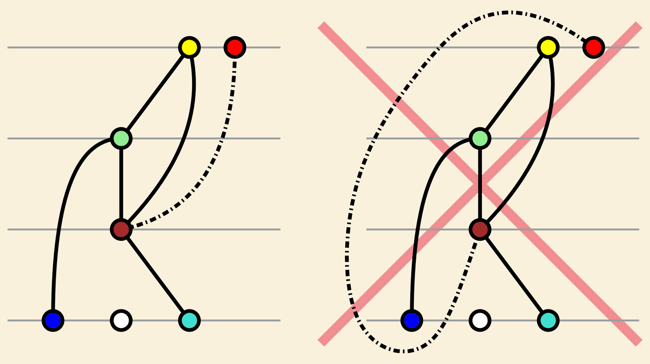
Level Planarity Is More Difficult Than We Thought

Simon D. Fink[1] • Matthias Pfretzschner[2] • Ignaz Rutter[2] • Peter Stumpf[3]

1 Technische Universität Wien
2 Universität Passau, Fakultät für Informatik und Mathematik
3 Charles University

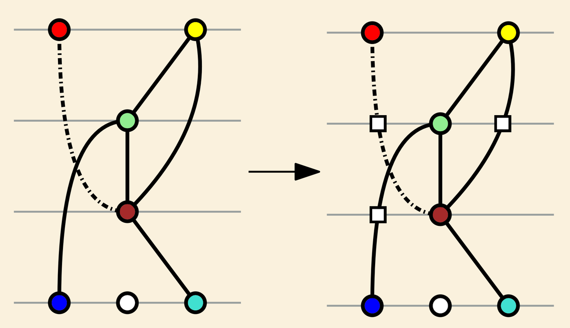
Level Planarity

Input:
Graph $G = (V, E)$
Leveling $\ell : V \to \mathbb{N}$



Problem:
Does G admit a planar drawing where
- each vertex v has y-coordinate $\ell(v)$
- edges are y-monotone

subdivide long edges to make inst. **proper**



Algorithms

Jünger et al. ('98, '02): involved $O(n)$ algorithm

subsequent slower but simpler algorithms

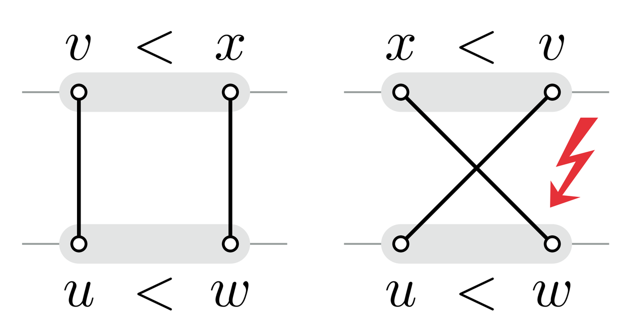
Randerath et al. 2001 [6]	Healy & Kuusik 2004 [3]	Harrigan & Healy 2007 [2]
$O(\|V\|^2)$	$O(\|V\|^3)$	$O(\|V\|^2)$

Counterexample

Randerath. et al [6]:

Equivalent 3-SAT Formulation:
- one variable $(a < b)$ for every ordered pair a, b of vertices on the same level
- constraints:

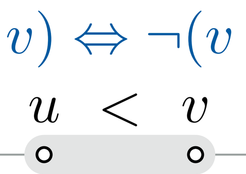
for independent edges vu, xw: $(v < x) \Leftrightarrow (u < w)$



