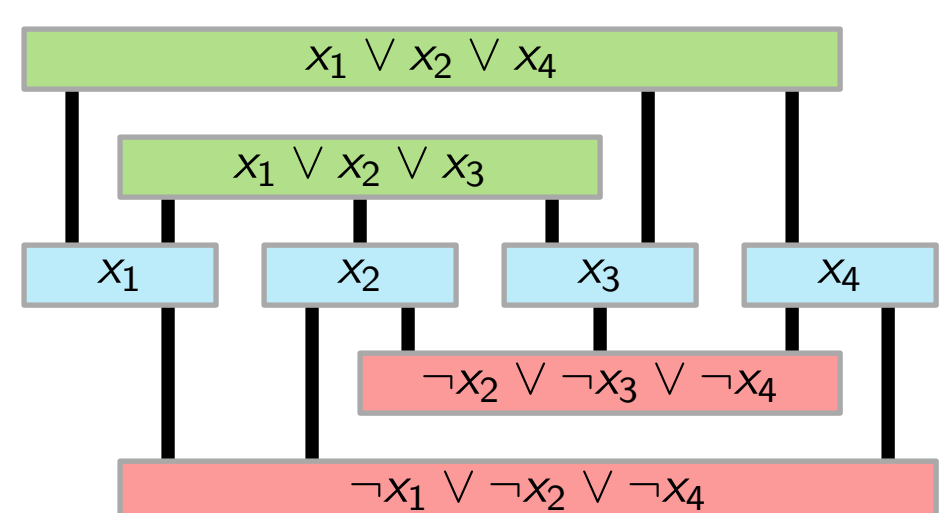


# From Planar via Outerplanar to Outerpath – Engineering NP-Hardness Constructions

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## Typical computational complexity question in graph drawing

**Input:** graph  $G$  with specific properties  
**Task:** decide whether  $G$  can be drawn with a specific drawing style  
**Result:** often a polynomial-time algorithm for a simple graph class and an NP-hardness reduction for a somewhat larger graph class



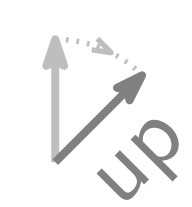
Planar Montone Rectilinear 3-SAT  
[de Berg, Khosravi; IJCGA'12]

## Example 1: Upward-planar 3-slope drawings

**Input:** upward-planar directed graph  $G$   
**Task:** decide whether  $G$  admits an upward-planar straight-line drawing where every edge has one of three distinct slopes  
**Result:** polynomial-time solvable for trees and cactus graphs, but NP-hard for outerplanar graphs [Klawitter, Zink; GD'21, JGAA'23]

Basic building blocks:

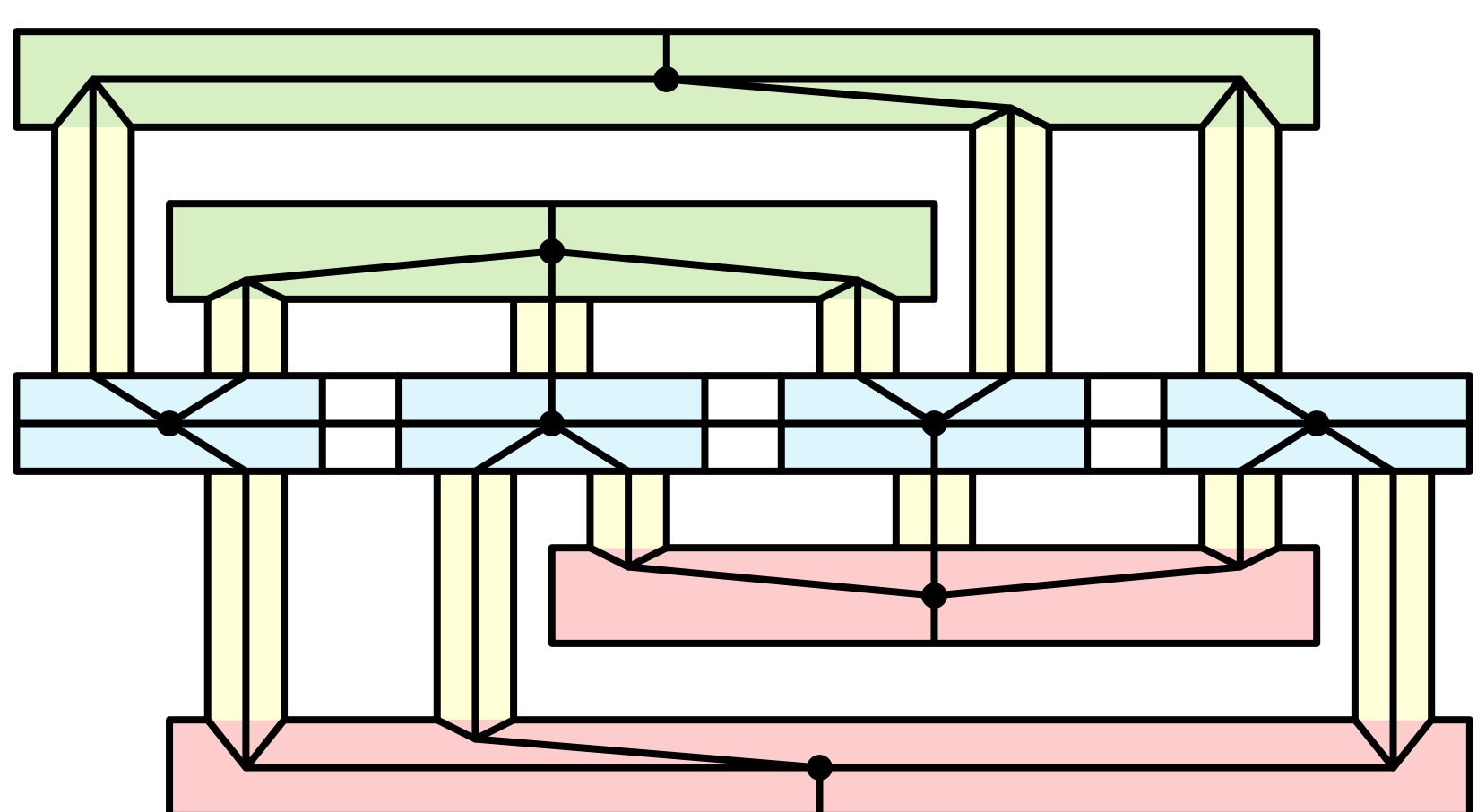
can only be drawn as unit squares with three slopes



