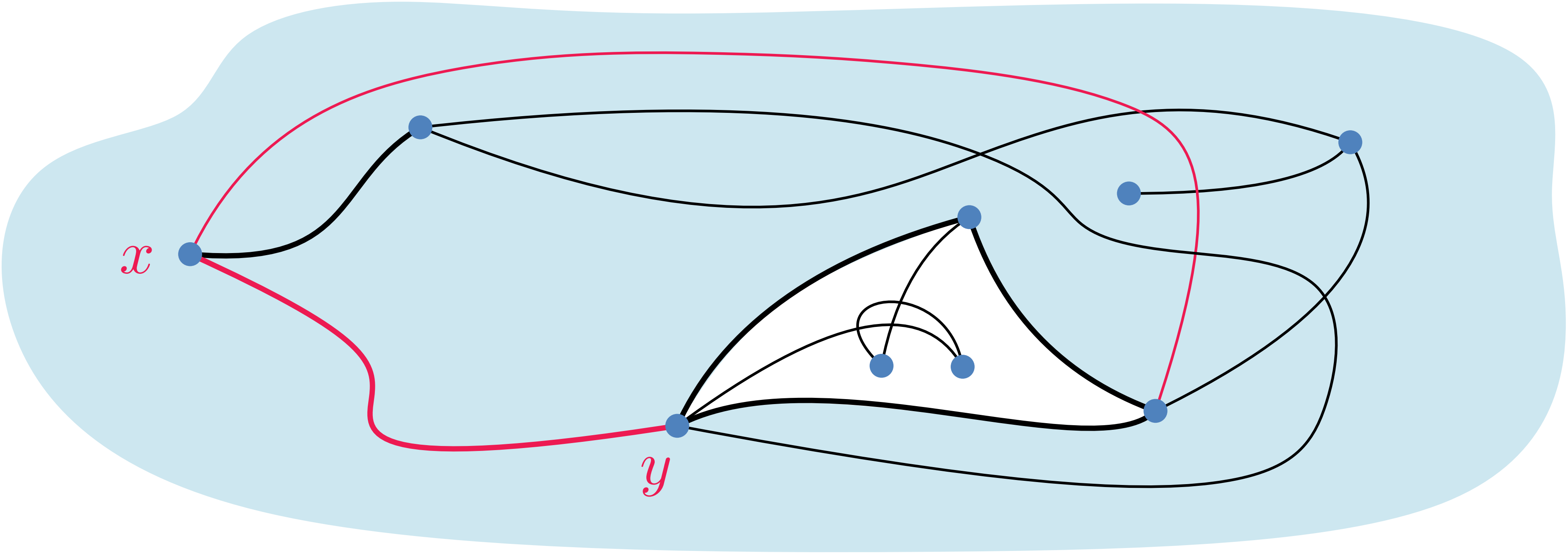


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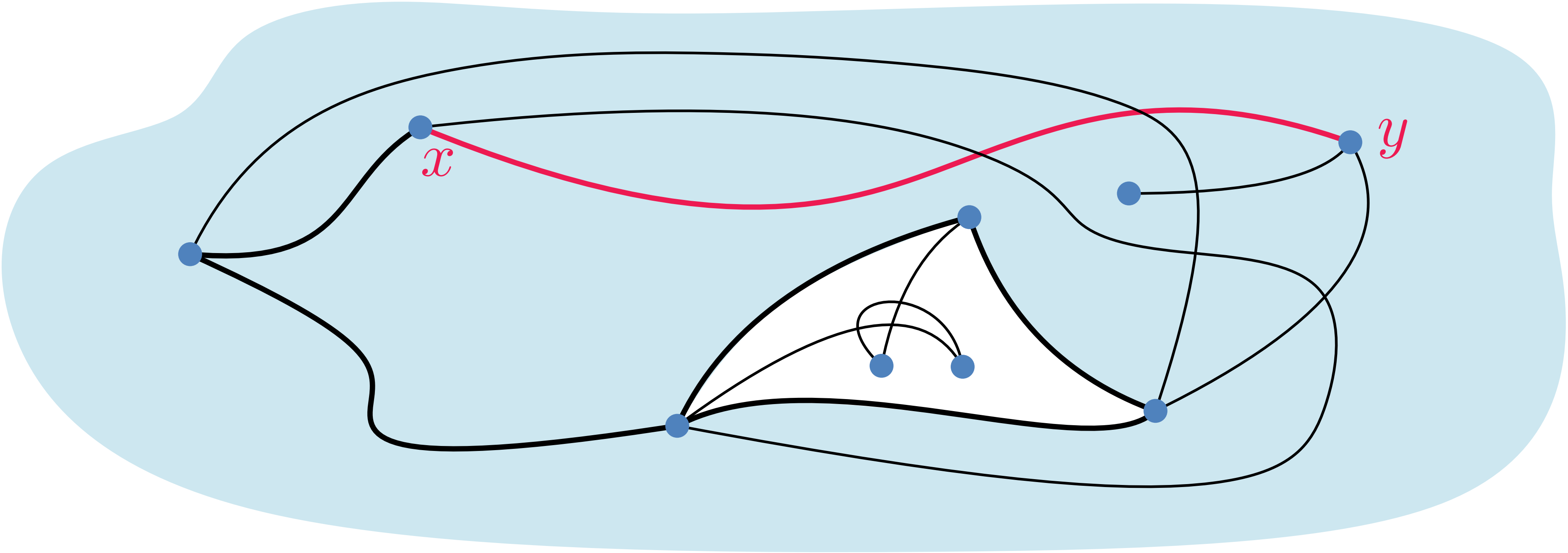


