Complexity of Positive Influence Domination on Partial Grids

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Abstract. Given an undirected graph, the Positive Influence Domination (PID) problem asks to specify a minimum cardinality subset of influencers among its vertices such that at least half the number of neighbors of each vertex are influencers. The underlying model can be used to determine an economical way of promoting (and keeping) good habits in society by interpreting vertices with individuals, their neighbors with their social contacts, and influencers with, e.g., healthy eaters. We show that the problem is NP-hard even when restricted to planar subcubic graphs. The same result turns out to apply for the so-called double total domination problem sharing a similar behavior on this graph class. We use this insight to derive NP-hardness of PID on the class of induced partial grids via a technique relying on orthogonal graph drawing. On complete $m \times n$ grids, we establish asymptotically tight primal and dual bounds, both of magnitude $\Theta(mn)$.

Keywords: Positive influence domination · NP-hardness · Partial grids.

1 Introduction

In 2009 Wang et al. [18] introduced the *Positive Influence Domination* (PID) problem, which asks, given an undirected simple graph, to find a minimum cardinality subset of so-called *influencing* vertices such that for each vertex of the graph a not necessarily strict majority of influencing vertices is present among its neighbors. Several closely related combinatorial optimization problems on graphs have been proposed in the form of *monopoly problems* [14], *α-domination* [6], and *signed domination* [5] to mention a few examples. Considering the aspect of temporal propagation of influence, the *Target Set Selection* problem [10] has been studied, too. A comprehensive comparison of many such problems can be found in [3,7]. Although these problems have slightly different requirements, e.g., concerning the influence which influencers themselves need to receive and deal with different notions of a majority, they have in common to be interpretable as a model for influence promotion/gain in social networks at a minimum cost.

This class of domination-type problems has been studied mainly from a theoretic view-point so far. For any scalar $\alpha \in (0, 1]$—steering the "strength" of a majority—the α-domination problem was shown to be NP-hard [6]. Its slight generalization *vector domination* [8] was shown to be fixed-parameter tractable on graphs excluding cycles of length 4 and on graphs of bounded degeneracy [16].

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The more constrained problem *total vector domination* [3], which carries PID as special case, was shown to be solvable in polynomial time on graphs of bounded branchwidth and bounded treewidth [9]—the same was shown to hold for vector domination. Appearing under manifold names in the literature, *k-tuple total domination* (see, e.g., [11]) is a particularly uniform special case of total vector domination, for which NP-hardness on the class of undirected path graphs (a special case of chordal graphs) was shown in [11]; NP-hardness was also shown on bipartite and split graphs [15].

Concerning complexity theoretic results on PID itself, it is shown in [19] that PID is APX-hard; this is also true more generally when, instead of a majority, at least a fixed proportion of neighbors is required to consist of influencers [3]. Recently, NP-hardness for the PID problem on bipartite and split graphs has been established [20]. NP-hardness was shown the first time in [19] by a reduction from vertex cover (via a proof in fact affirming this for graphs with a maximum degree of 6). The latter work shall also be our main motivation to identify more restrictive graph classes for which the hardness still applies.

The paper is organized as follows. After providing relevant notation in Secion 2, in Section 3 we show that PID and 2-tuple total domination are NP-hard problems on the class of planar subcubic graphs. The result is lifted to NP-hardness of PID on induced partial grids in Section 4, then leads to the study of bounds on complete grids in Section 5. Section 6 concludes the paper.

2 Notation and preliminaries

For two numbers r and s, we use the notation $[r:s] := \{r, r+1, \ldots, s-1, s\}$ and $[s] := [1:s]$. For $a, b \in \mathbb{Z}$, $b \neq 0$, denote by $\mathrm{mod}(a,b)$ the binary modulo operator (returning the unique number $r \in [0:b-1]$ satisfying $a = kb + r$ for some $k \in \mathbb{Z}$).

In our discussion, we assume graphs $G = (V, E)$ to be simple and undirected. For a vertex $v \in V$ we denote its neighborhood of adjacent vertices by $N(v)$ and its degree by $\deg(v) := |N(v)|$. A graph $G = (V, E)$ with $\max_{v \in V} \deg(v) \leq k$ is called *k-subregular*—or *subcubic* for $k = 3$. If further $\min_{v \in V} \deg(v) = k$, it is called *k-regular*—or *cubic* for $k = 3$. We say that G is k-connected, if G remains connected when fewer than k vertices are removed. Denote by $\mathcal{G}_3^{\mathrm{plan}}$ and $\mathcal{G}_{\leq 3}^{\mathrm{plan}}$ the class of planar cubic and planar subcubic graphs, respectively.

For the following we consider as a length-r path a graph $P_r = (V, E)$ with $V = [r]$ and $E = \{\{i, i+1\} : i \in [r-1]\}$. Denote by $P_m \Box P_n$ the Cartesian product of P_m and P_n, henceforth also called *complete $m \times n$ grid (graph)*. Its vertices are given by all pairs in $[m] \times [n]$; two such vertices are defined as adjacent if their first coordinates or second coordinates form an adjacent pair in their respective path of origin P_m or P_n. Let us think of $P_m \Box P_n$ as embedded in the plane, with vertices on integer-valued coordinates and unit-length straight, axes-parallel edges, oriented as the indices of an $m \times n$ matrix. By a *partial grid* we refer to an arbitrary (not necessarily induced) subgraph of a grid, in other words, a graph resulting from the removal of a nonnegative number of vertices and edges from $P_m \Box P_n$ for some specific $m, n \in \mathbb{N}$; here, a vertex-removal automatically removes all incident edges. A *unit disk graph* has vertices identifiable with points in the plane such that two vertices are adjacent iff the two compact unit disks centered at the respective points have nonempty intersection.

