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Minimizing Switches in Cased Graph Drawings

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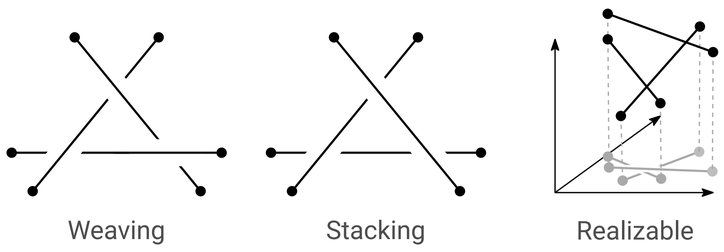
Models

Weaving Model
The top edge is chosen at each crossing independently.

Stacking Model
The top edge at each crossing is determined by a total edge order $e_1 < \cdots < e_m$.

Realizable Model
There exists an equivalent plane projection of line segments in $\mathbb{R}^3$.

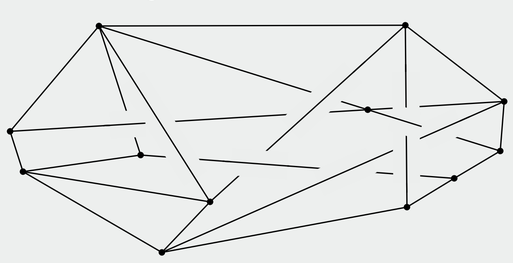


Weaving Stacking Realizable

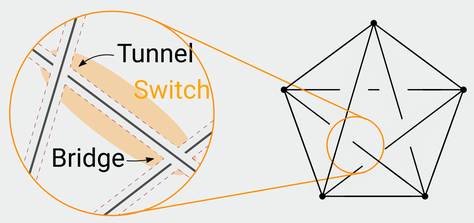
Cased Drawings

Introduced by Eppstein, Kreveld, Mumford, Speckmann [1]

Post-processing on drawing, where a halo is drawn around each edge

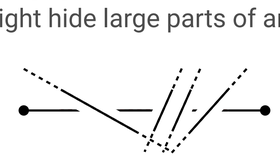


At each crossing, one edge is drawn on top.

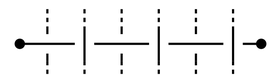


Optimization Goals

Minimize Tunnels
Too many tunnels might hide large parts of an edge.



Minimize Switches
Too many switches make it hard to follow along an edge.



Our problem:

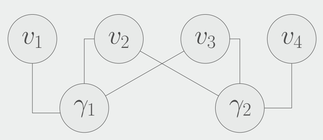
MinMaxSwitches
Given: Drawing of a graph
Goal: Find cased drawing with $\leq k$ switches per edge

MinMaxSwitches is NP-complete

Proof sketch. Reduction from NAE 3SAT. An NAE 3SAT instance I is satisfiable if and only if the reduction drawing δ_G is embeddable with $k = 1$ switches per edge.

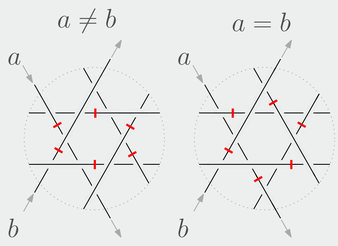
State $a \in \{$ bridge, tunnel $\}$ of a point on an edge: defined by the casing of the two neighboring crossings.

NAE 3SAT instance: $\gamma_1 = (v_1, \neg v_2, v_3)$
$\gamma_2 = (v_2, \neg v_3, v_4)$

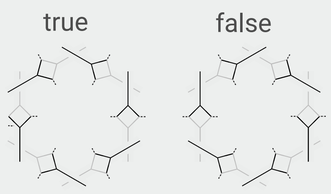


Crossing gadget: The following properties hold:
1. All edges have at least one switch.
2. The two input states a, b are independent.
3. Output states are always the inverse of input states.

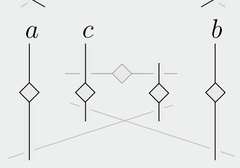
Symbol: — switch



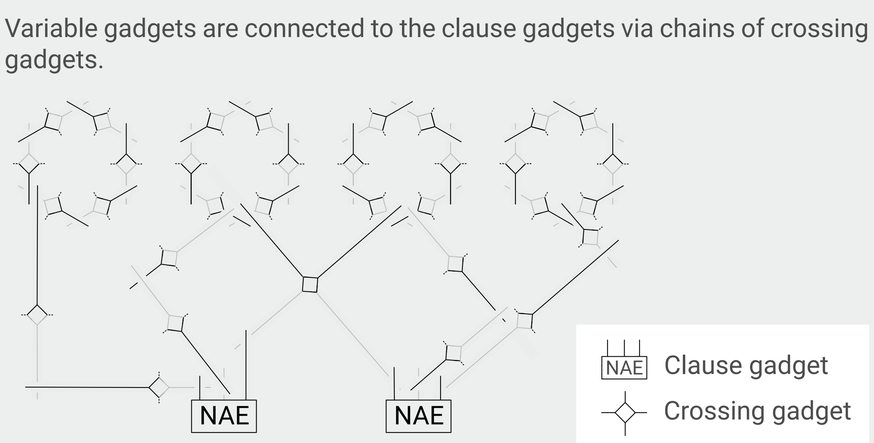
Variable gadget: Consists of a ring of crossing gadgets; as switches are used up by the crossing gadgets, there are only two possible casings representing true and false.