Planarity

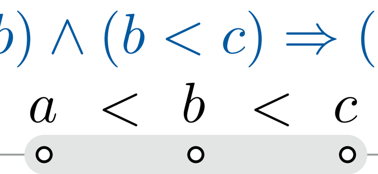
$+$

$(u < v) \Leftrightarrow \neg(v < u)$



Antisymmetry

$(a < b) \wedge (b < c) \Rightarrow (a < c)$



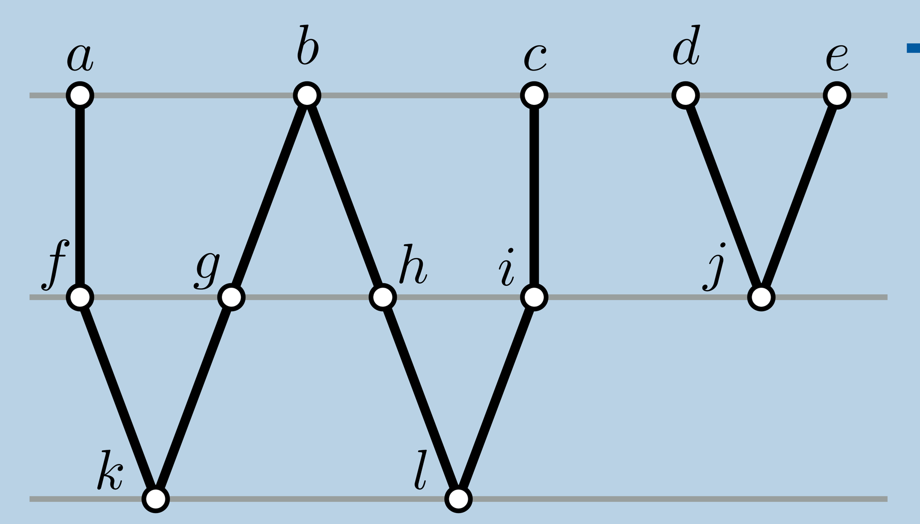
Transitivity

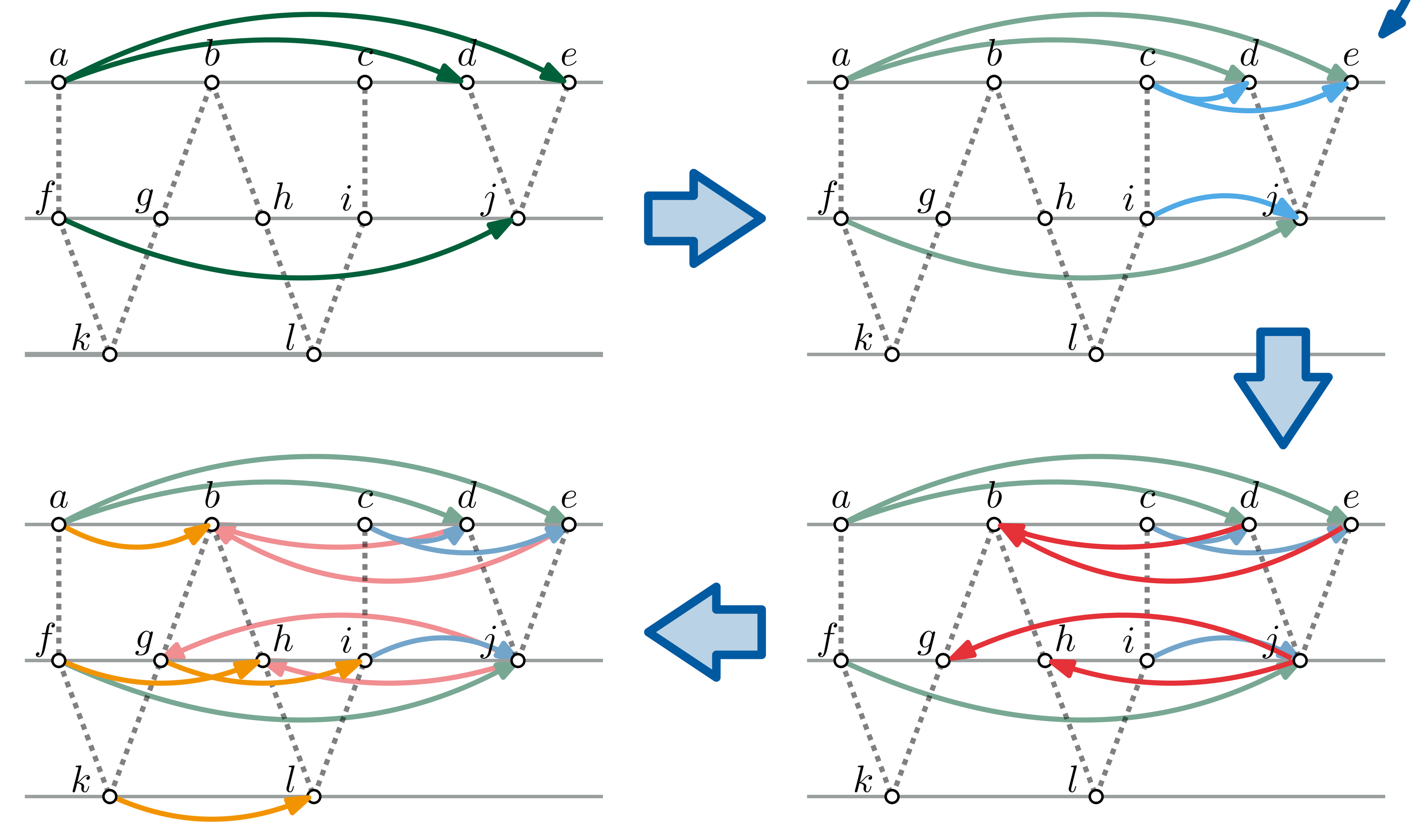
Theorem A [1, 6]
2-SAT formula w/o Transitivity satisfiable ⇔ graph is level planar