Although many of the subsequent variants of domination/independence are typically defined via subsets of vertices, let us state them for easier usability in a 0-1-encoded fashion: For a graph $G = (V, E)$, a function $f : V \to \{0, 1\}$ is *independent* if $f(v) + f(w) \leq 1$ for all $\{v, w\} \in E$. It is said to be (i) *positive influence dominating* [18] if $\sum_{w \in N(v)} f(w) \geq \lceil \deg(v)/2 \rceil$ for $v \in V$; and (ii) *double total dominating* if $\sum_{w \in N(v)} f(w) \geq 2$ for $v \in V$. When f is positive influence dominating we will often abbreviate this by saying that f is *PID-feasible* or a *PID-function.* When $k = 2$, k-tuple total domination mentioned in the introduction is precisely double total domination. For such $\{0, 1\}$-valued functions f let us call $\sum_{v \in V} f(v)$ the *(cumulative) weight* of f. Sometimes we call the f-value assigned to $v \in V$ the *weight* or *label* of v.

Given a graph $G = (V, E)$ and $k \in \mathbb{N}$, the problem of determining the existence of an independent $g : V \to \{0, 1\}$ with $\sum_{v \in V} g(v) \geq k$ is denoted as $\textsc{IndependentSet}(G, k)$. Similarly, the problem of determining the existence of $f : V \to \{0, 1\}$ with $\sum_{v \in V} f(v) \leq k$ being (i) positive influence dominating is denoted as $\textsc{PosInfluenceDom}(G, k)$; (ii) double total dominating is denoted as $\textsc{DoubleTotalDom}(G, k)$.

For fixed G, the smallest $k \in \mathbb{N}$, for which such a feasible function exists, is denoted as $\gamma_{\mathrm{PID}}(G)$ for (i), and $\gamma_{\times 2, \mathrm{t}}(G)$ for (ii).

3 Hardness on planar subcubic graphs

The findings of the current, respectively next section are inspired by the works [4], respectively [21] addressing the hardness of the so-called *signed dominating set* problem. In the latter problem, a minimum-weight function $f : V \to \{-1, 1\}$ has to be found such that $f(v) + \sum_{w \in N(v)} f(w) \geq 1$ for $v \in V$. Although the values assigned to vertices do not have the "neutral" option of zero as in the setting of positive influence domination and although the f-value of a vertex itself contributes to fulfilling its local "influence" threshold, we obtain comparable hardness results for $\textsc{PosInfluenceDom}$. In contrast to Damaschke [4] at that time, we can now fall back on a slightly stronger result (allowing to assume 2-connectedness) forming the cornerstone of our argumentation.

Theorem 1 ([13, Theorem 4.1(a)]). *The maximum independent set problem is NP-hard when restricted to planar* 2*-connected cubic graphs.*

It is easy to recognize that the subsequent Theorems 2–3 also hold when the concerned graph classes are strengthened by the additional assumption of connectedness. The assertions of the theorems ignore this insight, which also follows from the fact that solving the addressed problems can be done separately on each connected component.

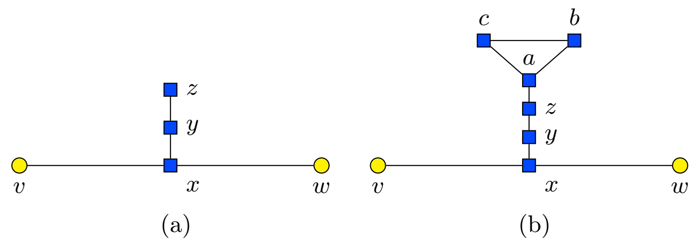

Fig. 1. Modifications of an edge $e = \{v, w\}$ used in the proof of (a) Theorem 2; and (b) Theorem 3. Subscripts e are omitted for the vertices x, y, z, a, b, and c.

Consider the planarity-preserving, polynomial-time operation $\mathsf{SplApd} : \mathcal{G}_3 \to \mathcal{G}_{\leq 3}$ replacing each edge $e = \{v, w\} \in E$ of $G = (V, E)$ with new edges $\{v, x_e\}$, $\{w, x_e\}$, $\{x_e, y_e\}$, $\{y_e, z_e\}$ on top of fresh vertices x_e, y_e, z_e; see Fig. 1a. Define $\mathcal{SP}^{\text{plan}}_{\leq 3}$ as the class of all graphs that are—up to isomorphy—contained in $\{\mathsf{SplApd}(G) : G \in \mathcal{G}_3\}$. Note that by detecting the vertices at distance at most 2 from the leaves of $G' \in \mathcal{SP}^{\text{plan}}_{\leq 3}$, we can always determine the underlying cubic graph, even when no explicit specification of the fresh vertices is provided.

Theorem 2. *When restricted to the class of planar subcubic graphs, the problem* PosInfluenceDom(G, k) *is NP-complete.*

Proof. More strongly we prove that the problem is NP-complete on $\mathcal{SP}^{\text{plan}}_{\leq 3}$, directly implying NP-completeness on $\mathcal{G}^{\text{plan}}_{\leq 3}$. The reduction is from maximum independent set on planar 2-connected cubic graphs; see Theorem 1.

The problem is in NP, as a guessed function can be checked for feasibility in linear time (in the size of G).

Let $G = (V, E)$ be a planar 2-connected cubic graph and let $G' := \mathsf{SplApd}(G)$, denoted as $G' = (V', E')$. We claim that G has an independent $g : V \to \{0, 1\}$ with $\sum_{v \in V} g(v) \geq k$ iff G' has a PID-feasible $f : V' \to \{0, 1\}$ with $\sum_{v' \in V'} f(v') \leq k'$ for some k' depending on G and k. In fact, if an independent g with

$$\sum_{v \in V} g(v) \geq k \tag{1}$$

exists, then the function

$$f(v') := \begin{cases} 1 - g(v') & \text{if } v' \in V, \\ 1 & \text{if } v' \in \{x_e, y_e : e \in E\}, \\ 0 & \text{if } v' \in \{z_e : e \in E\}, \end{cases}$$

is positive influence dominating on G': each vertex in V has three 1-labeled x-neighbors; all vertices in $\{x_e, y_e, z_e : e \in E\}$ have one 1-labeled neighbor; and, finally, the absence of an assigned 1-label by g to one of v and w (according to the independence of g) guarantees at least one additional 1-label assigned by f to one of the neighbors v or w of $x_{\{v,w\}}$.