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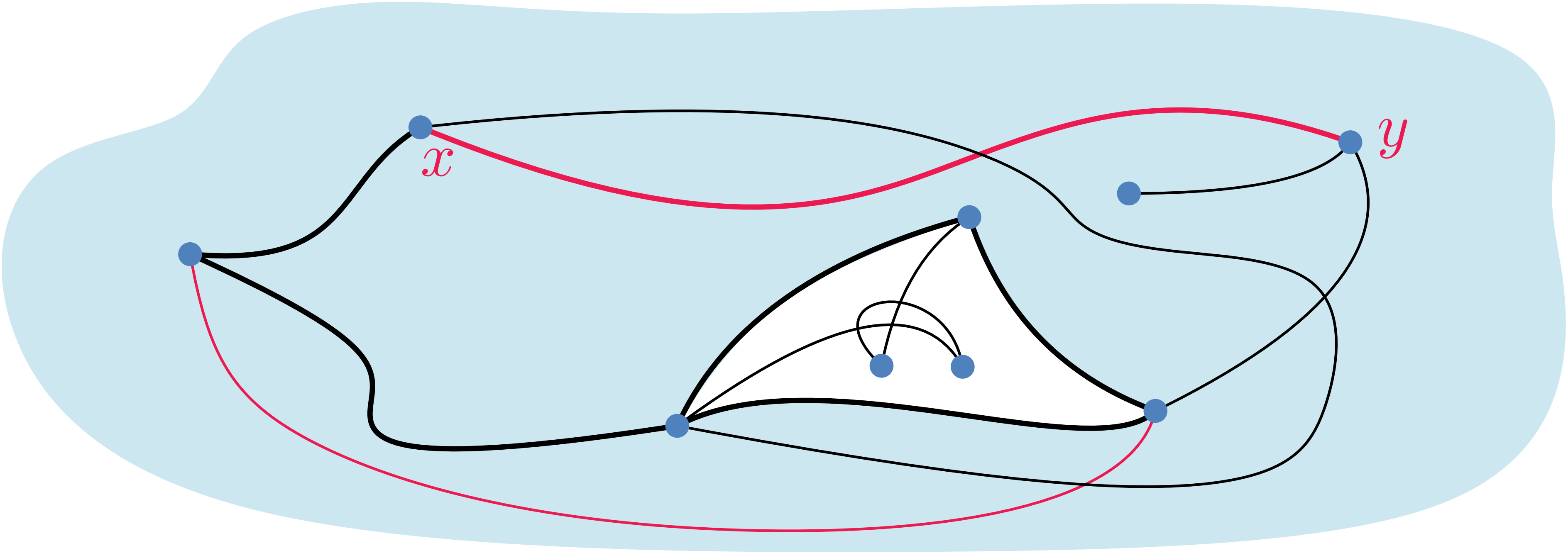


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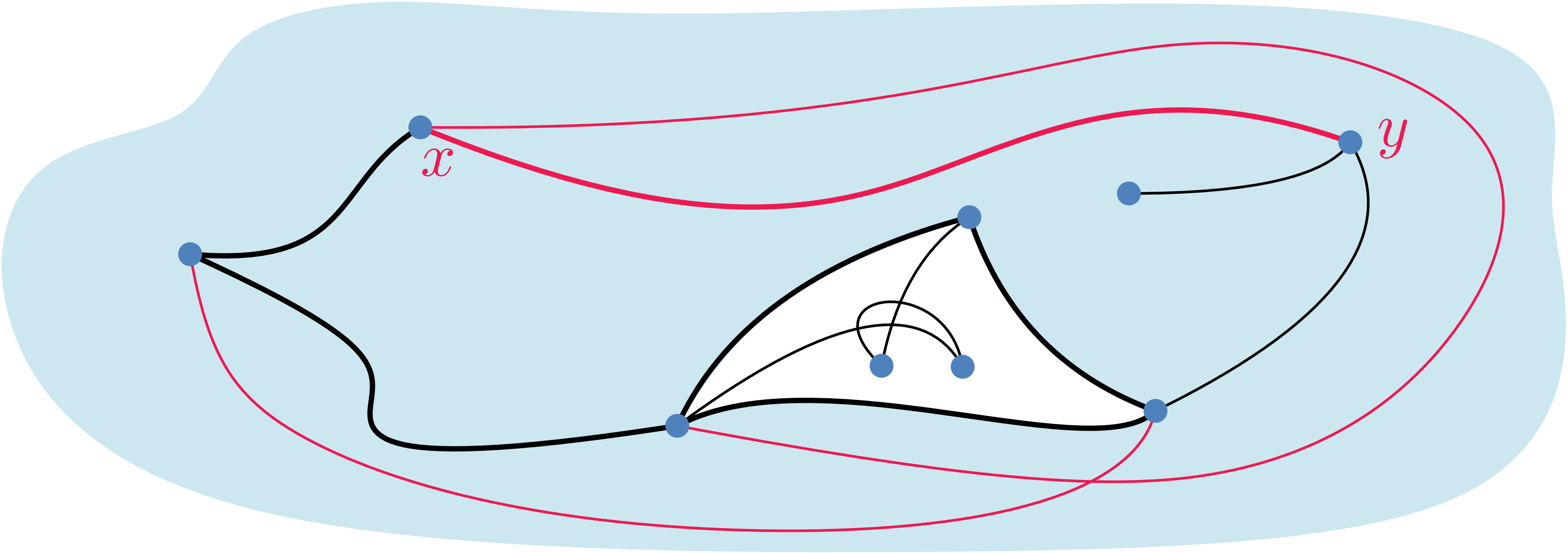