two parallel edges with fixed slope but flexible length

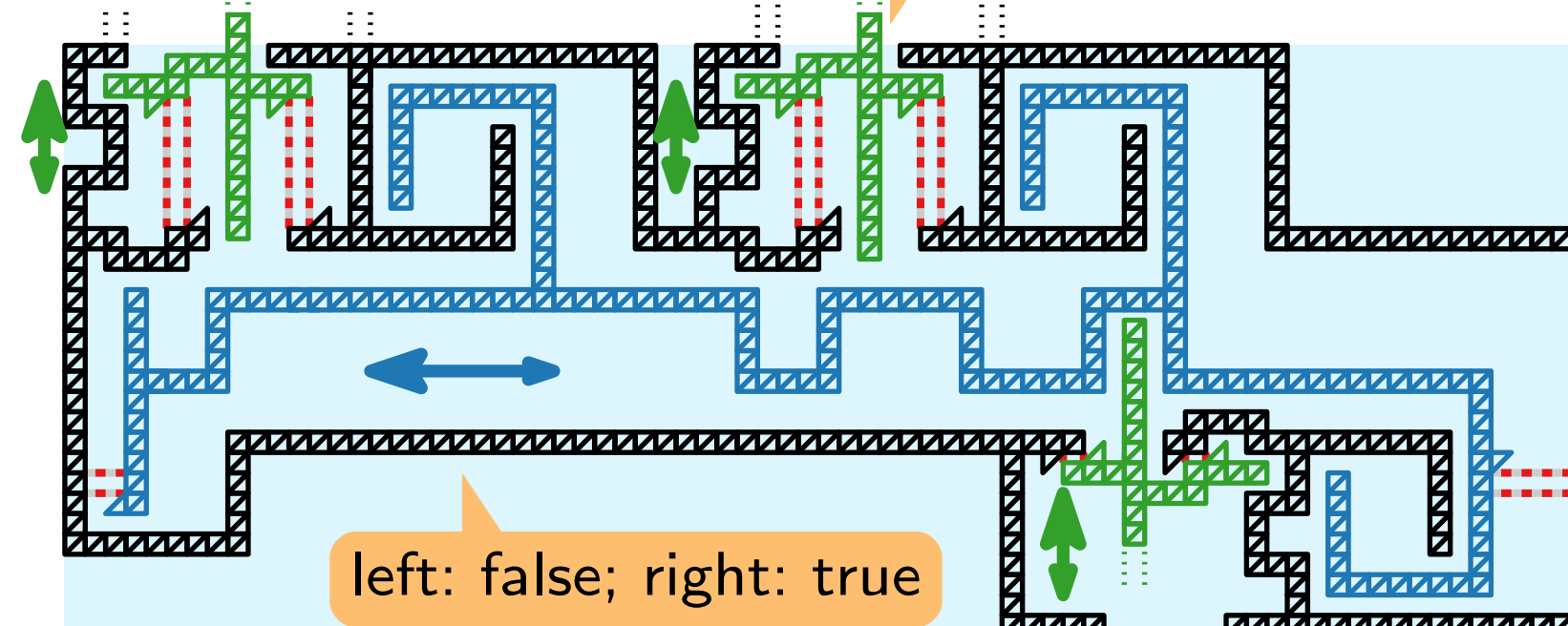
make planar

fill the planar structure with rigid and flexible parts to model variable and clause gadgets