We conclude that (below obtaining estimate (2) via (1))

$$\begin{aligned}\sum_{v'\in V'} f(v') &= \sum_{v\in V} f(v) + \sum_{v'\in V'\setminus V} f(v') = |V| - \sum_{v\in V} g(v) + \sum_{v'\in V'\setminus V} f(v')\\ &\leq |V| + \sum_{v'\in V'\setminus V} f(v') - k = |V| + 2|E| - k =: k'. \end{aligned} \tag{2}$$

The converse proof direction requires us to show that for a PID-function f on G' the circumstance $\sum_{v'\in V'} f(v') \leq k'$ allows constructing an independent g on G of cumulative weight at least $k = |V|+2|E|-k'$. We need the intermediate observation that each positive influence dominating $f : V' \to \{0,1\}$ can be updated on $V' \setminus V$ in a weight-preserving—or even weight-reducing—manner such that for each $e \in E$ we eventually have more canonically $f(x_e) = f(y_e) = 1$ and $f(z_e) = 0$.

This simple claim can be justified as follows: We recognize that always $f(y_e) = 1$ due to $\deg(z_e) = 1$. In case now $f(z_e) = 1$, we can overwrite $f(z_e) \leftarrow 0$ and if not already $f(x_e) = 1$, we update $f(x_e) \leftarrow 1$ to maintain the influence threshold of y_e—the cumulative weight does not increase by doing so. On the other hand, if initially $f(z_e) = 0$, necessarily we have already $f(x_e) = 1$.

We can therefore assume that a PID-feasible f with $\sum_{v'\in V'} f(v') \leq k'$, further satisfying the above canonicity, exists. Setting $g := 1 - f(\cdot)$ yields independence for g, as $f(v) + f(w) \geq 1$ implies $g(v) + g(w) \leq 1$. Furthermore, we have

$$\begin{aligned}\sum_{v\in V} g(v) &= |V| - \sum_{v\in V} f(v) = |V| - \sum_{v\in V} f(v) - \sum_{v'\in V'\setminus V} f(v') + \sum_{v'\in V'\setminus V} f(v')\\ &= |V| - \sum_{v'\in V'} f(v') + 2|E| \geq |V| - k' + 2|E|.\end{aligned}$$

□

Corollary 1. PosInfluenceDom(G, k) *is NP-complete on planar subcubic* bipartite *graphs.*

Proof. The vertices of $G' \in \mathcal{SP}^{\text{plan}}_{\leq 3}$ with an underlying cubic graph $G = (V, E)$ are bi-partitioned by $V \cup \{y_e : e \in E\}$ and $\{x_e, z_e : e \in E\}$. □

Similarly, we obtain the following result.

Theorem 3. DoubleTotalDom(G, k) *is NP-complete even when restricted to planar subcubic graphs.*

Proof. We fall back on a leaves-free class of subcubic graphs resulting from the replacement of the edges of a cubic graph by the gadgets described in Fig. (1b). Analogously to the proof of Theorem 2, we can here carry out the argumentation with $k' := |V| + 6|E| - k$ and $f(v') := 1 - g(v')$ if $v' \in V$ and otherwise, if $v' \in \{x_e, y_e, z_e, a_e, b_e, c_e : e \in E\}$, $f(v') := 1$. For the converse proof direction the degree of freedom for the choice of f-values is now considerably lower than in the last proof, as, to be feasible for DoubleTotalDom, all fresh vertices necessarily must be 1-labeled. □

4 Hardness on induced partial grids

In this section we lift the result in Theorem 2 to the class of induced partial grids. We start with the following preparatory observation.

Lemma 1. *Consider a path $P = (v, x_1, x_2, \ldots, x_{4\ell-1}, x_{4\ell}, w)$ for some $\ell \in \mathbb{N} \cup \{0\}$ and $f : \{v, w\} \cup \{x_j : j = 1, \ldots, 4\ell\} \to \{0, 1\}$ such that each vertex x_j, $j = 1, \ldots, 4\ell$, has at least one 1-labeled neighbor. Suppose further that $f(v) = f(w) = 1$. Then, $\sum_{j=1}^{4\ell} f(x_j) \geq 2\ell$, where the lower bound 2ℓ can be attained.*

Proof. Examine f on the induced subpaths $(x_1, x_2, x_3, x_4), \ldots, (x_{4\ell-3}, x_{4\ell-2}, x_{4\ell-1}, x_{4\ell})$. By exhaustion it turns out that each of them, independently of the f-assignment on their neighboring paths, must have a cumulative f-weight of at least 2. Thus, $\sum_{j=1}^{4\ell} f(x_j) \geq 2\ell$. This weight is attainable for $f(x_j) := 0$, when $j \equiv 1 \vee j \equiv 2 \pmod 4$; $f(x_j) := 1$, otherwise. □

We will consider such inter-vertex paths for any two adjacent vertices v and w. The paths' lengths will again be zero or multiples of 4 but can vary depending on the edge $e = \{v, w\}$. The following lemma affirms that, after inserting a number of splitting vertices divisible by 4 on each edge, a PID-function of the same quality can be achieved (up to an additional constant originating from the number of splitting vertices). The core argument of the proof is that the labels on the splitting vertices can be chosen such that the constellations of neighboring labels remains invariant around all original vertices. The fresh labels can further be feasibly chosen such that the weight does not exceed the half of the count of splitting vertices. The converse proof direction requires a more technical canonization process in the style of the proof of Theorem 2.

Lemma 2. *Let $k \in \mathbb{N} \cup \{0\}$. Further, let $G = (V, E)$ be a subcubic graph, and $L_e \in \mathbb{N} \cup \{0\}$ be divisible by 4, for $e \in E$. Consider $G' = (V', E')$ resulting from replacing each $e = \{v, w\} \in E$ by an undirected path $(v, p_e^1, \ldots, p_e^{L_e}, w)$ of length $L_e + 1$, i.e., the former edge e is split by L_e fresh vertices. Then, the following assertions are equivalent.*

(i) A PID-feasible f on G with $\sum_{v \in V} f(v) \leq k$ exists.
(ii) A PID-feasible f' on G' with $\sum_{v \in V'} f'(v) \leq k + \sum_{e \in E} L_e/2$ exists.