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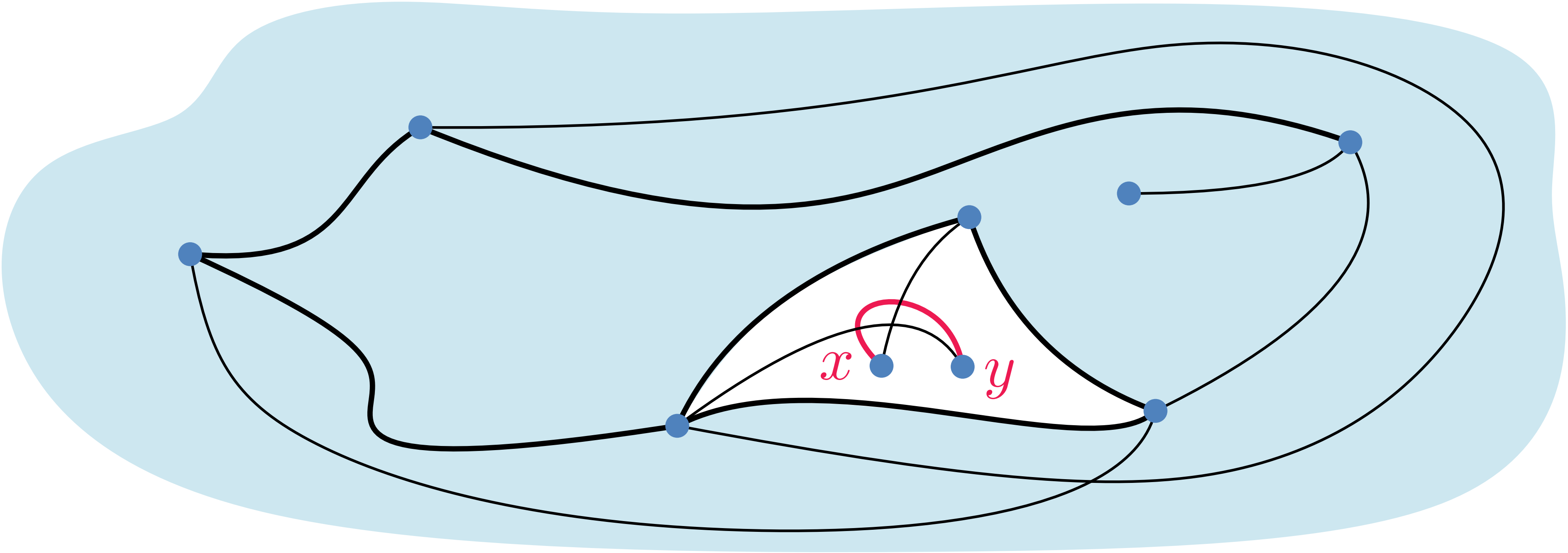


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$\Rightarrow$ add $xy$ to $D'$, redraw edges crossing $xy$ in same face

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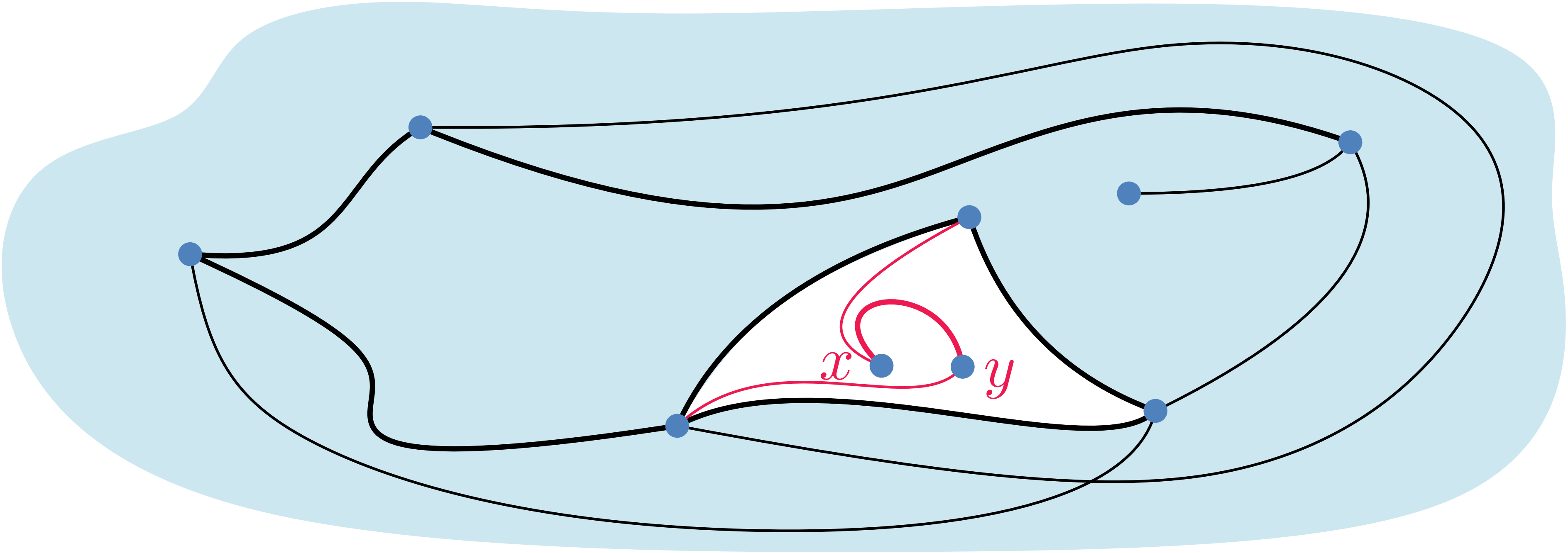


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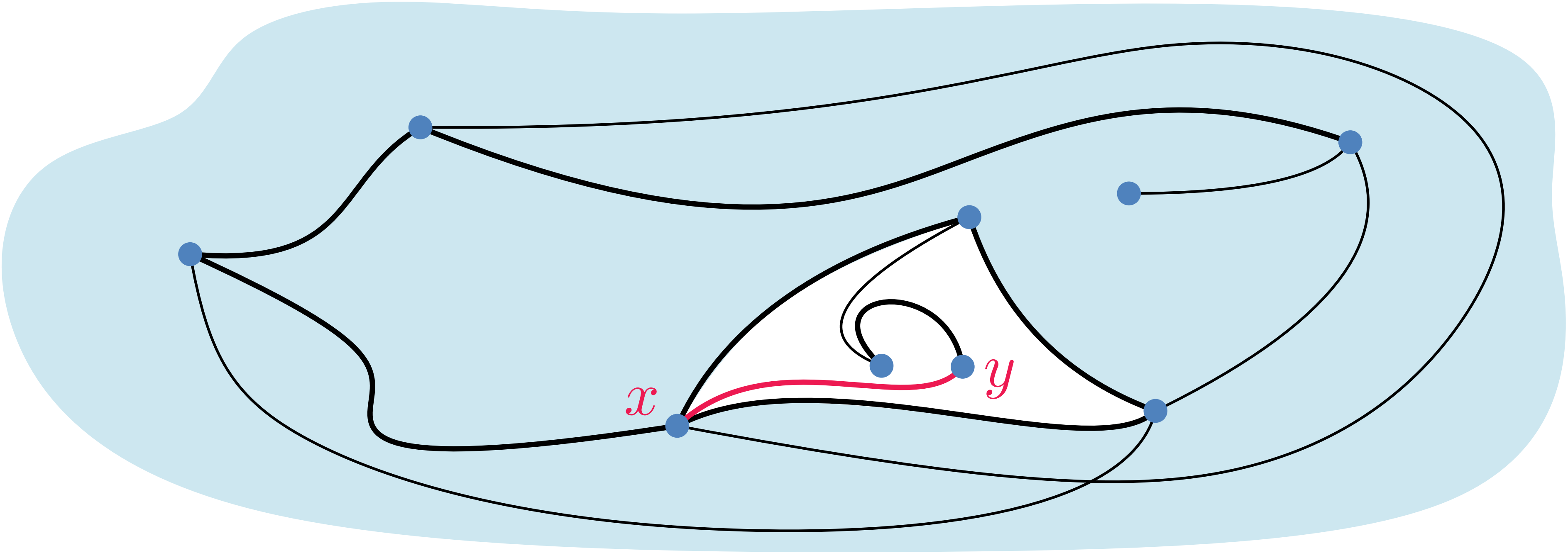