Proof [6]: greedily pick and assign arbitrary equivalence classes, but perform transitive closure when necessary

Claim: resulting embedding is level planar

proper level graph G



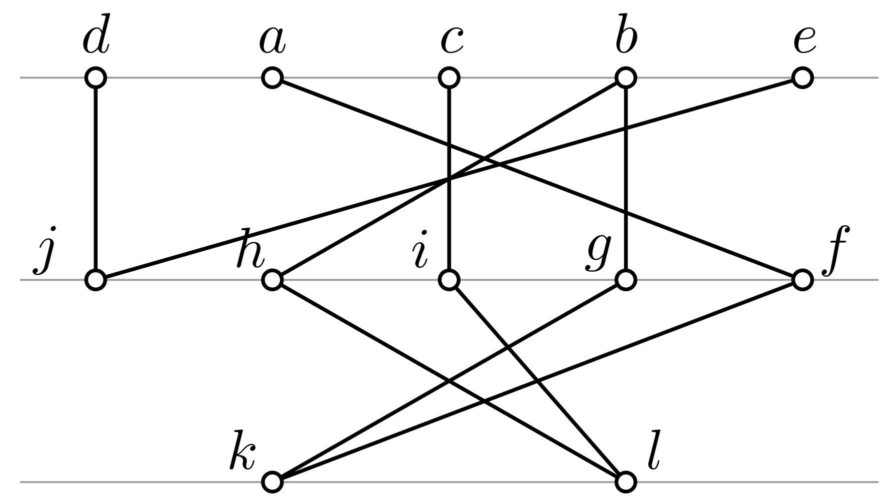


- $a < d < b$ transitively forces $a < b$
- $i < j < g$ transitively forces $i < g$
- planarity forces $a < b \Leftrightarrow f < h \Leftrightarrow k < l \Leftrightarrow g < i$