Proof. See Appendix, proof 2. □

Next, let us show how the previous result is related to partial grids. In advance we provide the notion of an $\mathcal{R}$-*curve* referring to a rectilinear polygonal chain made up of integral-length line segments forming a simple curve (meaning that it is injectively parameterizable, or in other words not self-intersecting). The argumentation relies on the following auxiliary Lemma due to Valiant [17].

Lemma 3 (adapted from [17, Theorem 2]). *A planar graph G with degree at most 4 can be embedded in polynomial time in the plane using $O(|V|^2)$ area such that the following properties hold: (i) Embedded vertices possess integer-valued coordinates; (ii) each embedded edge is an $\mathcal{R}$-curve; and (iii) no edge-crossings occur, i.e., any two embedded edges do not intersect except possibly on jointly incident embedded vertices.*

Theorem 4. PosInfluenceDom(G,k) *is NP-complete on the class containing all induced partial grids.*

Proof. From the proof of Theorem 1 we know that PosInfluenceDom(G,k) is NP-complete on $\mathcal{SP}^{\text{plan}}_{\leq 3}$. We will rely on a reduction from the latter.

Starting from $G' = (V', E') \in \mathcal{SP}^{\text{plan}}_{\leq 3}$ consider also its underlying cubic graph $G = (V, E)$, i.e., $\mathsf{SplApd}(G) = G'$. The first goal will be to find a particular embedding for G' by adapting an embedding for G. This is accompanied by an illustration in Fig. 2 and works as follows:

Using Lemma 3, planarly draw in $\mathbb{R}^2$ the embedding of G such that embedded vertices have integer-valued coordinates and all its embedded edges are $\mathcal{R}$-curves; to prevent an information-theoretic blow-up of the data, make sure the coordinates of the embedded edges are bounded as in Lemma 3. Afterwards, subject the drawing in $\mathbb{R}^2$ to the geometric scaling $(x, y) \mapsto (8x, 8y)$. For each edge $\{v, w\}$ of G, prolong by two units the associated (now scaled) $\mathcal{R}$-curve $\rho_{\{v,w\}}$ by the following procedure consisting of two steps:

Step 1. Fix an arbitrary endpoint $p = p_{\{v,w\}} \in \{v, w\}$ of the edge and consider the length-4 line subsegment $s(p, \{v, w\}; 2, 6)$ of $\rho_{\{v,w\}}$ starting at Manhattan distance 2 from p and ending at Manhattan distance 6 from p. Let us translate $s(p, \{v, w\}; 2, 6)$ by one unit in direction orthogonal to the extension direction of $s(p, \{v, w\}; 2, 6)$ itself. The now modified $\rho_{\{v,w\}}$ consists of three disconnected components, which we join by two unit-length line segments, yielding our definite form of $\rho_{\{v,w\}}$. The length of the original $\mathcal{R}$-curve $\rho_{\{v,w\}}$ was hence artificially increased by two units by performing a local detour geometrically reminding of a rectangular U-turn. This guarantees that all present $\mathcal{R}$-curves are now of length congruent 2 modulo 8. The absence of crossing edges is clearly maintained.

Step 2. Determine on $\rho_{\{v,w\}}$ the point at geodesic distance 5 from p (according to the geometry of the curve $\rho_{\{v,w\}}$ itself). Let us regard it as the embedded "splitting" vertex $x_{\{v,w\}}$ of G'. Then, append to the $x_{\{v,w\}}$-embedding a fresh length-2 straight path, meant to host the embeddings of $y_{\{v,w\}}$ and $z_{\{v,w\}}$, in a way ensuring that one coordinate-entry of $y_{\{v,w\}}$ is divisible by 8. Eventually, we have found an embedding with $\mathcal{R}$-curves for all edges of G'.

Note that for each edge $e = \{v, w\}$ of G, the number of lattice points of $\mathbb{Z} \times \mathbb{Z}$ which are covered by the $\mathcal{R}$-curve of G' connecting $x_{\{v,w\}}$ and v is divisible by 4 when this number renounces counting the embedded endpoints $x_{\{v,w\}}$ and v.

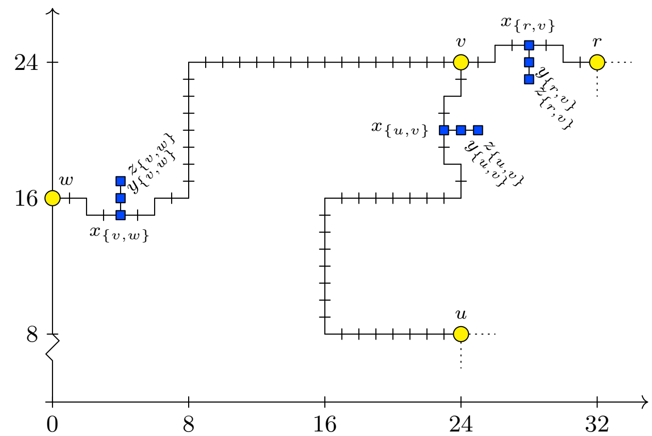

Fig. 2. Up to the detours (visible as lengthy U-turns) and the paths between the quadratic vertices, the drawing corresponds to an initial $\mathcal{R}$-curve embedding scaled by a factor of 8. The Euclidean distance between u and v, e.g., is assumed to have mutated from initially 2 to eventually 16. The splitting vertices x_e, $e \in E$, lie at geodesic distance 5 from an endpoint according to the respective $\mathcal{R}$-curve.