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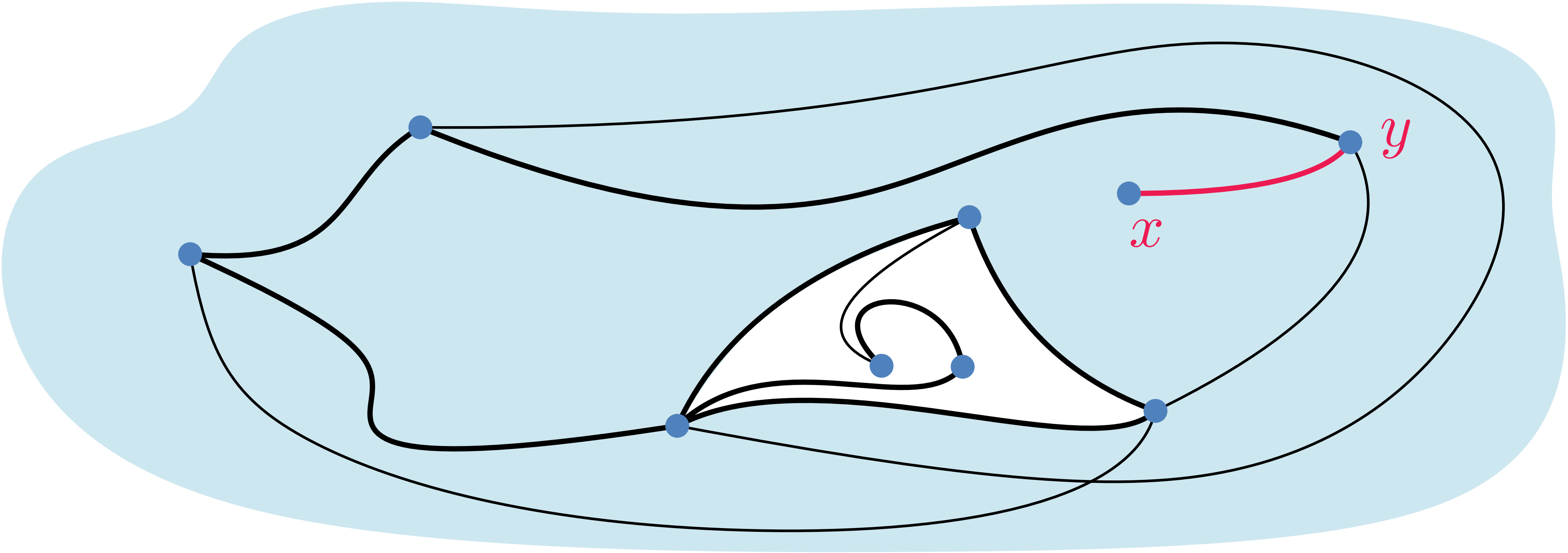


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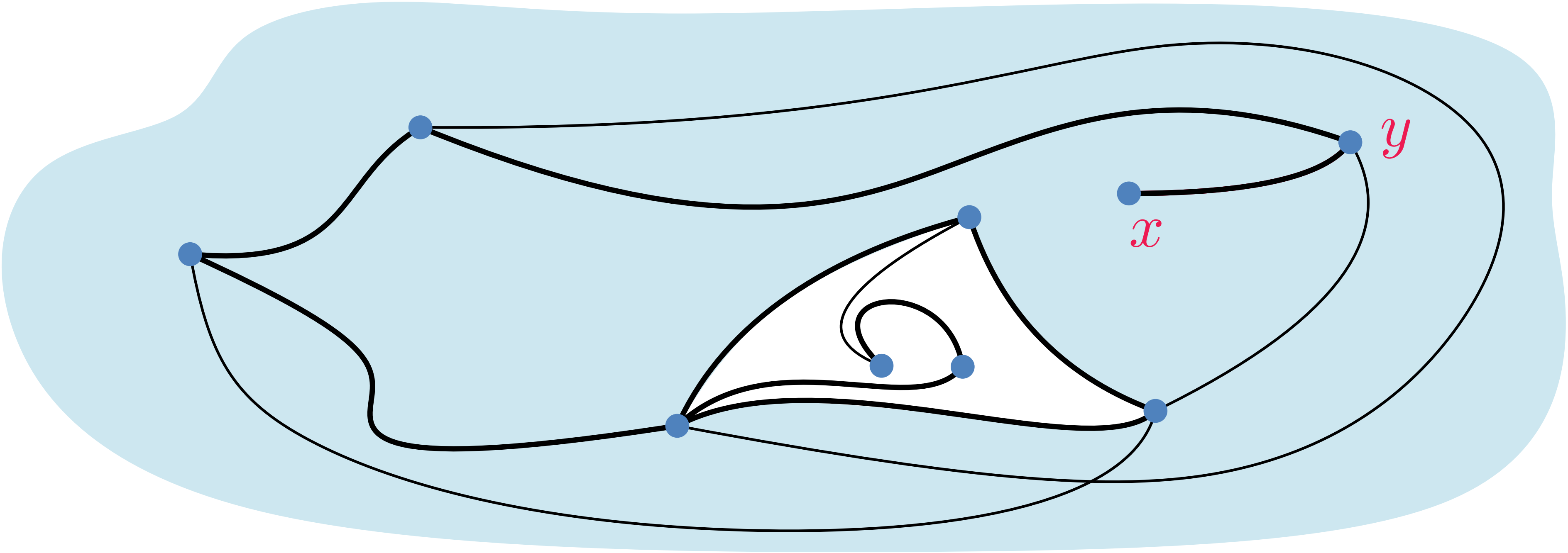