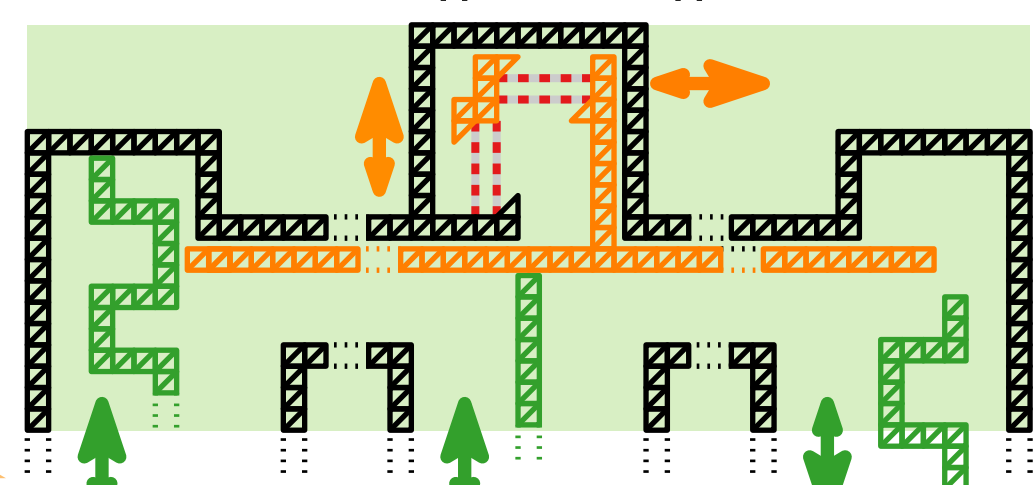
Variable gadget:

extend to the clause gadgets



Clause gadget:

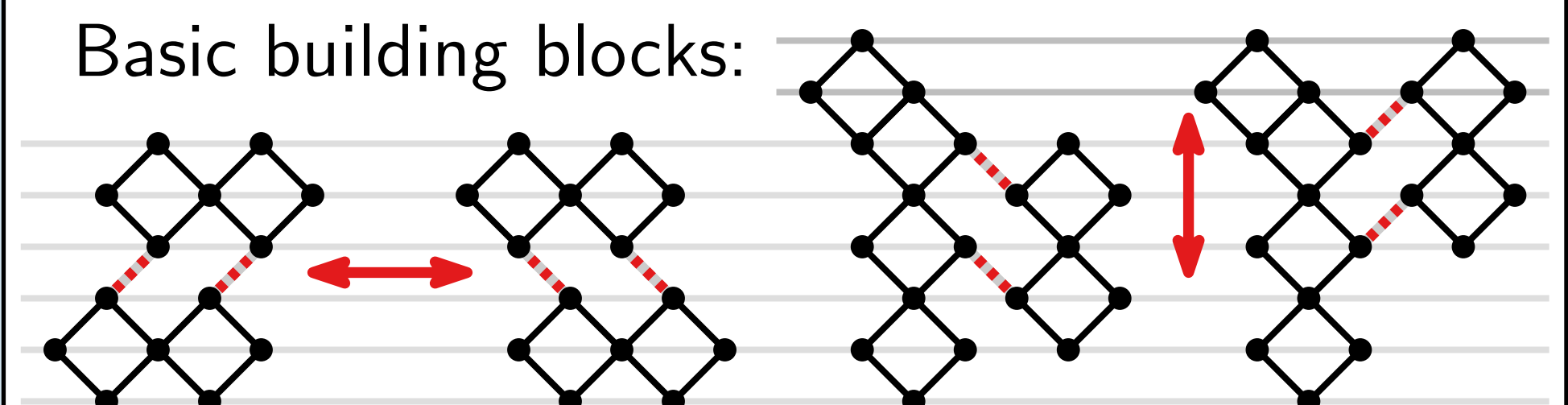
only if all three variables are false, there is no planar drawing



## Example 2: Proper level-planar 2-slope drawings

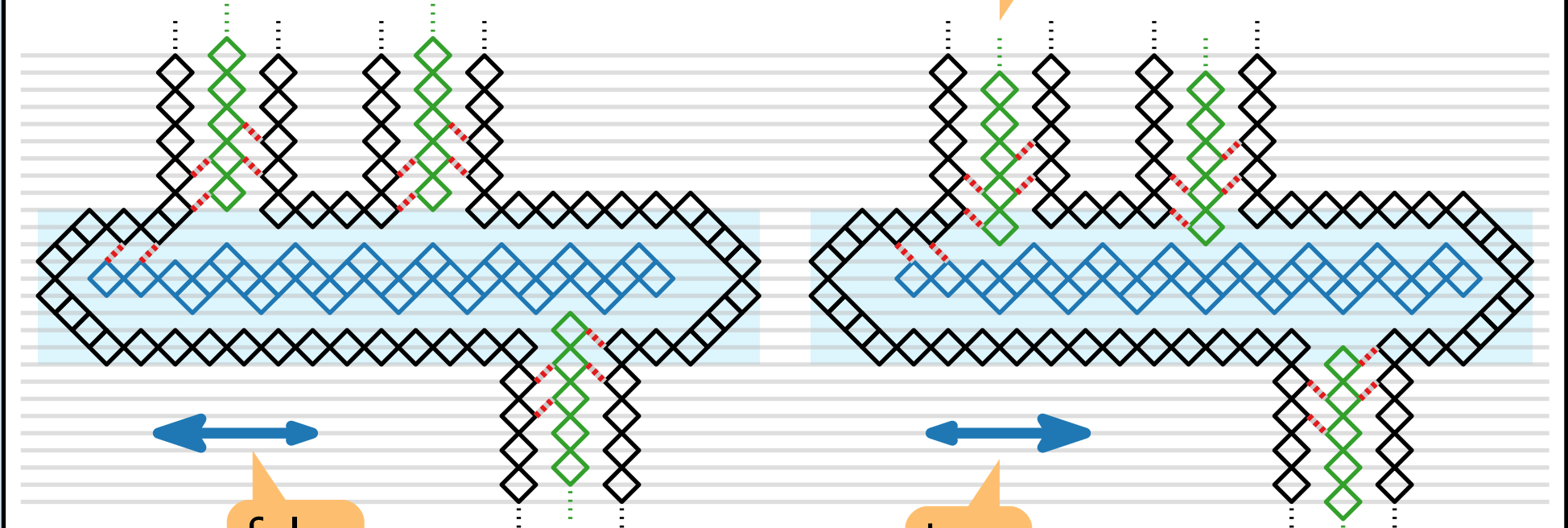
**Input:** planar embedded undirected graph  $G$   
**Task:** decide whether  $G$  admits a proper level-planar straight-line drawing where every edge has one of two distinct slopes  
**Result:** polynomial-time solvable if a leveling is given [Brückner, Krisam, Mchedlidze; GD'19, Algorithmica'22], but otherwise NP-hard even for outerplanar graphs [Kraus; bachelor thesis'20]