The same is true for the $\mathcal{R}$-curve connecting $x_{\{v,w\}}$ and w. On these $x_{\{v,w\}}$-incident $\mathcal{R}$-curves let us consider these intermediate $\mathbb{Z} \times \mathbb{Z}$ points as embedded fresh subdivisors of the combinatorial edges of G; call this new graph $\tilde{G}$. In particular, as they lie at Manhattan distance 1, no combinatorial subdivision of the edges $\{x_e, y_e\}, \{y_e, z_e\}$ occurs. $\tilde{G}$ is a (subcubic planar) induced partial grid and has the form of the transformed graphs from Lemma 2 with

$$\begin{aligned} 4 \leq L_{\{v,x_e\}} = L_{\{w,x_e\}} \equiv 0 \pmod 4, \\ 0 = L_{\{x_e,y_e\}} = L_{\{y_e,z_e\}} \equiv 0 \pmod 4, \end{aligned}$$

for all $e = \{v, w\} \in E$. Thus, by Lemma 2, the existence of a feasible f for PosInfluenceDom(G, k) is equivalent to the existence of a feasible f' on the induced partial grid $\tilde{G}$ of weight at most $k + \sum_{e=\{v,w\} \in E} L_{\{v,x_e\}}/2 + L_{\{w,x_e\}}/2$. □

The latter result has implications for another class on which domination-type problems are often considered.

Corollary 2. PosInfluenceDom(G, k) *is NP-complete on the class of unit disk graphs.*

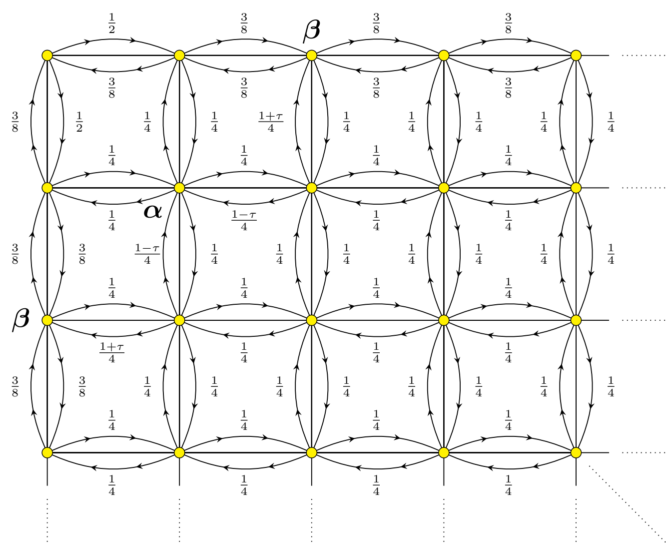

Fig. 3. The top-left area of an $m \times n$ grid subject to the discharging process. Coefficients attached to the arrows indicate the proportion of transferred "charge". Vertices in the set A or B from the proof of Theorem 6 are marked by α or β, respectively. Assuming exemplarily that v is the upper β-marked vertex, the situation $f(v) = 1$ implies that the two neighbors of v on the grid's upper boundary receive additional charge of $3/8$ whereas the interior neighbor just $1/4$; if $f(v) = 0$, a zero-charge transfer to neighbors applies.

5 Bounds on complete grids

We focus now on giving tight bounds for γ_{PID} and $\gamma_{\times 2,\mathrm{t}}$ on *complete* grids $P_m \Box P_n$. For this graph class the specifications of $\gamma_{\times 2,\mathrm{t}}$ are strictly more constrained than those of γ_{PID}. In fact, tighter lower bounds for $\gamma_{\times 2,\mathrm{t}}$ will be found.

Theorem 5. *For $m, n \geq 6$ we have*

$$\frac{mn}{2} + \frac{m+n}{4} - \frac{3}{2} \leq \gamma_{\mathrm{PID}}(P_m \Box P_n) \leq \frac{mn}{2} + \frac{m+n}{2} + \frac{3 - |\operatorname{mod}(n,4) - 1|}{2}. \quad (3)$$

Proof. As any double total dominating function on grids is automatically positive influence dominating, the upper bound of [12] for $\gamma_{\times 2,\mathrm{t}}$ is valid for γ_{PID}, too; the latter is stated as the right-hand side of (3). For $\gamma_{\times 2,\mathrm{t}}$ it constitutes the currently tightest upper bound up to exceptional congruencies of m, n modulo 4, where it worsens by 1 the older bound in [2].

Let us prove the lower bound by using the so-called *discharging method*, arguably best known due to the proof of Appel and Haken [1] of the Four Color Theorem. We transform a double total dominating $f : [m] \times [n] \to \{0, 1\}$ to a function $g : [m] \times [n] \to \mathbb{Q}$ in a weight-preserving manner, i.e., $\sum_{v \in [m] \times [n]} f(v) =$

$\sum_{v\in[m]\times[n]} g(v)$ and prove the bound for g: To obtain g, we simultaneously subtract from each f-value of a vertex this f-value and entirely redistribute this quantity ("charge"), according to a specific convex combination, among all its neighbors.

Let A be the set of vertices having precisely two degree-4 neighbors (lying at Manhattan distance 2 from a corner of the grid) and B be the set of vertices having only one degree-4 neighbor and being at distance 2 from a corner (for both see Fig. 3).

In the following, we directly state the function g resulting from a specific discharging process sketched in Fig. 3. Apart from fetching it from Fig. 3, more formally, the proportionality-scalar of the charge-transfer from w to v can be read off the respective case for v in (4) from the coefficient of the unique summand associated to the index w. Denote by D_t the preimage $\{v \in [m]\times[n] : \deg(v) = t\}$. For the parameter $\tau \in \{0, 1\}$ consider

$$g^\tau(v) := \begin{cases} \sum_{w\in N(v)\cap D_3} \frac{f(w)}{4} + \sum_{w\in N(v)\cap D_4} \frac{(1-\tau)f(w)}{4} & \text{if } v \in A, \\ \sum_{w\in N(v)\cap D_3} \frac{3f(w)}{8} + \sum_{w\in N(v)\cap D_4} \frac{(1+\tau)f(w)}{4} & \text{if } v \in B, \\ \sum_{w\in N(v)\setminus D_4} \frac{3f(w)}{8} + \sum_{w\in N(v)\cap D_4} \frac{f(w)}{4} & \text{if } v \in (D_2 \cup D_3) \setminus B, \\ \sum_{w\in N(v)\cap D_4} \frac{f(w)}{4} & \text{if } v \in D_4 \setminus A. \end{cases} \tag{4}$$