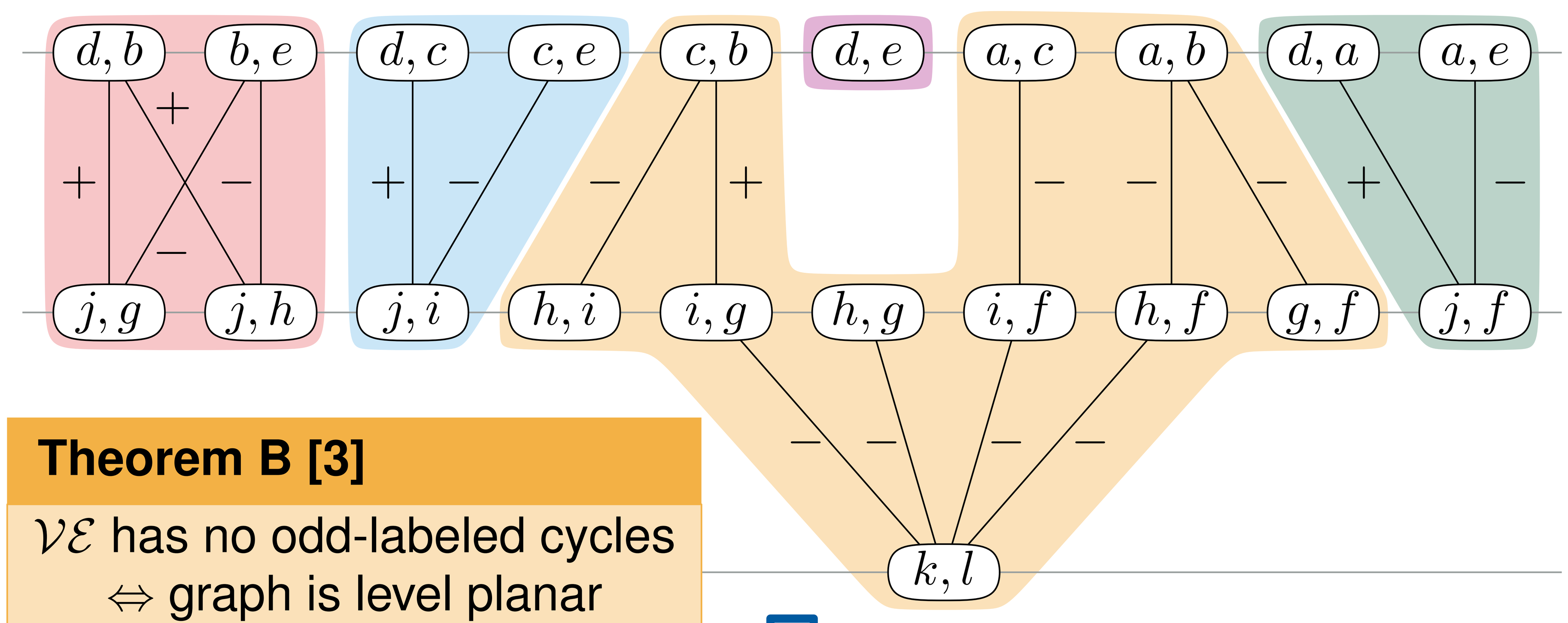
contradiction! → false-negative answer

Harrigan & Healy [2]:
(analogously for Healy & Kuusik [3])

start with arbitrary drawing $\mathcal{L}$:



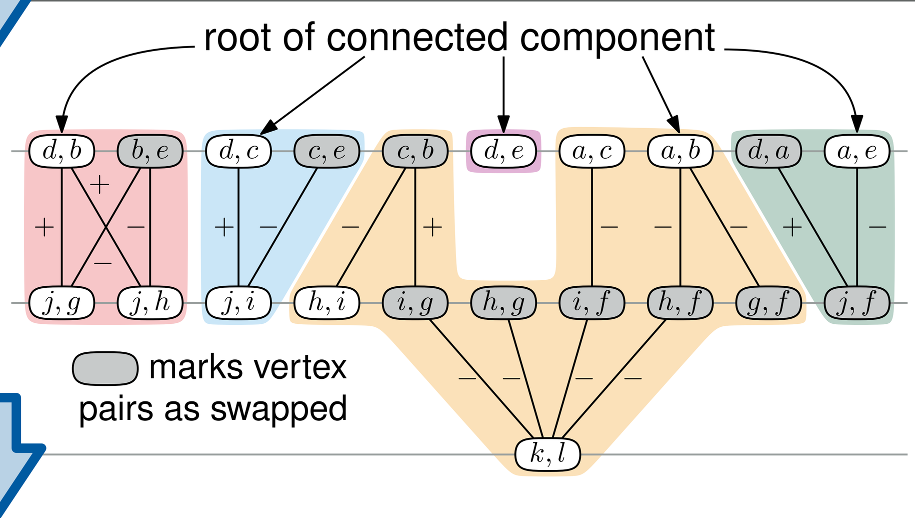
Construct **Vertex-Exchange Graph** $\mathcal{VE}$:
- one vertex for every ordered pair of vertices on the same level
- for independent edges vu, xw: connect vertices v, x and u, w; if they cross in $\mathcal{L}$ label edge $+$, else label it $-$