Basic building blocks:



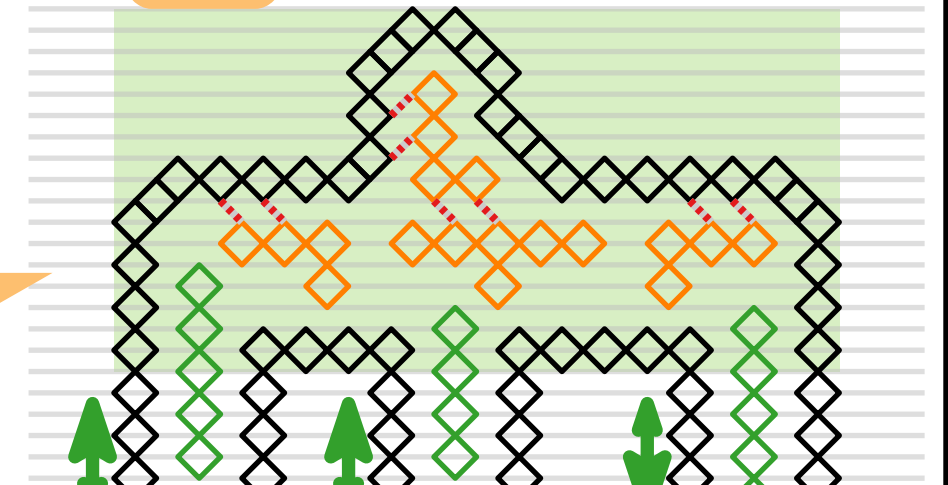
Variable gadget:

extend to the clause gadgets



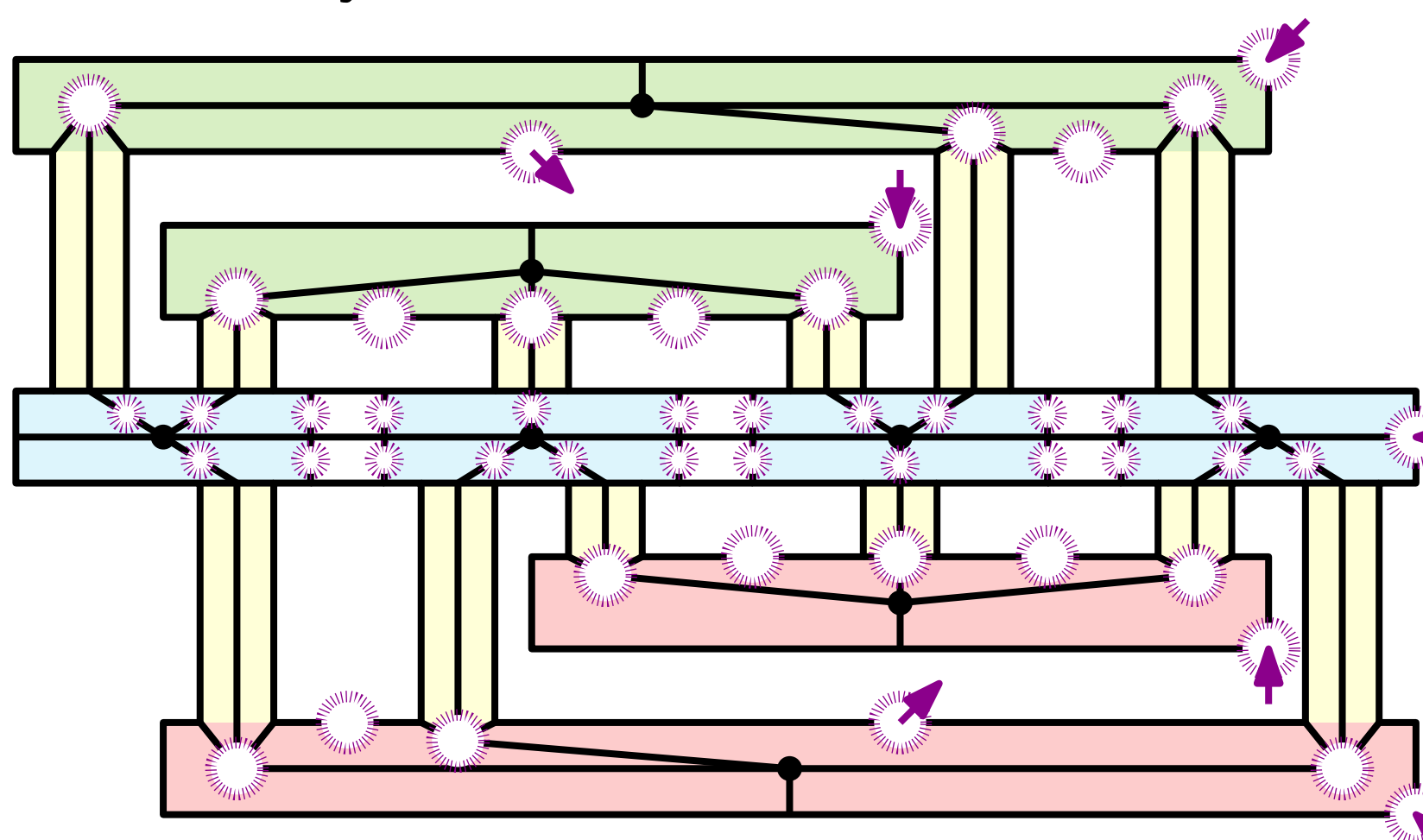
Clause gadget:

the three orange pieces can be arranged without overlap if at least one variable is true

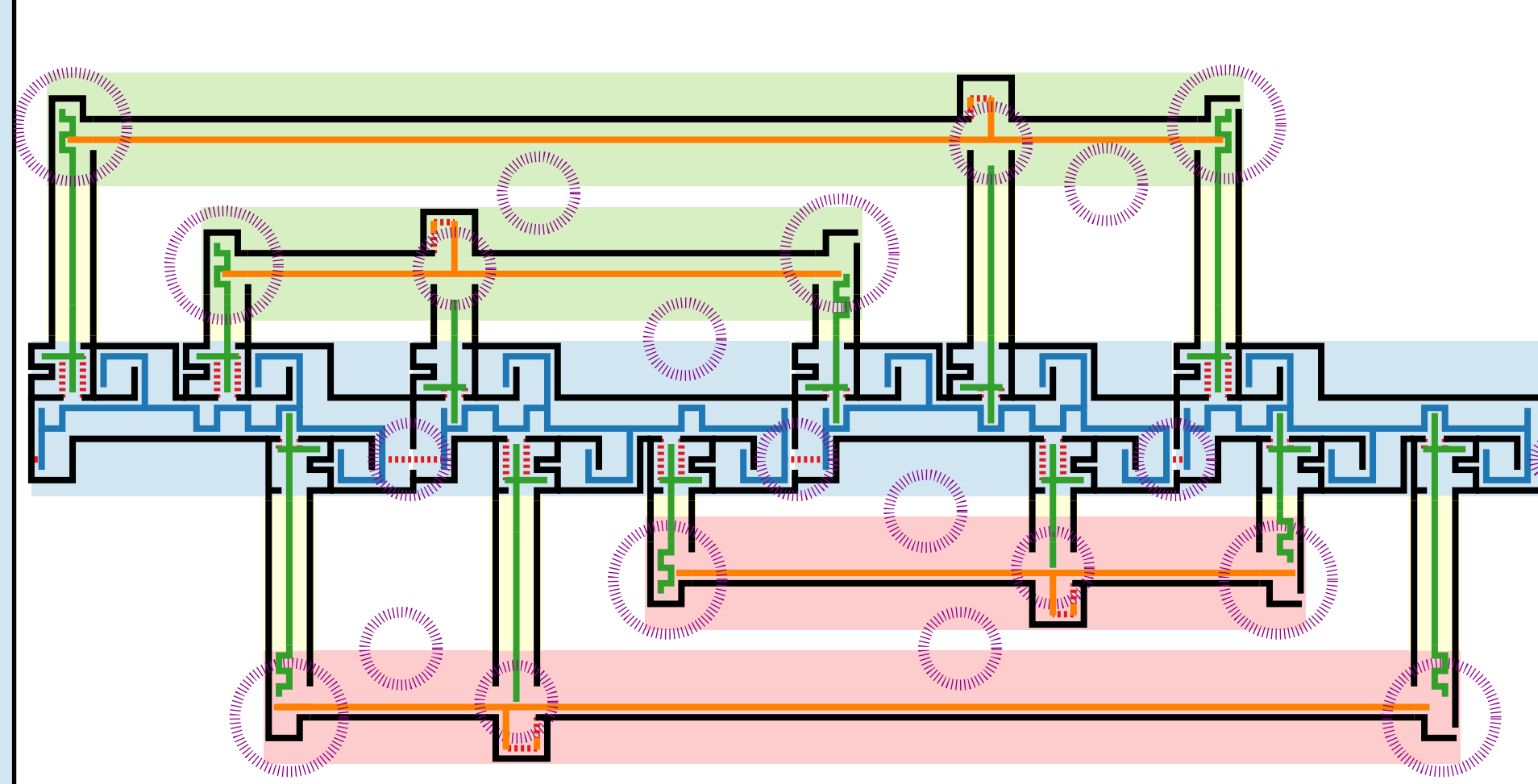


make outerplanar (Def: all vertices lie on the outer face; weak dual is a forest)