The parameter τ is introduced just to be able to prove Theorem 6 as well via (4). In this proof, we are interested only in the discharging process associated to $g := g^0$. As f is a PID-function, we have

$$\begin{aligned} \sum_{v\in[m]\times[n]} g(v) &= \sum_{v\in D_4} g(v) + \sum_{v\in D_3} g(v) + \sum_{v\in D_2} g(v) \\ &\geq \frac{1}{2}\,|D_4| + \frac{5}{8}\,|D_3| + \frac{3}{8}\,|D_2| \\ &= \frac{(m-2)(n-2)}{2} + \frac{5}{8}(2m-4+2n-4) + \frac{3}{8}\cdot 4 \\ &= \frac{mn}{2} + \frac{m+n}{4} - \frac{3}{2}. \end{aligned}$$

Here, the three sums have been estimated from below one by one with the following justification: By feasibility of f, every $v \in D_4$ has at least two neighbors with an f-value of 1; this simply allows to conclude that $g(v) \geq 1/4 + 1/4$, for $v \in D_4$. The estimate for each vertex in D_3 holds due to the fact, that again two neighbors with an f-value of 1 must exist leading in the lowest possible case to an inflow of charge $1/4 + 3/8 = 5/8$. The charge inflow for a corner must be at least $3/8$. □

Almost verbatim following the strategy of the proof of Theorem 5, using $g := g^1$ from (4), we obtain the following result.

Theorem 6. *For $m, n \geq 6$ we have*

$$\frac{mn}{2} + \frac{m+n}{4} + 1 \leq \gamma_{\times 2,\mathrm{t}}(P_m \Box P_n) \leq \frac{mn}{2} + \frac{m+n}{2} + \frac{3 - |\operatorname{mod}(n,4) - 1|}{2}. \quad (5)$$

Proof. For the sake of completeness the Appendix contains a full proof 6. $\square$

Despite the simplicity of the discharging approach, the bound (9) on $\gamma_{\times 2,\mathrm{t}}$ slightly strengthens—the difference is precisely 1—the lower bound given in [12]. However, the more striking insight lies here in the possibility to give a considerably shorter proof for the lower bound than in the latter work.

6 Conclusion

We showed that already the class of planar subcubic graphs is expressive enough such that PosInfluenceDom(G, k) and DoubleTotalDom(G, k) become NP-complete problems when restricted to them. The employed reductions in Sect. 3 suffer a loss of characteristics, which raises two open questions: Is it possible to determine a small k (perhaps even $k = 3$) such that the hardness result of Theorem 2 still applies for k-subregular graphs in the setting of 2-*connected* planarity? Can we adapt the reductions to work for 3-regular instead of 3-subregular graphs?

Using a geometry-accented argument, we observed that NP-completeness for the former problem is inherited by the class of induced partial grids. Concerning the complete $m \times n$ grids we showed how to derive a lower bound for γ_{PID} by a short argument, which applied to the setting of $\gamma_{\times 2,\mathrm{t}}$ slightly tightens the known lower bound.

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A Appendix

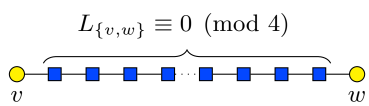

Fig. 4. Splitting each edge of the initial graph by a number of vertices divisible by 4.

Lemma 4. *Let $k \in \mathbb{N} \cup \{0\}$. Further, let $G = (V, E)$ be a subcubic graph, and $L_e \in \mathbb{N} \cup \{0\}$ be divisible by 4, for $e \in E$. Consider $G' = (V', E')$ resulting from replacing each $e = \{v, w\} \in E$ by an undirected path $(v, p_e^1, \ldots, p_e^{L_e}, w)$ of length $L_e + 1$, i.e., the former edge e is split by L_e fresh vertices. Let $U = \{(0, 1, 1, 0), (0, 0, 1, 0), (0, 0, 1, 1), (0, 1, 1, 1)\}$. If there exists a PID-feasible f' with $\sum_{v \in V'} f'(v) = K$, then there exists a PID-feasible $f'' : V' \to \{0, 1\}$ satisfying $(f''(v), f''(p_e^1), f''(p_e^{L_e}), f''(w)) \notin U$, for $e = \{v, w\} \in E$, and $\sum_{v' \in V'} f''(v') \leq K$.*

Proof. An existing labeling f'' with the required properties is given by the output of Algorithm 1. In the subsequent let us prove its soundness.

PID-feasibility. The periodicity-4 pattern are designed to meet PID-feasibility certainly on the non-endpoint vertices of the inter-vertex path. On the two endpoints of the length-L_e pattern, PID-feasibility is given due to the increased label of v (and w).

Correctness and termination. The fact that constellations from U are successfully entirely removed and that the algorithm terminates is explained as follows: Initially $|M^{f''}_{(0,1,1,0)}|$ is a finite number. Each iteration of the loop removes a single occurrence of a type-$(0, 1, 1, 0)$ constellation and does not cause a novel one of this type somewhere else in the graph to be handled in a later iteration—in fact, the overwriting process only increases labels for vertices in V. Similarly, the second while loop terminates: By the increased labels in the overwriting process in the iterations, no novel type-$(0, 0, 1, 1)$ constellations of are caused. Furthermore, no novel occurrences of constellations eliminated in Step 1 are caused. The same can be observed for the third while loop, where no novel type-$(0, 0, 1, 1)$ constellations are caused. Additionally, none of the constellations eliminated in the prior two loops are caused. Finally, termination of the last loop is justified analogously. We observe that no constellation eliminated in prior loops is caused by the overwriting processes of the last loop, too.

No excess of weight. We show that in every loop, each iteration does not increase the weight.