Theorem B [3]
$\mathcal{VE}$ has no odd-labeled cycles ⇔ graph is level planar

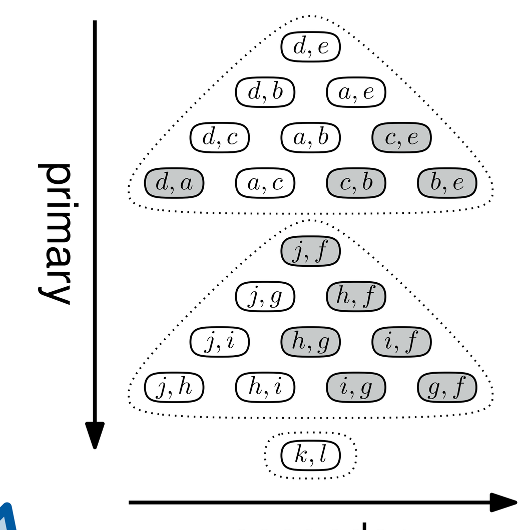
Algorithm 1 [2]

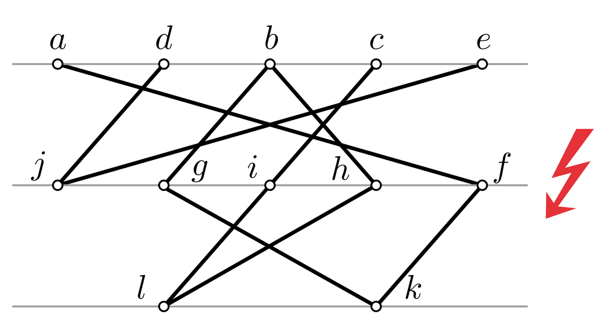
1. **for** *each connected component in* $\mathcal{VE}$ **do**
2. fix order of arbitrary root pair;
3. check for odd-labeled cycles via DFS;
4. swap pairs with odd-labeled root-path;



Algorithm 2 [2]

1. **for** *each pair, ordered first by level and then by asc. distance of vertices in* $\mathcal{L}$ **do**
2. **if** *first-encountered pair for con. comp.* **then**
3. fix according to $\mathcal{L}$;
4. **else if** *neither edges to later pairs nor match with remainder of comp.* **then**
5. exchange in $\mathcal{L}$;
6. **else if** *edge to later pair with label '$-$'* **then**
7. exchange in $\mathcal{L}$;





non-planar drawing

Summary: We have counterexamples to
- the embedder by Randerath [6]
- the correctness proof of Theorem A [6] (relies on the embedder)
- the embedder by Harrigan & Healy [2] / Healy & Kuusik [3]
- the correctness proof of Theorem B [3] (assumes that eq. classes can always be combined without contradiction)

To the best of our knowledge, this leaves
- no correct *simple* embedding algorithm
- no correct embedder implementation

Disclaimer: Theorem A and B can still be used for simple and correct quadratic tests for Level Planarity, as [1] shows their equivalence to the Hanani-Tutte Level Planarity characterization.

1. G. Brückner, I. Rutter, and P. Stumpf. Level-Planarity: Transitivity vs. Even Crossings. *Electron. J. Comb.*, 29(4), 2022.
2. M. Harrigan, P. Healy, Practical level planarity testing and layout with embedding constraints. *Graph Drawing '07, Lecture Notes in Computer Science*, 4875:62-68, 2007.
3. P. Healy, A. Kuusik, Algorithms for multi-level graph planarity testing and layout. *Theor. Comput. Sci.* 320(2–3):331–344, 2004.
4. M. Jünger, S. Leipert, Level planar embedding in linear time. *J. Graph Algorithm. Appl.* 6(1):67–113, 2002.
5. M. Jünger, S. Leipert, P. Mutzel, Level planarity testing in linear time, *Graph Drawing, Lecture Notes in Computer Science*, 1547:224–237, 1998.
6. B. Randerath, E. Speckenmeyer, E Boros, P. L. Hammer, A. Kogan, K. Makino, B. Simeone, and O. Cepek. A satisfiability formulation of problems on level graphs. *Electron. Notes Discret. Math.*, 9:269–277, 2001.

more infos & our interactive imple-mentation of [2]