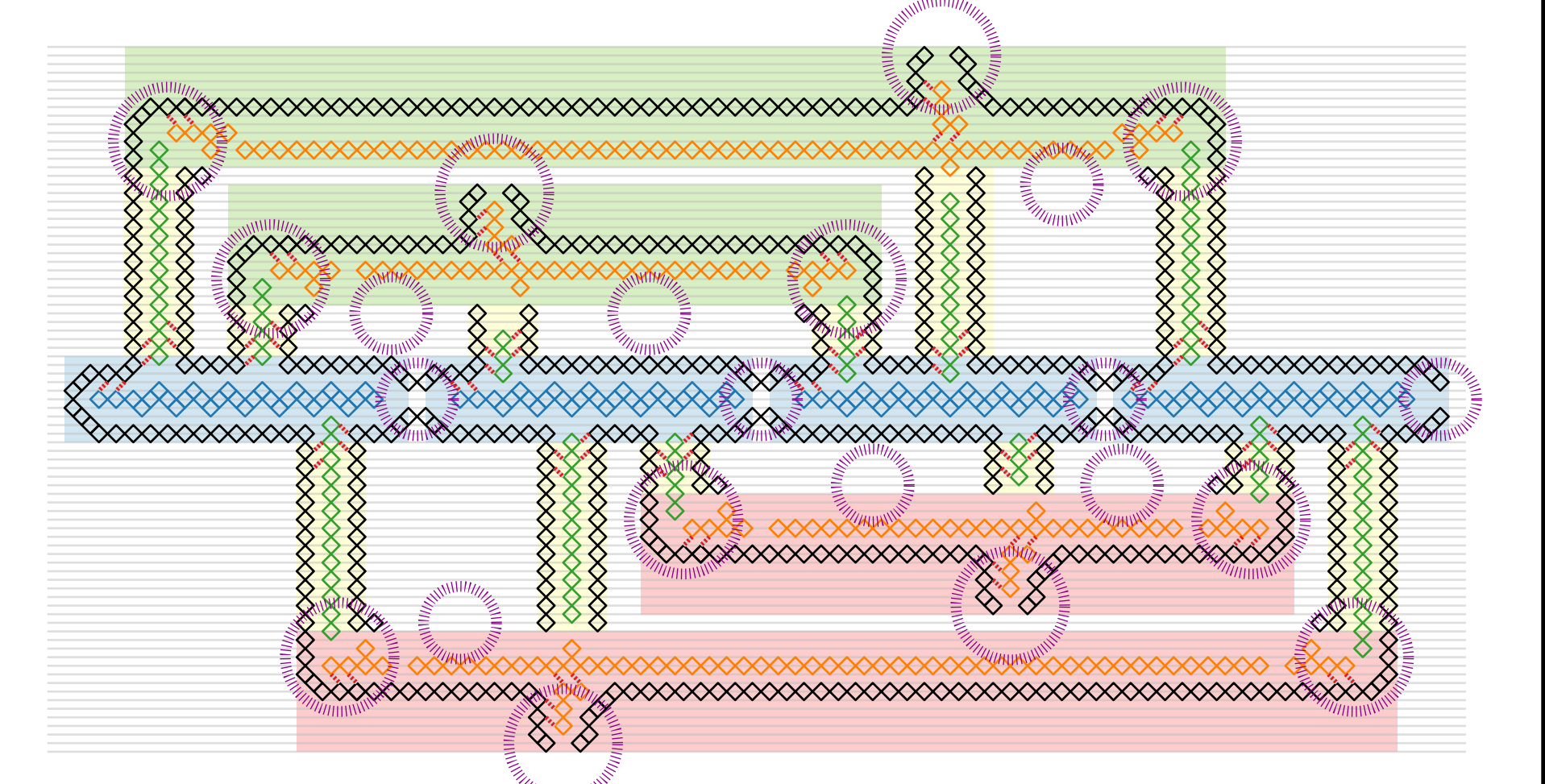
leave gaps (but keep connectivity)  
→ every vertex lies on the outer face



NP-hardness for directed outerplanar graphs  
[Klawitter, Zink; GD'21, JGAA'23]