Loop 1: Note that before overwriting, $(p_e^3, \ldots, p_e^{L_e-2})$ is a path of length divisible by 4, hence, according to Lemma 1 its weight will be at least $(L_e - 4)/2$ regardless of its assigned values by f'. Necessarily $f'(p_e^2) = f'(p_e^{L_e-1}) = 1$

procedure *Canonize*$(V, E; (L_e)_{e \in E}; f')$
  $f'' \leftarrow f'$ // create a working copy of f'
  $M^{f''}_{(q_1,q_2,q_3,q_4)} := \{\{v,w\} \in E : (f''(v), f''(p_e^1), f''(p_e^{L_e}), f''(w)) = (q_1, q_2, q_3, q_4)\}$
  while $M^{f''}_{(0,1,1,0)} \neq \emptyset$ **do**
    Pick (v,w) from $M^{f''}_{(0,1,1,0)}$.
    Overwrite $F''_{(v,w)} \leftarrow (1,0,0,1,\ldots,1,0,0,1)$.
    Overwrite $f''(v) \leftarrow 1$, $f''(w) \leftarrow 1$.
  while $M^{f''}_{(0,0,1,0)} \neq \emptyset$ **do**
    Pick (v,w) from $M^{f''}_{(0,0,1,0)}$.
    Overwrite $F''_{(v,w)} \leftarrow (0,0,1,1,\ldots,0,0,1,1)$.
    Overwrite $f''(v) \leftarrow 1$.
  while $M^{f''}_{(0,0,1,1)} \neq \emptyset$ **do**
    Pick (v,w) from $M^{f''}_{(0,0,1,1)}$.
    Overwrite $F''_{(v,w)} \leftarrow (1,0,0,1,\ldots,1,0,0,1)$.
    Overwrite $f''(v) \leftarrow 1$.
  while $M^{f''}_{(0,1,1,1)} \neq \emptyset$ **do**
    Pick (v,w) from $M^{f''}_{(0,1,1,1)}$.
    Overwrite $F''_{(v,w)} \leftarrow (1,0,0,1,\ldots,1,0,0,1)$.
    Overwrite $f''(v) \leftarrow 1$.
  return f''

Algorithm 1: Constellations forbidden by the set U are eliminated from f'.

implying that the weight of f' on the entire length-L_e path is at least $L_e/2+2$, consequently, $f'(v) + f'(w) + \sum_{i=1}^{L_e} f'(p_e^i) \geq L_e/2+2$. On the other hand, by construction $f''(v) + f''(w) + \sum_{i=1}^{L_e} f''(p_e^i) = L_e/2+2$, i.e., the same or even a better weight is obtained while maintaining PID-feasibility.

Loop 2: Note that before overwriting, $(p_e^4, \ldots, p_e^{L_e-1})$ is a path of length divisible by 4, hence, according to Lemma 1 its weight will be at least $(L_e-4)/2$ regardless of the values of f'. Necessarily $f'(p_e^2) = f'(p_e^3) = 1$ implying that the weight of f' on the entire length-L_e path is at least $L_e/2+1$, consequently, $f'(v)+f'(w)+\sum_{i=1}^{L_e} f'(p_e^i) \geq L_e/2+1$. The loop produces hence an equal-quality or even better update in terms of weight.

For the last two loops let us state the same argumentation compactly.

Loop 3: Recognize that the f'-weight of $(p_e^4, \ldots, p_e^{L_e-1})$ will be at least $(L_e-4)/2$. Necessarily $f'(p_e^2) = f'(p_e^3) = 1$ implying that the weight of f' on the entire length-L_e path is at least $L_e/2+1$, consequently, $f'(v) + f'(w) + \sum_{i=1}^{L_e} f'(p_e^i) \geq L_e/2+1$ and the in-loop updates yield no worse alternative f''.

Loop 4: Recognize that the f'-weight of $(p_e^3, \ldots, p_e^{L_e-2})$ will be at least $(L_e-4)/2$. Necessarily $f'(p_e^2) = 1$ implying that the weight of f' on the entire length-

L_e path is at least $L_e/2+1$, consequently, $f'(v)+f'(w)+\sum_{i=1}^{L_e} f'(p_e^i) \geq L_e/2+1$ and the in-loop updates yield no worse alternative f''. $\square$

We restate Lemma 2 and provide its proof.

Lemma 2. *Let $k \in \mathbb{N} \cup \{0\}$. Further, let $G = (V, E)$ be a subcubic graph, and $L_e \in \mathbb{N} \cup \{0\}$ be divisible by 4, for $e \in E$. Consider $G' = (V', E')$ resulting from replacing each $e = \{v, w\} \in E$ by an undirected path $(v, p_e^1, \ldots, p_e^{L_e}, w)$ of length $L_e + 1$, i.e., the former edge e is split by L_e fresh vertices. Then, the following assertions are equivalent.*

(i) A PID-feasible f on G with $\sum_{v\in V} f(v) \leq k$ exists.
(ii) A PID-feasible f' on G' with $\sum_{v\in V'} f'(v) \leq k + \sum_{e\in E} L_e/2$ exists.

Proof. In particular values of f' on the undirected paths will be considered (such inter-edge paths are visualized in Fig. 4). Formally, as each undirected path can be traversed in two directions, and as labels have to be assigned to the vertices on the path, let us encode the labels via $F'_{(v,w)} \in \{0, 1\}^{L_{\{v,w\}}}$ determining hence the chronological ordering of the vertex labels (excluding those of v and w) encountered when traversing the path from v to w in G'. Implicitly we assume that $F'_{(w,v)}$ is automatically coherently specified by the reversal of $F'_{(v,w)}$. With a slight abuse of notation, by writing $F'_{\{v,w\}}$ we refer to $F'_{(v,w)}$.

(i) $\Longrightarrow$ (ii): Starting from a feasible f on G, defining $f'(v) := f(v)$, for $v \in V$, the idea is then, for all $e = \{v, w\} \in E$, to choose the remaining f'-values on $(v, p_e^1, \ldots, p_e^{L_e}, w)$ ensuring feasibility, having cumulative weight $L_e/2$, and satisfying

$$f'(v) = f'(p_e^{L_e}) \wedge f'(p_e^1) = f'(w). \tag{6}$$

The latter property emulates for vertices $v \in V$, seen as vertices from G', the neighboring label constellation around each vertex according to f on G.