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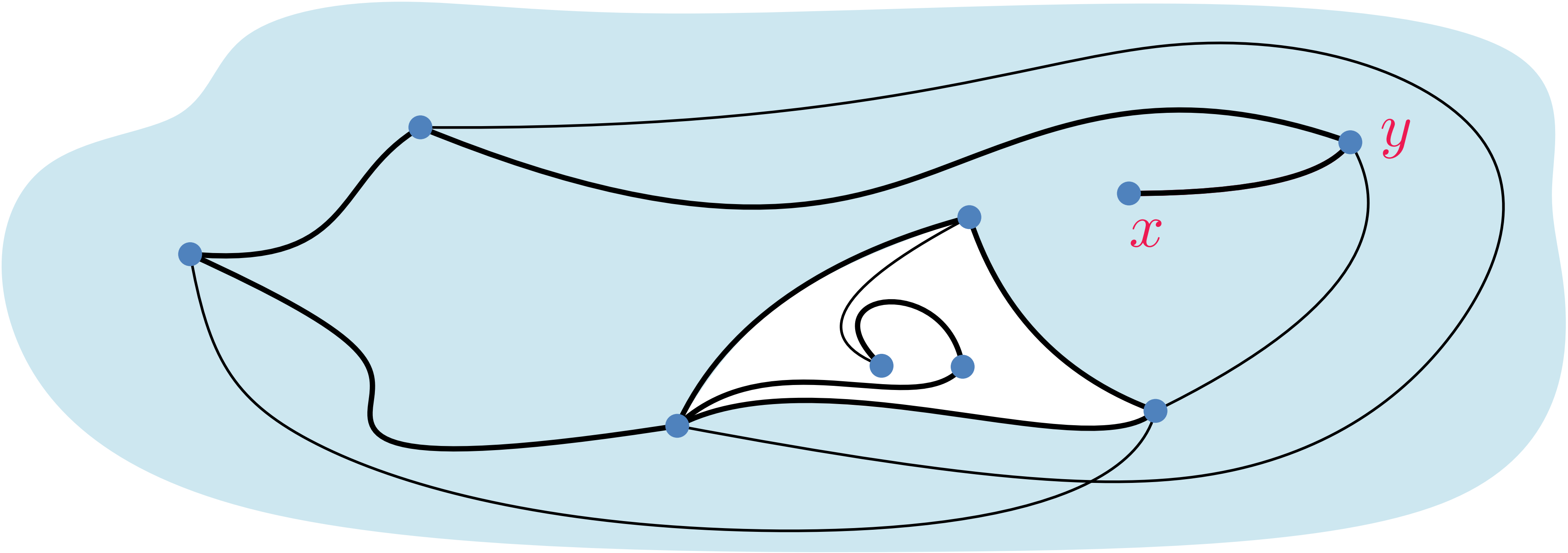


**Corollary.** If $G$ is connected then every maximal uncrossed subdrawing of $G$ is connected.

# Maximum Uncrossed Subdrawings

*maximum uncrossed subdrawing* $D'$ *of a graph* $G$:

number of edges of $D'$ is maximum over all uncrossed drawings of $G$

$h(G)$: number of edges in $D'$

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**Observations.**

Let $G=(V,E)$ with $|V|=n$ and $|E|=m$. Then

$$h(G) = m - \mathrm{ecr}(G)$$

$$\mathrm{unc}(G) \geq \left\lceil \frac{m}{h(G)} \right\rceil$$

$$h(G) \leq 3n-6,\ \mathrm{unc}(G) \geq \left\lceil \frac{m}{3n-6} \right\rceil$$

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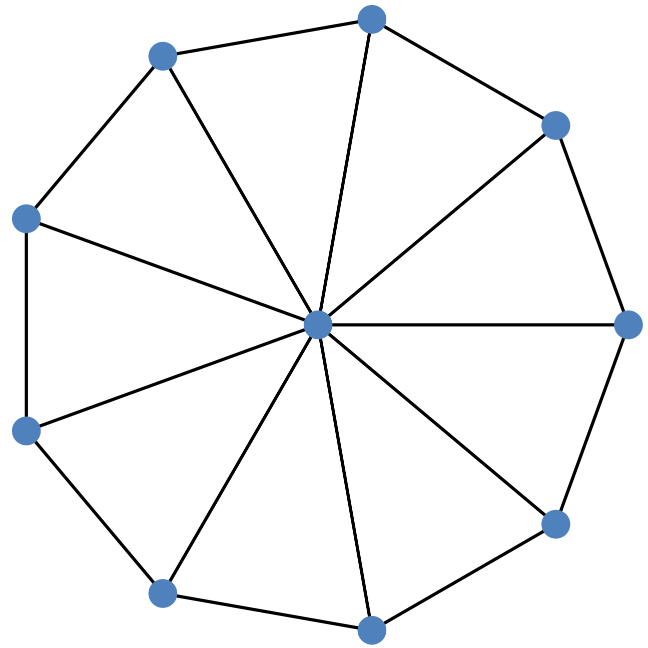
**Theorem** [Ringel, 1963]**.**

1. $h(K_n) = 2n - 2$ for every $n \geq 4$.
2. Every uncrossed subdrawing of $K_n$ with $2n - 2$ edges forms a *wheel graph* $W_n$.