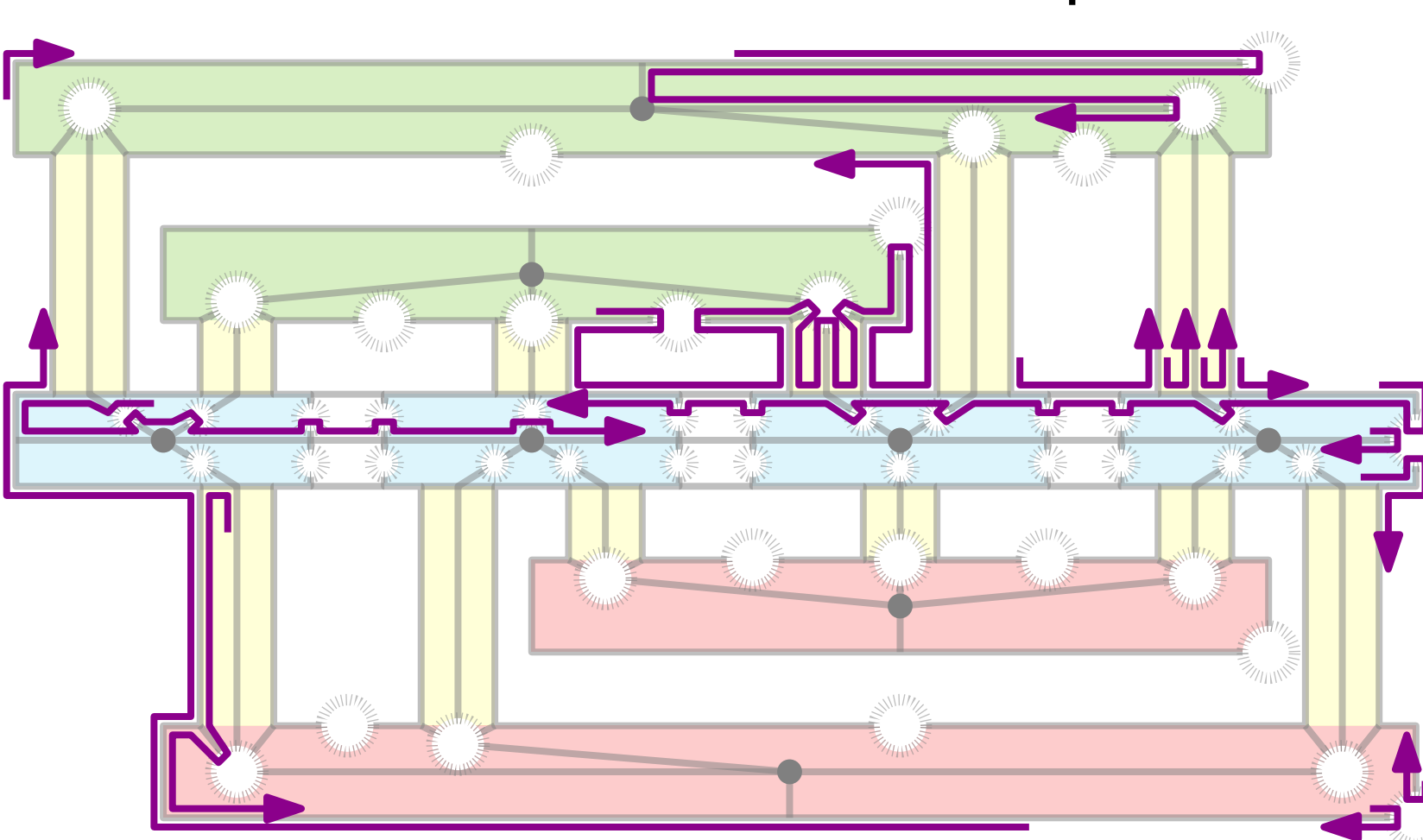


NP-hardness for outerplanar graphs  
[Kraus; bachelor thesis '20]



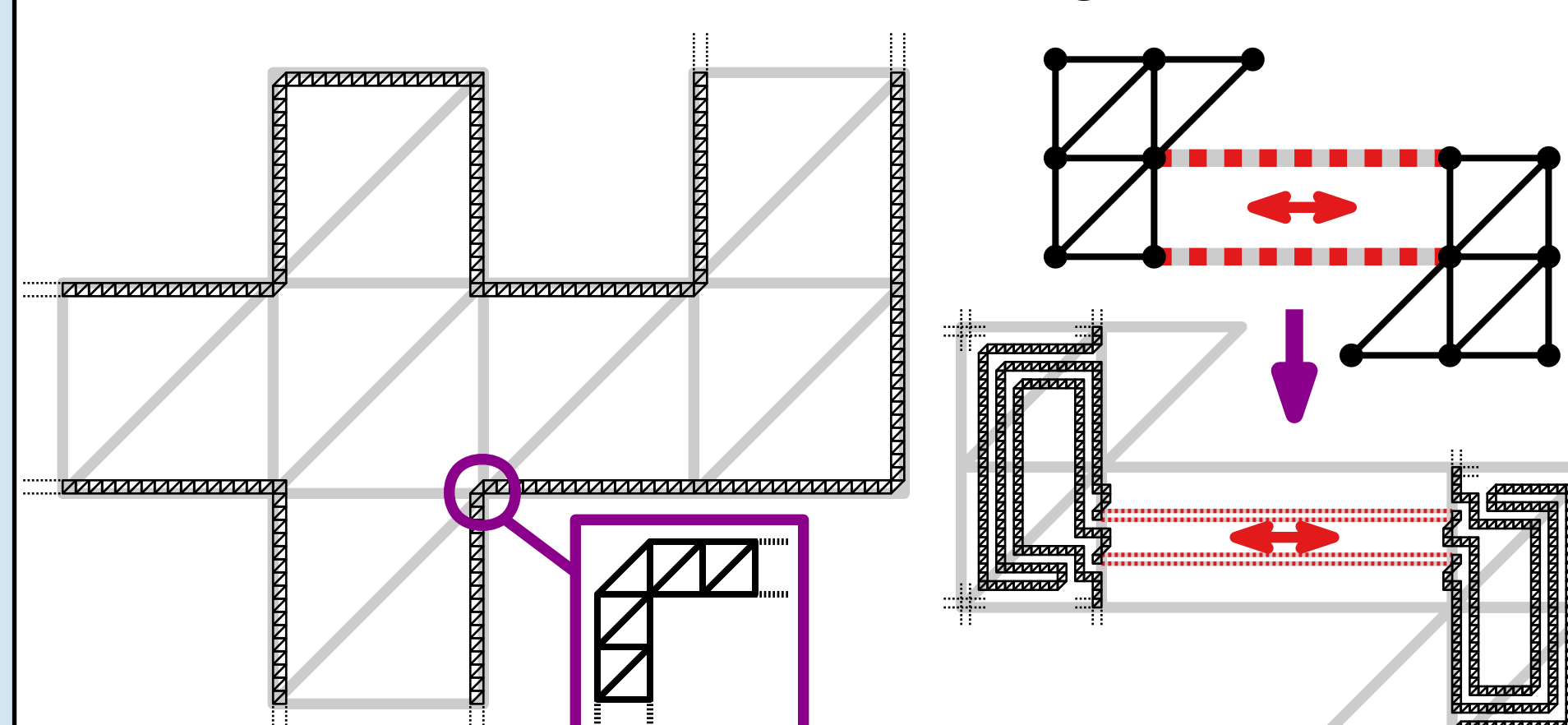
make outerpath (Def: outerplanar where the weak dual is a path)