The choice of the following length-L_e labeled paths ensures such a behavior (note that together with the labeled endpoints $f(v)$, $f(w)$ PID-feasibility is guaranteed on such a path, i.e., at least one 1-labeled neighbor is present for the path's vertices):

$$F'_{(v,w)} := \begin{cases} (0,1,1,0,\ldots,0,1,1,0,\ldots,0,1,1,0) & \text{if } f(v) = 0 \wedge f(w) = 0, \\ (1,1,0,0,\ldots,1,1,0,0,\ldots,1,1,0,0) & \text{if } f(v) = 0 \wedge f(w) = 1, \\ (0,0,1,1,\ldots,0,0,1,1,\ldots,0,0,1,1) & \text{if } f(v) = 1 \wedge f(w) = 0, \\ (1,0,0,1,\ldots,1,0,0,1,\ldots,1,0,0,1) & \text{if } f(v) = 1 \wedge f(w) = 1. \end{cases}$$

The definition of f' on V results hence from the older values of f on V while on the paths' fresh vertices the labels are inferrable from the definition of F'_e according to the above case distinction, which also shows that a weight of only $\sum_{e\in E} L_e/2$ is added to the cumulative weight of f on V.

(ii) $\Longrightarrow$ (i): Due to Lemma 4 we can not only assume f' to be PID-feasible but also without loss of generality $(f'(v), f'(p_e^1), f'(p_e^{L_e}), f'(w)) \notin U$. For all edges

$e = \{v, w\}$, up to symmetry breaking (reversed quadruples) we can therefore only have $(f'(v), f'(p_e^1), f'(p_e^{L_e}), f'(w)) \in T \dot{\cup} W$ with

$$T := \{(1,1,1,1), (0,1,0,1), (0,0,0,0)\},$$
$$W := \{(1,0,1,1), (1,0,0,1), (0,0,0,1)\}.$$

The ultimate goal will be to modify f' towards a more useful version f'' being as well PID-feasibile, having a weight not greater than f' on G' and satisfying for all edges $\{v, w\} \in E$ the special property

$$f''(v) = f''(p_e^{L_e}) \wedge f''(p_e^1) = f''(w). \tag{7}$$

Initially set $f'' := f'$. Immediately we notice that all edges whose behavior is captured by one of the scenarios in T satisfies (7); no updates will hence be needed.

On the other hand, an update leaving the values of f'' unchanged on v and w suffices for the scenarios covered by W:

$$F_e'' \leftarrow \begin{cases} (1,0,0,1,\ldots,1,0,0,1) & \text{if } (f'(v), f'(p_e^1), f'(p_e^{L_e}), f'(w)) = (1,0,1,1), \\ (1,0,0,1,\ldots,1,0,0,1) & \text{if } (f'(v), f'(p_e^1), f'(p_e^{L_e}), f'(w)) = (1,0,0,1), \\ (1,1,0,0,\ldots,1,1,0,0) & \text{if } (f'(v), f'(p_e^1), f'(p_e^{L_e}), f'(w)) = (0,0,0,1). \end{cases} \tag{8}$$

Note that the updated values on $p_e^1, \ldots, p_e^{L_e}$ meet the bound $L_e/2$ (see Lemma 1) and yield an on-par or better weight than the initial values of f'.

After carrying out the updates on all edges, we end up with a labeling f'' fulfilling (7). Therefore, if we contract the inter-vertex paths to the original edges of G, then the constellation of neighboring f''-labels will be unchanged for all vertices $v \in V$. Consequently, the restriction $f := f''|_V$ is the claimed existing labeling. As the weight of f does not include an additional weight $L_e/2$ per edge e, we finally conclude

$$\sum_{v \in V} f(v) \leq \sum_{v' \in V'} f''(v) - \sum_{e \in E} L_e/2 \leq \sum_{v' \in V'} f'(v) - \sum_{e \in E} L_e/2 \leq k.$$

$\square$

We restate Theorem 6 and provide its proof.

Theorem 6. *For $m, n \geq 6$ we have*

$$\frac{mn}{2} + \frac{m+n}{4} + 1 \leq \gamma_{\times 2,\mathrm{t}}(P_m \Box P_n) \leq \frac{mn}{2} + \frac{m+n}{2} + \frac{3 - |\operatorname{mod}(n,4) - 1|}{2}. \tag{9}$$

Proof. Again, as in Theorem 5 the right-hand side of (9) stems from [12].

For the lower bound, use the definitions of the proof of Theorem 5 and repeat its argumentation via the discharging method—this time using the premise $g :=$

g^1 with g^τ defined as in (4): As f is a valid double total dominating function we have

$$\begin{aligned}\sum_{v\in[m]\times[n]} g(v) &= \sum_{v\in D_4} g(v) + \sum_{v\in D_3\setminus B} g(v) + \sum_{v\in B} g(v) + \sum_{v\in D_2} g(v)\\ &\geq \frac{1}{2}\,|D_4| + \frac{5}{8}\,|D_3\setminus B| + \frac{6}{8}\,|B| + \frac{6}{8}\,|D_2|\\ &= \frac{1}{2}(m-2)(n-2) + \frac{5}{8}(2m-8+2n-8) + \frac{6}{8}\cdot 8 + \frac{6}{8}\cdot 4\\ &= \frac{mn}{2} + \frac{m+n}{4} + 1.\end{aligned}$$

Here, the four sums have been estimated from below one by one with the following justification: By feasibility of f, every $v \in D_4$ has at least two neighbors with an f-value of 1; while for $v \in D_4 \setminus A$ this simply allows to conclude that $g(v) \geq 1/4 + 1/4$, for $v \in A \subseteq D_4$ we have to recognize that an inflowing charge of $1/4 + 1/4$ originates solely from the two degree-3 neighbors of v (due to the fact that both neighbors of a corner these are forced to attain an f-value of 1). The estimate for each degree-3 vertex not contained in B holds due to the fact, that again two neighbors with an f-value of 1 must exist leading in the lowest possible case to an inflow of charge $1/4 + 3/8 = 5/8$; concerning the estimate for the vertices in B this lowest case is slightly higher, namely $3/8 + 3/8 = 6/8$. By the aforementioned particularity of a corner, its charge inflow must be precisely $3/8 + 3/8 = 6/8$. $\square$