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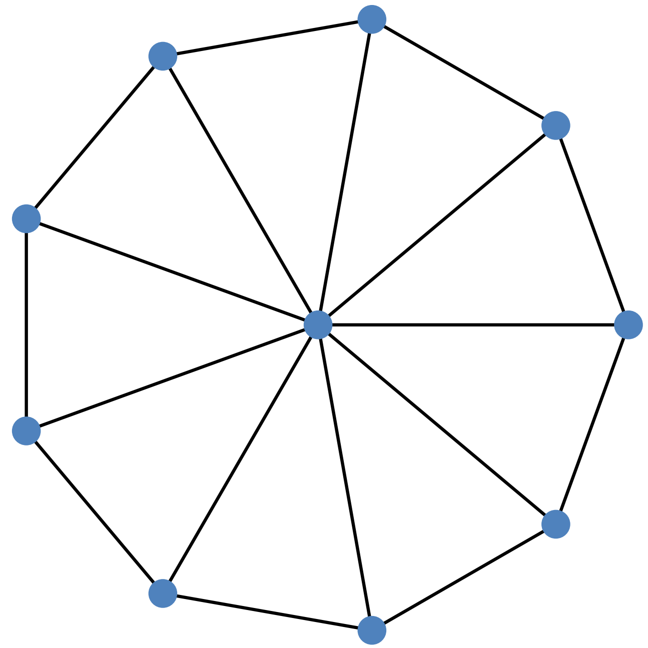

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**Corollary.** $\mathrm{ecr}(K_n) = \binom{n}{2} - 2n + 2$

$$\mathrm{unc}(K_n) \geq \left\lceil \frac{\binom{n}{2}}{2n-2} \right\rceil$$

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**Theorem** [Mengersen, 1978]. For every $m \leq n$, we have

$$h(K_{m,n}) = \begin{cases} 2m - n - 2, & \text{for } m = n \\ 2m + n - 1, & \text{for } m < n < 2m \\ 2m + n, & \text{for } 2m \leq n \end{cases}$$

$\Rightarrow$ value of $\mathrm{ecr}(K_{m,n})$, lower bound for $\mathrm{unc}(K_{m,n})$

# Uncrossed Conjectures for $K_n$ and $K_{m,n}$

**Conjecture 1.** $\mathrm{unc}(K_n) = \theta_o(K_n)$ if $n \neq 4$

**Conjecture 2.** $\mathrm{unc}(K_{m,n}) = \theta_o(K_{m,n})$ if $\min\{m,n\} \neq 2$

**Theorem** [Guy, Nowakowski, 1990]**.**
For any integer $n$, resp. any two integers $m, n$ with $m \leq n$, we have

$$\theta_o(K_n) = \begin{cases} \lceil \frac{n+1}{4} \rceil, & \text{for } n \neq 7 \\ 3, & \text{for } n = 7 \end{cases} \quad \text{resp.} \quad \theta_o(K_{m,n}) = \lceil \frac{mn}{2m+n-2} \rceil$$

# Determining $\mathrm{unc}(K_n)$

**Theorem 1.** For every positive integer $n$, it holds that

$$\mathrm{unc}(K_n) = \begin{cases} \lceil \frac{n+1}{4} \rceil, & \text{for } n \notin \{4, 7\} \\ 3, & \text{for } n = 7 \\ 1, & \text{for } n = 4. \end{cases}$$

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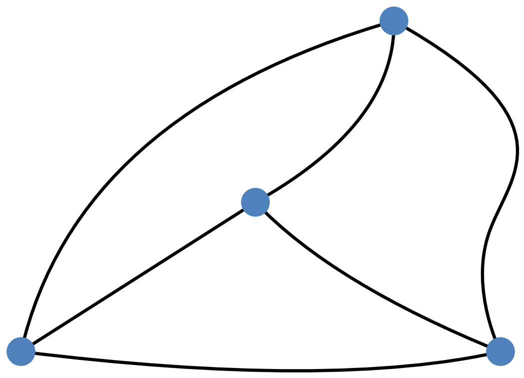
**Special Cases.**

$\mathrm{unc}(K_4) = 1$:



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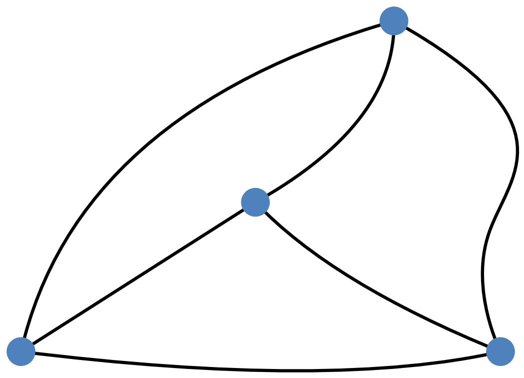

$\mathrm{unc}(K_7) = 3$: by [Hliněný, Masařík, 2023]

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**General upper bound.**

$\mathrm{unc}(K_n) \leq \theta_o(K_n) = \lceil \frac{n+1}{4} \rceil$ for $n \neq 7$: by [Guy, Nowakowski, 1990]

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uncrossed subdrawings $D'_1, \ldots, D'_k$ of $K_n$
s.t. each edge of $K_n$ in is exactly one $D'_i$

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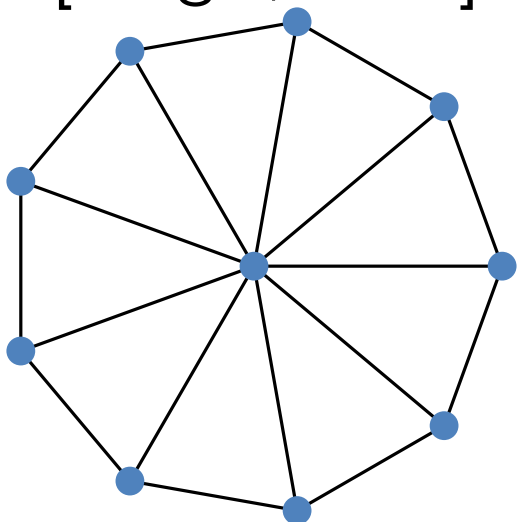

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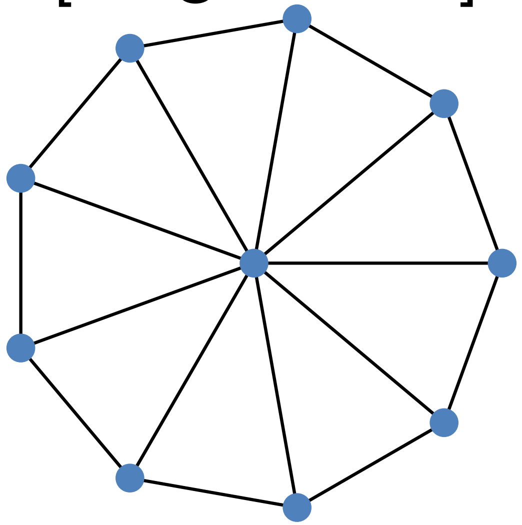

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**Corollary.** Conjecture 1 is true.