trace outer face (but don't close the cycle)  
→ weak dual becomes a path



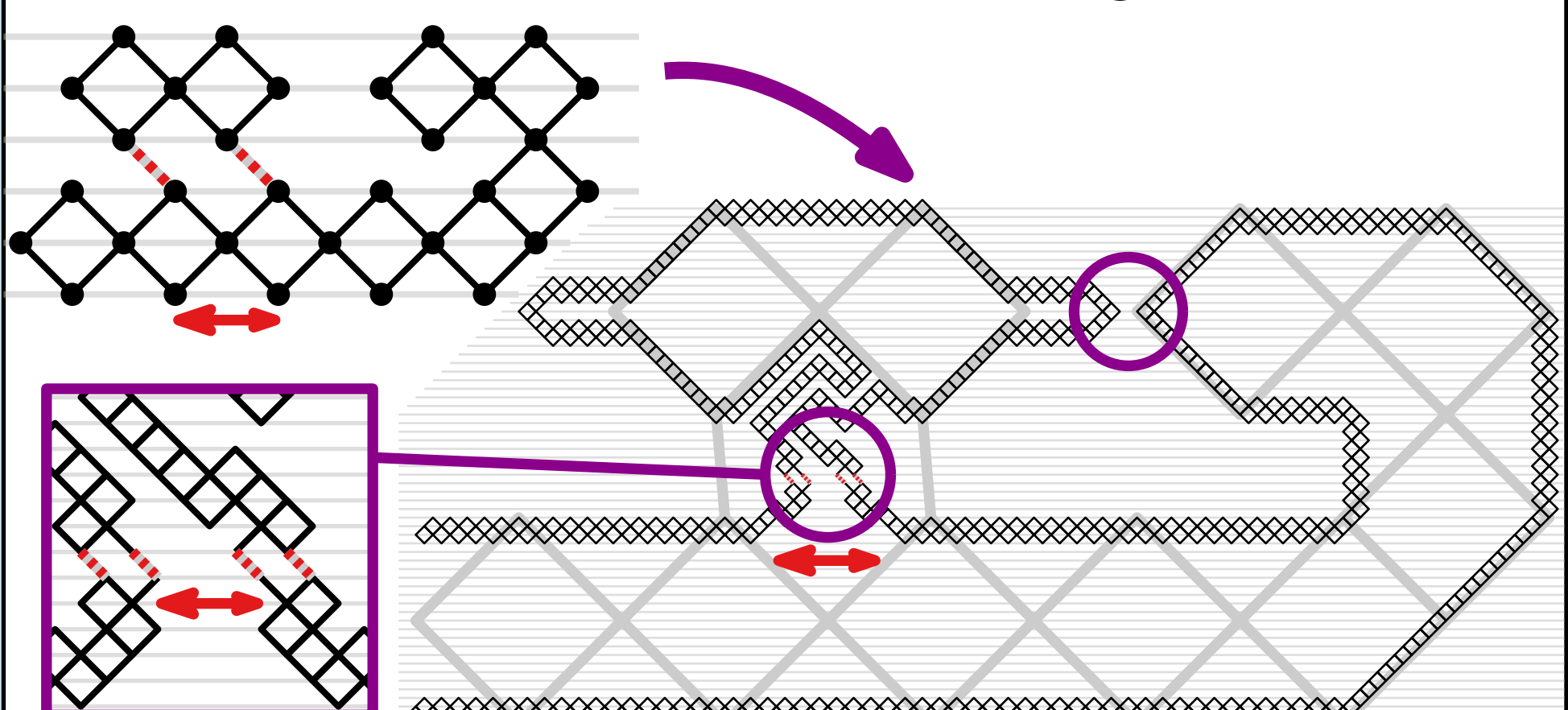
**Our contribution:**

Deciding whether an outerpath admits an upward-planar 3-slope drawing is NP-hard.



**Our contribution:**

Deciding whether an outerpath admits a proper level-planar 2-slope drawing is NP-hard.



examples?  
more