Clause gadget: Only possible with 1 switch per edge if one input state is different.



Final reduction: Variable gadgets are connected to the clause gadgets via chains of crossing gadgets.

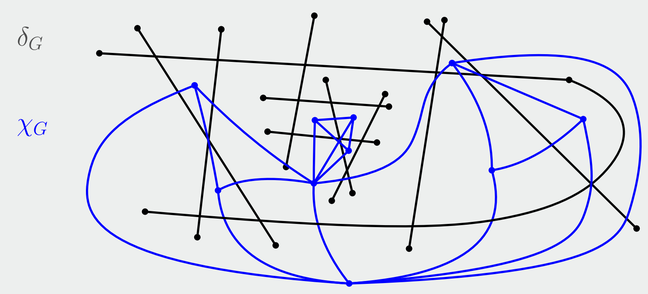


The reduction is valid in the weaving/realizable model only. □

MinMaxSwitches is FPT

As the problem is only influenced by the structure of edge crossings, parameters on the original graph are unlikely to yield tractability.

Therefore, we consider the *cell adjacency graph* $\chi_G = (\mathcal{F}, E')$ over the faces $\mathcal{F}$ of δ_G



Two faces $f_1, f_2 \in \mathcal{F}$ are connected in χ_G if they share a crossing in δ_G.

We consider the *vertex cover number* of χ_G and show that MinMaxSwitches is FPT w.r.t this parameter.

$X \ldots$ vertex cover of χ_G

Proof sketch: Checking if $k = 0$ is polynomial-time solvable [1]. Assume w.l.o.g. $k \geq 1$, which allows the following reduction rule:

Reduction rule: Remove all edges e with at most two crossings in δ_G.

Without proof we claim that the following statements hold after exhaustive application of the reduction rule:

Claim 1. Each cell has at least three neighbors, all of which are in X.

Claim 2. Every triple $x_1, x_2.x_3 \in X$ has at most 2 cells adjacent to all three cells x_1, x_2, x_3.

Claim 1 and 2 imply that there are at most $2|X|^3 + |X|$ cells. Next we bound the total number of edges.

Claim 3. Each cell of δ_G with degree d in χ_G has at most $4d$ edges on its boundary.

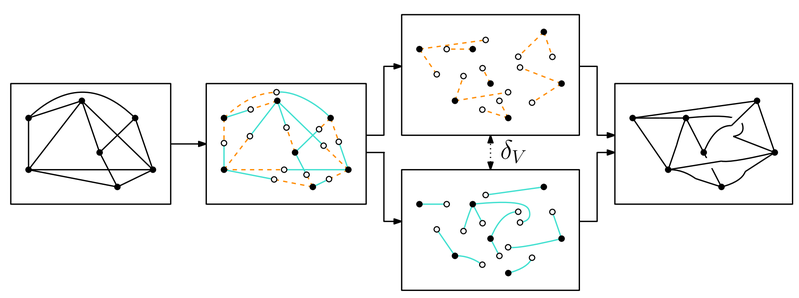
Since the maximum degree of cells outside of X is bounded by $|X|$ and the maximum degree of cells in X by $|X|^3 + |X|$; this gives us a kernel of size $O(|X|^4)$. □

What if the drawing is not fixed?

A graph G is k**-switch-planar** if it admits a cased drawing with at most k switches per edge. We obtain a full characterization of this notion.

Theorem. A graph G is 0-switch-planar if and only if it is biplanar, i.e. its edge set can be partitioned into two planar subgraphs.
Proof sketch. A biplanar graph can be partitioned into two planar edge sets. Assign all tunnels to the first set, all bridges to the second set.

Theorem. Any graph G is 1-switch-planar.
Proof sketch. (see figure) Subdivide all edges once, partition the subdivided edges into two sets E_1, E_2. Draw both sets – consisting of sets of stars – crossing-free with the same vertex positions. Rejoin the edges, assign all tunnels to E_1, all bridges to E_2, leading to at most one switch per edge.



[1] David Eppstein, Marc Van Kreveld, Elena Mumford, and Bettina Speckmann. Edges and switches, tunnels and bridges. *Computational Geometry*, 42(8):790–802, October 2009