# Determining $\mathrm{unc}(K_{m,n})$

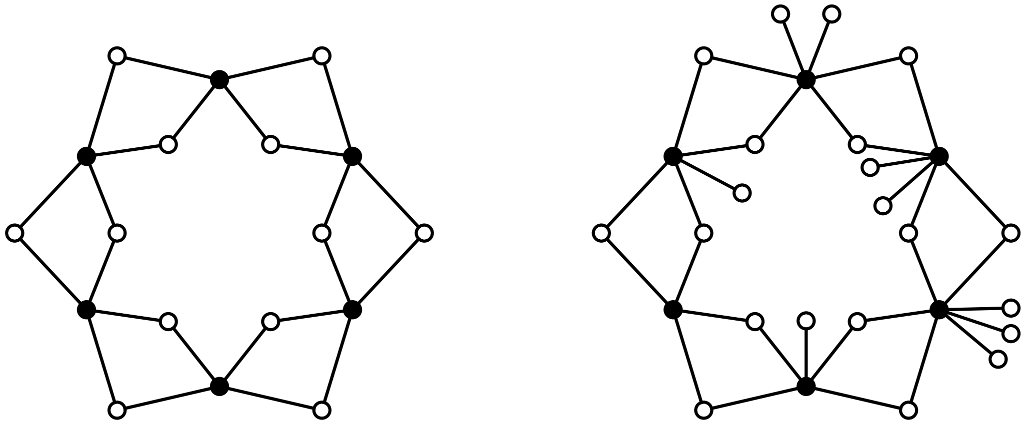
**Theorem 2.** For all positive integers $m$ and $n$ with $m \leq n$, it holds that

$$\mathrm{unc}(K_{m,n}) = \begin{cases} \left\lceil \frac{mn}{2m+n-2} \right\rceil, & \text{for } m \leq n \leq 2m-2 \\ \left\lceil \frac{mn}{2m+n-1} \right\rceil, & \text{for } n = 2m-1 \\ \left\lceil \frac{mn}{2m+n} \right\rceil, & \text{for } 6 \leq 2m \leq n \\ 1, & \text{for } m \leq 2. \end{cases}$$

**Proof Idea.** similar path as for $\mathrm{unc}(K_n)$, more complicated

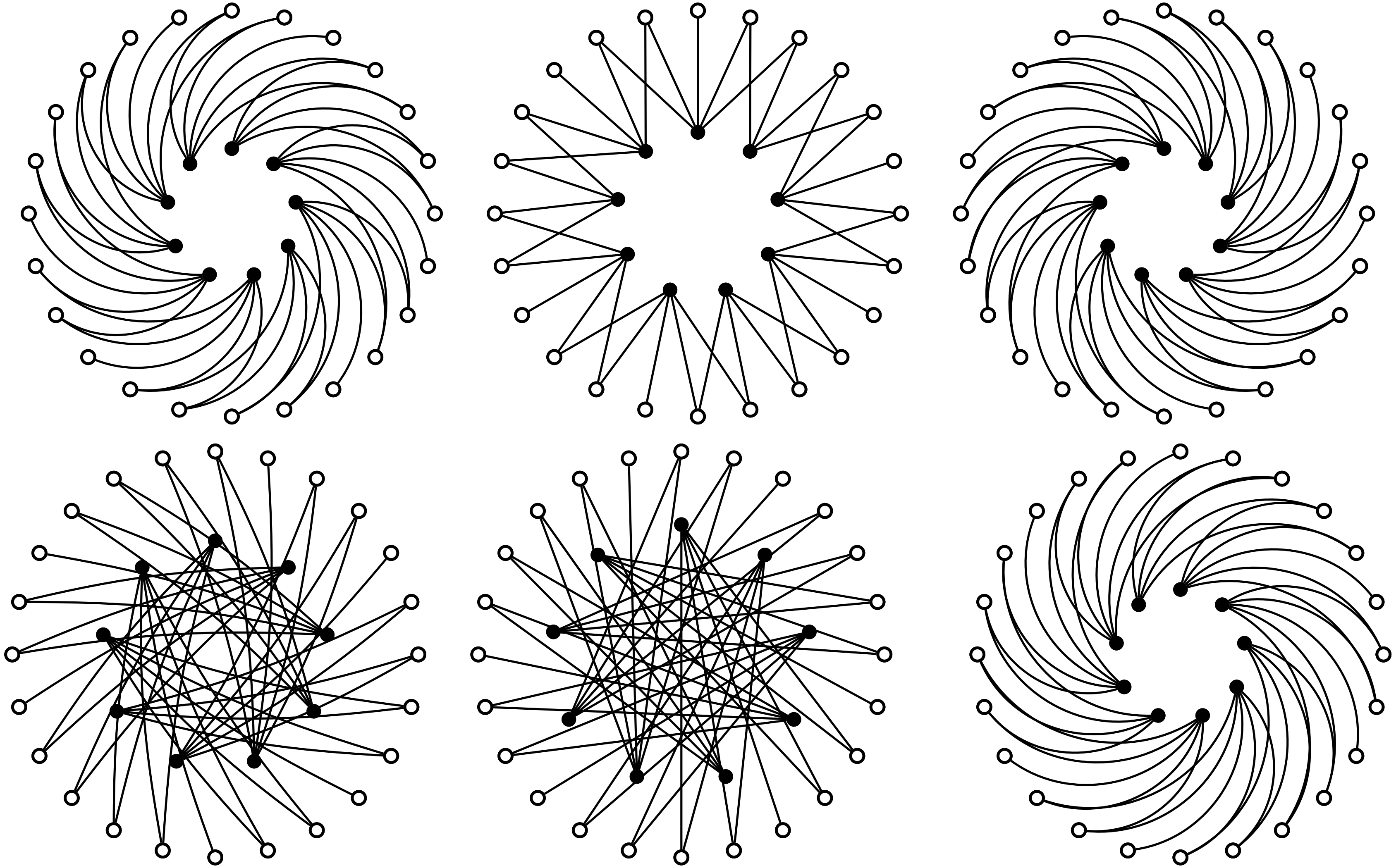
use uncrossed subdrawings called *double-cycles (with leaves)*:

not outerplanar, but all missing black-white edges can be added



# Determining $\mathrm{unc}(K_{m,n})$

six double-cycles with leaves that cover $K_{9,24}$:



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**Corollary.** Conjecture 2 is not true.

For example: $\mathrm{unc}(K_{4,7}) = 2$ and $\theta_o(K_{4,7}) = 3$

# Bounding $h(G)$ and $\mathrm{unc}(G)$ for General Graphs

**Theorem 3.** Every connected graph $G$ with $n \geq 3$ vertices and $m \geq 0$ edges satisfies $h(G) \leq \left(3n - 5 + \sqrt{(3n-5)^2 - 4m}\right) / 2$

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**Corollary.** Every connected graph $G$ with $n \geq 3$ vertices and $m \geq 0$ edges satisfies

$$\mathrm{unc}(G) \geq \left\lceil \frac{m}{\left(3n - 5 + \sqrt{(3n-5)^2 - 4m}\right) / 2} \right\rceil$$

$\Rightarrow$ First nontrivial lower bound for $\mathrm{unc}(G)$

always at least as good as trivial bound $\left\lceil \frac{m}{3n-6} \right\rceil$

# Computational Problems

*Input for all problems:* A graph $G$ and a positive integer $k$.

EDGECROSSINGNUMBER
*Question:* Does $G$ have edge crossing number $\mathrm{ecr}(G) \leq k$?

MAXIMUMUNCROSSEDSUBDRAWING
*Question:* Does $G$ have an uncrossed subdrawing with at least $k$ edges?

UNCROSSEDNUMBER
*Question:* Does $G$ have uncrossed number $\mathrm{unc}(G) \leq k$?

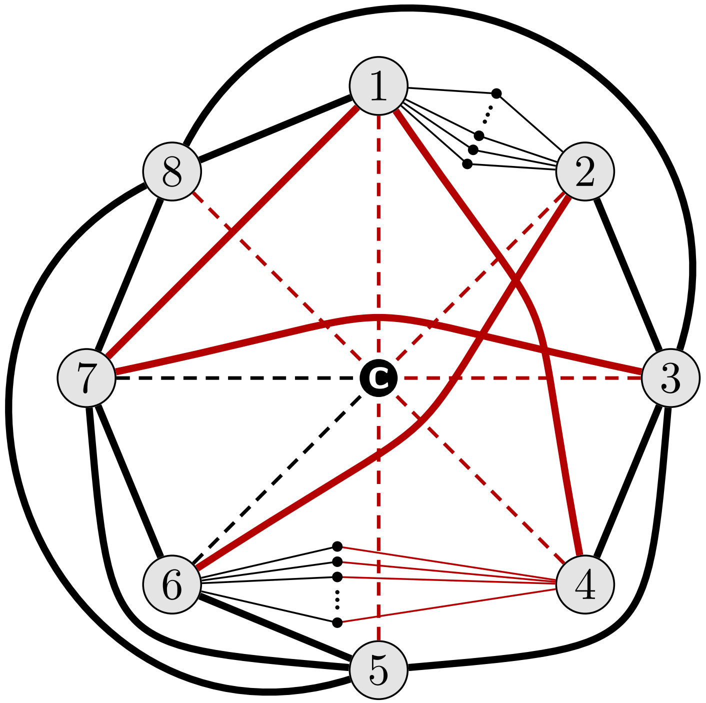
OUTERTHICKNESS
*Question:* Does $G$ have outer-thickness $\theta_o(G) \leq k$?

MAXIMUM OUTERPLANAR SUBGRAPH [Yannakakis, 1978]
*Question:* Is there an outerplanar subgraph of $G$ with at least $k$ edges?

# Computational Complexity Results

**Theorem 4.** The EDGECROSSINGNUMBER problem is NP-complete.

**Proof.** Reduction from MAXIMUMUNCROSSEDSUBDRAWING




**Corollary.** The MaximumUncrossedSubdrawing problem is NP-complete.



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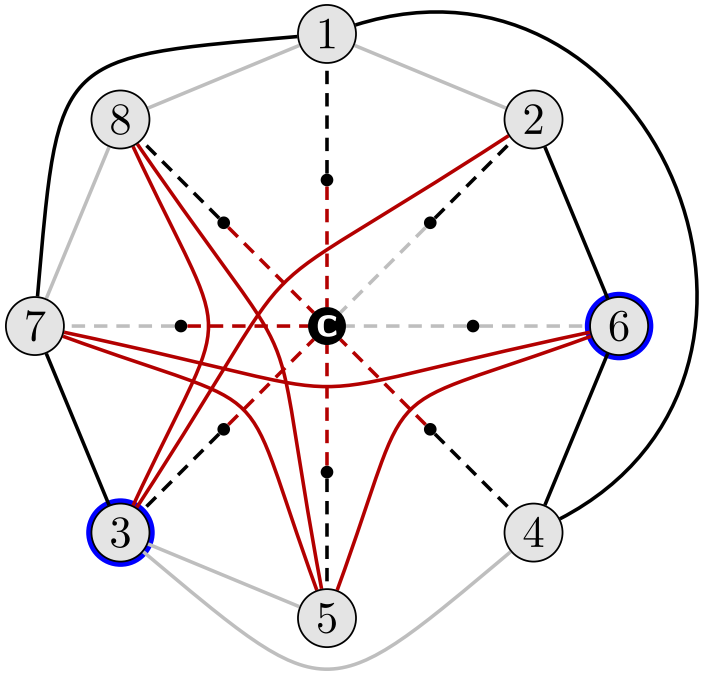
# Computational Complexity Results

**Theorem 4.** If the OuterThickness problem is NP-complete, then also the UncrossedNumber problem is NP-complete.



# Conclusion

**Results.**

- derived exact values of $\mathrm{unc}(K_n)$ and $\mathrm{unc}(K_{m,n})$
- improved lower bound on $\mathrm{unc}(G)$ for general graphs $G$
- showed that NP-completeness of UncrossedNumber is implied by NP-completeness of OuterThickness
- proved NP-completeness of EdgeCrossingNumber and Maximum UncrossedSubdrawing

**Open Problems.**

- reslove the computational complexity of OuterThickness
- determine the uncrossed number for relevant graph classes
- Prove that for every $k$, there is a graph $G$ with $\theta_o(G) - \mathrm{unc}(G) \geq k$

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